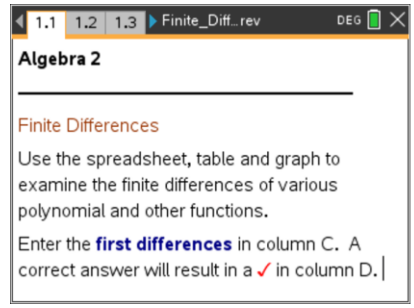


## Math Objectives

- Students will be able to recognize that the first set of finite differences for a linear function will be constant.
- Students will be able to recognize that the second set of finite differences for a quadratic function will be constant.
- Students will be able to determine the relationship between the constant set of first differences and the slope of a linear function.
- Students will be able to recognize that it is not possible to make inferences about the type of function from a set of data without further information.
- Students will construct viable arguments and critique the reasoning of others (CCSS Mathematical Practice).

## Vocabulary

- finite differences
- constant
- degree
- linear, quadratic, and cubic functions
- rate of change
- slope

## About the Lesson

- This lesson involves an investigation into the sets of finite differences for linear and quadratic functions
- As a result, students will:
  - Calculate the first differences in a set of ordered pairs of a linear function.
  - Relate the set of first differences in a linear function to the slope.
  - Calculate the first and second differences in a set of ordered pairs of a quadratic function.
  - Relate the sign of the second differences to the concavity of the quadratic function.
  - Investigate a polynomial function where the second differences in a set of ordered pairs appear to be constant but are not.
  - Investigate a function where none of the differences in a set of ordered pairs are constant.

## TI-Nspire™ Navigator™ System

- Use Screen Capture/Live Presenter to demonstrate the procedure for this activity and to assess student progress
- Use Quick Poll to assess students' understanding of the concepts in this activity.

**TI-Nspire™ Technology Skills:**

- Download a TI-Nspire document
- Open a document
- Move between pages
- Click a slider
- Enter a value in a spreadsheet

**Tech Tips:**

- Make sure the font size on your TI-Nspire handheld is set to Medium.

**Lesson Materials:**

*Student Activity*
Finite_Differences_Student.pdf
Finite_Differences_Student.doc

*TI-Nspire document*
Finite_Differences.tns

Visit [www.mathnspired.com](http://www.mathnspired.com) for lesson updates and tech tip videos.

## Discussion Points and Possible Answers

**Tech Tip:** Replace the zero with the value of the first difference in cell C1. Press enter and a check mark will show in column D for a correct answer. Continue for the rest of the first differences. The next page graphs the set of ordered pairs and a graph of the function. Use the slider to step through the finite differences calculations. The following pages graph the set of ordered pairs and a graph of the function.

**Tech Tip:** If students experience difficulty finding the slider, tell them to look in the upper left corner.

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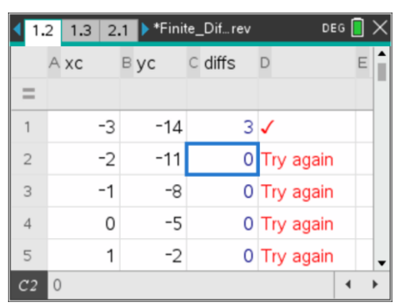
**TI-Nspire Navigator Opportunity: *Screen Capture/Live Presenter***
**See Note 1 at the end of this lesson.**

**Teacher Tip:** When working with data, it is important to know what type of function is represented before making conclusions about the function. Although a function may seem to fit the data, it is not sufficient evidence of a particular function. Unless context suggests a model, or the type of function is given, you cannot make mathematical conclusions from a finite set of data points. If you know the data are from a cubic, the third set of finite differences will be constant, but if you know that as *x* increases by 1, the third set of differences is constant for a given set of values with no other information. The function is not uniquely defined.

**Move to page 1.2.**



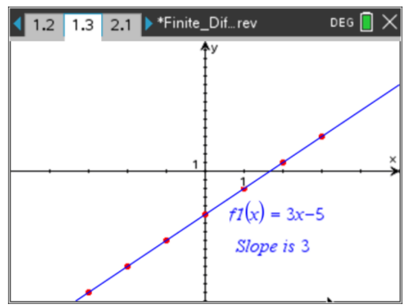

On page 1.2 there is a table of points (*xc, yc*) for a linear function. An interesting property for some functions is called the Finite Differences Method. The set of first differences is $y_2 - y_1, y_3 - y_2, y_4 - y_3, \ldots$ (the value of *y* minus the previous value of *y*) when the *x*-values increase by the same amount. In column C, enter the values of the first differences. A message will tell you if you are correct.

