NEWTON'S LAW OF COOLING

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(Please feel free to email me questions