1. What do you notice about the set of first differences?

   **Answer:** The set of first differences is a constant.

**Teacher Tip:** Be certain that students understand how to find the set of differences. Many students will want to subtract $y_1 - y_2$. While subtracting in that order will yield constant differences, the later connections to the slope of the line and the leading coefficient of the quadratic will not be as easy to determine if students subtract in that way. Another possible connection that students may have studied earlier is the notion of sequences. Using the slider on the following pages, you can look at the differences and demonstrate that the second and third differences are zero. This may lead you to talk about the first time the differences are constant.

**Move to page 1.3.**




This is the graph of the linear function with the set of ordered pairs shown on page 1.2. The slope and equation are also given.

2. a. What do you notice about the set of first differences, the slope and the equation?

**Answer:** The set of first differences is the slope of the graph (when the increment of the *x*-value is 1). Because the equation of the line is given in slope-intercept form, the slope is the coefficient of the first degree term in the equation.

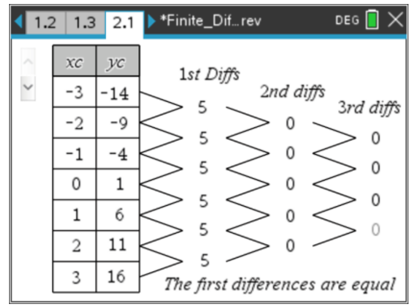
b. Use the linear equation **f**($x$) = $mx + b$ to explain the relationship between the set of first differences and the slope. (Hint: Consider how the value of the function changes for any $x$ and $x + 1$.)

**Answer:** The ordered pairs would be ($x$, $mx + b$) and ($x + 1$, $m(x + 1) + b$). The difference in the *y*-values will be $m(x + 1) + b - (mx + b)$. This simplifies to $m$. The slope of the line will be the change in *y*-values over the change in corresponding *x*-values or $\frac{m(x+1)+b-(mx+b)}{(x+1)-x}$, which also simplifies to $m$. For a linear equation, the set of first differences are constant, and the constant is the slope.

**Teacher Tip:** Students might need to investigate specific linear equations to really understand the connection before they are able to generalize their findings.

**Move to page 2.1.**

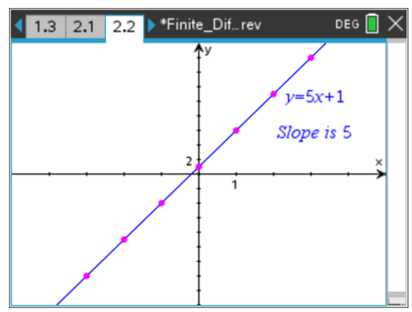
Click the slider in the upper left corner of the screen until the differences are equal.




**Tech Tip:** In order for the graphs on the next pages to show, students must click the slider until the differences are equal. If they click and remove them, the graphs will disappear.

**Move to page 2.2.**

This is the graph of the linear function with the set of ordered pairs shown on page 2.1.



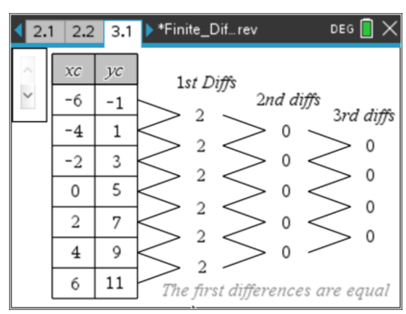

3. Is the slope related to the differences?

   **<u>Answer:</u>** Yes, with an *x*-value increment of 1, the slope is equal to the set of first differences.

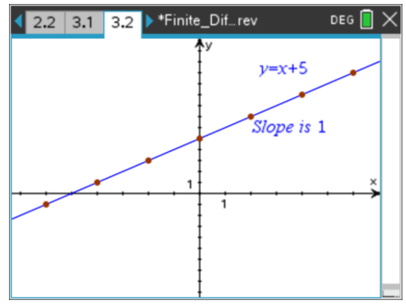
**Teacher Tip:** In order for the finite differences to determine the degree of the polynomial function, the *x*-values must be shown in equal increments. When the increment of the *x*-values is 1, the constant first difference is also the slope of the line.

**Move to page 3.1.**

Click the slider in the upper left corner of the screen again until you get the message that the differences are equal.



**Move to page 3.2.**

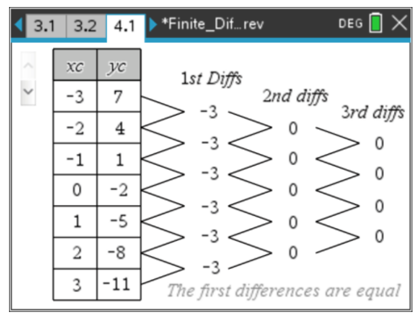



4. What is the relationship between the first set of differences (when the *x*-value increases by something other than 1) and the slope of the line? Explain.

   **<u>Answer:</u>** Because the *x*-value increase is not 1, the slope of the line is determined by rise (set of first differences) over the run (*x*-value differences).

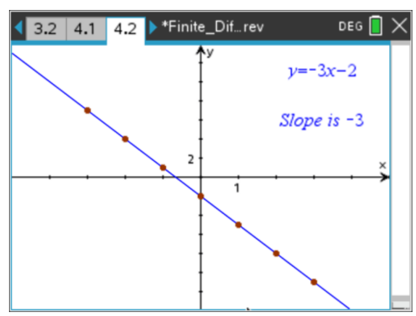
> **Teacher Tip:** By using an increment of something different than 1, this reinforces the idea of slope = $\frac{\Delta y}{\Delta x}$.

**Move to page 4.1.**




Click the slider in the upper left corner of the screen again until you get the message that the differences are equal

**Move to page 4.2.**




5. When the first set of differences is negative, what impact does that have on the graph?

   **<u>Answer:</u>** The slope of the line is negative and therefore goes down from left to right.

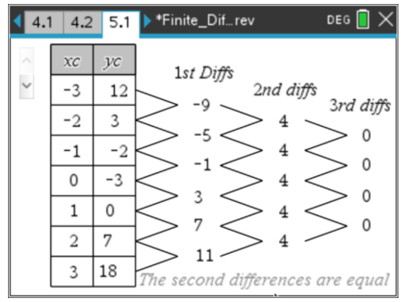
6. Looking at linear data, Meredith subtracted $y_1 - y_2$ and found the constant set of first differences to be 5. Owen subtracted $y_2 - y_1$ and found the constant set of first differences to be –5. What is the slope of the line, assuming that the *x*-values are increasing by 1? Explain why the order in which the subtraction is performed is important.

   **<u>Answer:</u>** The slope of the line is –5. In the formula for the slope, the rise and run must both be calculated in the same direction. In these examples $x_2 - x_1 = 1$. So, in order for the first difference to be equal to the slope, you must evaluate $y_2 - y_1$.

**Teacher Tip:** In order for the finite differences to be constant on the next problems, the student must click the slider  to the second or third differences.

**Move to page 5.1.**

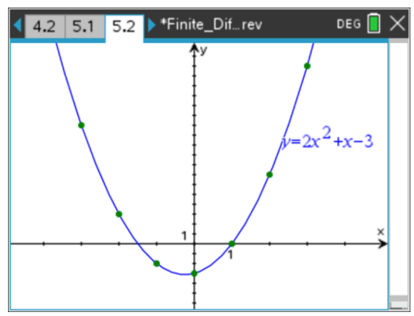



Click the slider in the upper left corner of the screen until the differences are equal.

7. What do you notice about the set of first differences? Second differences?

   **<u>Answer:</u>** The set of first differences is not constant but does increase by a constant amount. The set of second differences is constant.

**Move to page 5.2.**




This is the graph of the quadratic function with the set of ordered pairs shown on page 5.1.

8. a. With an *x*-value increase of 1, what seems to be the relationship between the second differences and $a$, the leading coefficient in the equation?

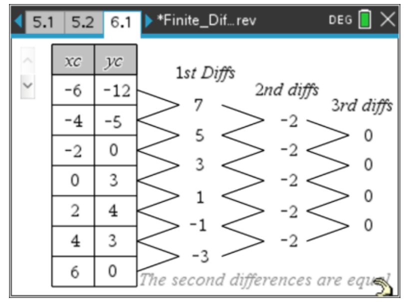
   **<u>Answer:</u>** The value of the constant second differences, 4, is equal to double the leading coefficient in the quadratic equation, 2, when the $x$-value increases by 1.

   b. Tanesia made a conjecture that the rate of change for the quadratic function is a linear function. Does her conjecture seem reasonable? Why or why not?

   **<u>Sample answer:</u>** Some students might think her conjecture is reasonable because if the second differences are constant, the function that generated them would be linear. Some might have trouble thinking about rate of change varying. This is an opportunity to refer to the graph and explore what would have to be true if the rate of change were constant.

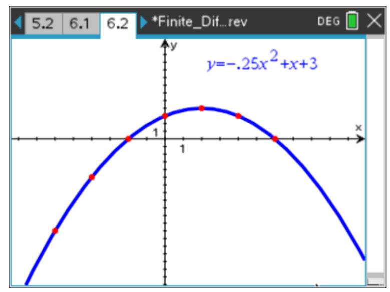
**Teacher Tip:** Ask students to describe the rate of change of the quadratic and to think about how the rate of change is changing.

**Move to page 6.1.**




Click the slider in the upper left corner of the screen until the differences are equal.

**Move to page 6.2.**

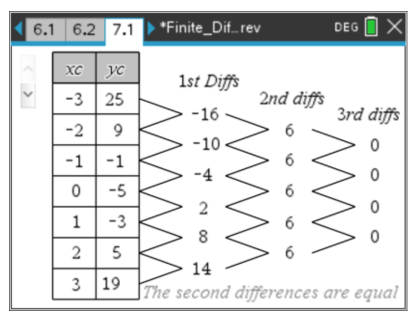



This is the graph of the quadratic function with the set of ordered pairs shown on page 6.1.

9. With an *x*-value increase of 2, what is the relationship between the second differences and *a*, the leading coefficient in the equation?

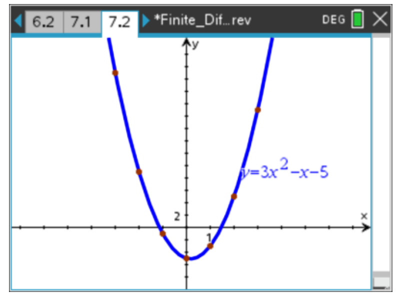
   **<u>Answer:</u>** The value of the set of constant second differences is equal to double the leading coefficient in the quadratic equation (as before), but also multiplied by the increment squared.

**Move to page 7.1.**




Click the slider in the upper left corner of the screen until the differences are equal.

**Move to page 7.2.**




This is the graph of the quadratic function with the set of ordered pairs shown on page 7.1.

10. Regardless of the *x*-value increase, what is the relationship between the second differences and *a*, the leading coefficient in the equation?

    **Answer:** The difference is equal to $2a(\Delta x)^2$. The second difference will have the same sign as the leading coefficient *a*.

11. Revisit the three quadratics from pages 5.2, 6.2, and 7.2. Remember what you learned earlier about the relationship between the sign of the leading coefficient *a* and the direction in which the quadratic opens.

    a. Make a prediction about the relationship between the constant difference and the sign of the leading coefficient, *a*.

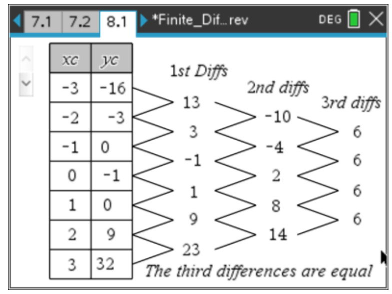
    **Sample answer:** When the quadratic opens down, the set of constant differences will be negative; when the quadratic opens up, the set of constant differences will be positive.

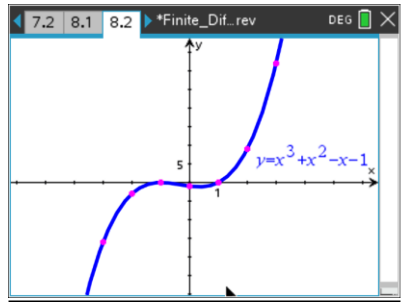
    b. Use the graph of the function and what you know about the rate of change of a quadratic function to explain why your prediction is reasonable.

    **Answer:** For any *x*, when the curve opens up, the rate of change is always increasing; the second difference can be thought of as the "rate of change" of the rate of change of the quadratic function. When this number is positive, the curve will open up. In earlier work, students learned that when $a > 0$, the quadratic opens up. Likewise, when the curve opens down, the rate of change is always decreasing, and the "rate of change" of the rate of change is negative, which matches earlier work that suggested when $a < 0$, the quadratic opens down.

> **Teacher Tip:** Students may need to sketch or estimate (using points and intervals from the graph) how the rate of change behaves for a quadratic. It might also help to have them think about velocity and acceleration, where acceleration is the rate at which the velocity is changing. For example, if you brake hard while driving, the velocity of the car changes rapidly; if you brake gently, the velocity slowly decreases.

**Move to pages 8.1 and 8.2.**




The points on page 8.1 are from a cubic equation.

12. Predict how many subtractions it will take until the differences are constant. Check your prediction by clicking the slider until the differences are equal.

    **Sample answer:** It might take three subtractions until the differences are constant. It almost seems as if the first differences are quadratic, the second are linear, and the third differences then have to be constant.

13. Which set of finite differences would be a constant for a polynomial of $n^{th}$ degree? Explain your reasoning.

    **Sample answer:** The set of $n^{th}$ differences will be constant for a polynomial of $n^{th}$ degree. One way to think is to work backwards; it seems as if when the first set of differences is a constant, they came from a first degree equation (linear); when the second differences are constant, the first differences came from a linear or first degree equation, so it might make sense to think that if you kept going up from the constant, $n – 1$ sets of differences, you would have started with a function of degree $n$.

**Teacher Tip:** In general, the coefficient of highest power, $x^n$ , is the constant $n^{th}$ finite difference divided by $n!$, divided by the $n^{th}$ power of the change in the $x$-values.

14. Summarize your results from the investigation.

    a. For a linear function, if the first set of differences is a positive constant, the graph has a positive slope. $\left(\text{slope} = \frac{\text{the constant}}{\Delta x}\right)$

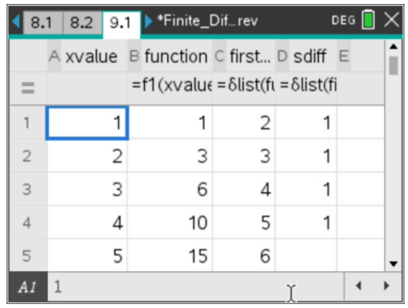
    b. For a linear function, if the first set of differences is a negative constant, the graph has a negative slope. $\left(\text{slope} = \frac{\text{the constant}}{\Delta x}\right)$

c. For a quadratic function, if the second set of differences is a positive constant, the graph <u>opens upwards. $\left(a = \frac{\text{constant}}{2\cdot(\square x)^2}\right)$</u>

d. For a quadratic function, if the second set of differences is a negative constant, the graph <u>opens downwards. $\left(a = \frac{\text{constant}}{2\cdot(\square x)^2}\right)$</u>

**Move to page 9.1.**

15. a. Suppose the terms of a sequence are given as 1, 3, 6, 10, 15, 21, …. Make a conjecture about the next three elements in the sequence. Explain the rule you are using.




**<u>Sample answer:</u>** Students may notice that the difference in the elements is increasing by 1: 1 + 2 = 3, 3 + 3 = 6, 6 + 4 = 10, etc. Therefore, they would expect the next three elements to be 21 + 7 = **28**, 28 + 8 = **36**, 36 + 9 = **45.**

b. The function $\mathbf{f}(x) = \frac{-x^7}{504} + \frac{x^6}{18} - \frac{23x^5}{36} + \frac{35x^4}{9} - \frac{967x^3}{72} + \frac{239x^2}{9} - \frac{178x}{7} + 10$ gives the values of the sequence from part 15a. when the *x*-values of 1 to 6 are substituted into the equation. Input 7, 8, and 9 in the *x-value* column of the spreadsheet and complete the table below.

**<u>Answers:</u>** (bolded)

| *x*-value | Function (f(*x*) from above) | First Differences | Second Differences |
|---|---|---|---|
| 1 | 1 | 2 | 1 |
| 2 | 3 | 3 | 1 |
| 3 | 6 | 4 | 1 |
| 4 | 10 | 5 | 1 |
| 5 | 15 | 6 | **1** |
| 6 | 21 | **7** | **–9** |
| **7** | **28** | **–2** | **–59** |
| **8** | **26** | **–61** | |
| **9** | **–35** | | |

c. Look at the set of first and second differences for these values in the completed table. How do the results compare to your prediction in part 15a?

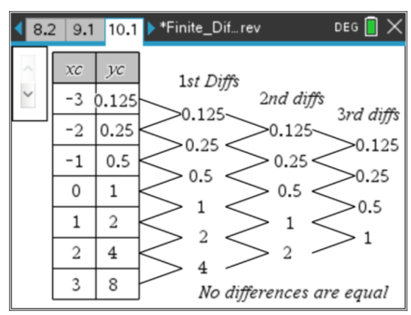
**Sample answer:** It appears for the first 7 terms that the set of second differences will be constant and equal to 1, but the 8$^{th}$ and 9$^{th}$ values do not follow this pattern.

> **Teacher Tip:** Students should recognize that appearances in patterns may be misleading. An interesting geometric pattern they might explore is the number of regions obtained by connecting points on a circle (1 point, 1 region; 2 points, 2 regions; 3 points, 4 regions; 4 points, 8 regions; 5 points, 16 regions; 6 points, 31 regions; 7 points, 57 regions). A detailed explanation can be found on several Web sites, (e.g., www.math.toronto.edu/mccann/assignments/199S/regions.pdf.)

> **Teacher Tip:** The equation was generated by creating a system of 7 equations in 8 unknowns using the seven ordered pairs (1, 1); (2, 3); (3, 6); (4, 10); (5, 15); (6, 21); and (7, 28), then specifying the constant term as 10. Other values for the constant term will produce other equations and thus generate other sequences.

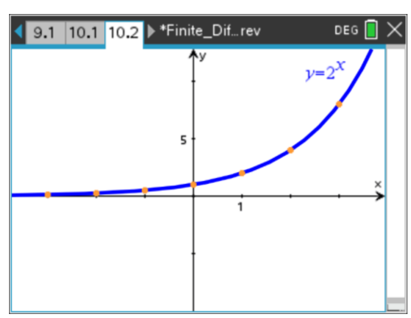
> **Teacher Tip:** In case students get the idea that all functions have sets of finite differences that are constants, this lesson is extended for non-polynomial functions. Question 16 looks at an exponential function, $y = 2^x$ .

**Move to pages 10.1 and 10.2.**




16. a. What is different about the sets of finite differences and the graph of this function, compared to the others you have looked at in this activity?

**Answer:** The difference is that sets of finite differences do not become a constant. In fact, it should be obvious that the same set of finite differences is generated each time. This is not a polynomial function but an exponential function.



b. Explain why the pattern of differences repeats in the way it does.

**Sample answer:** The sequence is generated by multiplying by 2 to obtain each succeeding element, so for all $y_{n+1} - y_n$ the differences will be $2^{n+1} - 2^n = 2^n(2-1) = 2^n$. This will happen over and over again for every set of differences.

**Teacher Tip:** Another discussion is to relate all of the differences to the notion of rate of change The first differences give the rate of change, the second differences are the rate of change of the first differences, etc. Students may continue the study of finite differences in calculus.

**Teacher Tip:** Using real data for linear or quadratic scenarios, the differences may not ever be constant.

**TI-Nspire Navigator Opportunity: *Quick Poll***
**See Note 2 at the end of this lesson.**

## Wrap Up

At the end of the discussion, students should understand:

- That the set of first differences of a linear function is a constant.
- That the set of second differences of a quadratic function is a constant.
- When looking at linear data and when the $x$ -value increases by 1, the value of the constant set of first differences is the slope of the line.
- Not all sets of differences eventually become constant, and some patterns that appear to do so may be misleading.

## Assessment

Sample Questions:

1. What degree of polynomial has the fifth set of finite differences as a constant?
   a. Linear: degree 1
   b. Quadratic: degree 2
   c. Cubic: degree 3
   d. Quartic: degree 4
   e. Quintic: degree 5

2. Given polynomial data, if the set of first differences is –3, what can you tell about the polynomial?
    a. The polynomial is linear with a positive slope.
    b. The polynomial is linear with a negative slope.
    c. The polynomial is quadratic and opens upwards.
    d. The polynomial is quadratic and opens downwards.

## TI-Nspire Navigator

**Note 1**

**Questions 1–5, *Screen Capture/Live Presenter:*** You may want to use *Live Presenter* to demonstrate the correct procedure for entering data and for using the sliders. You could also use *Screen Capture* to show pages 2.2, 3.2, and 4.2 at the same time to discuss the similarities found with linear functions.

**Note 2**

**End of Lesson: *Quick Poll:*** You may want to assess students' understanding using *Quick Poll* with questions like the sample ones above.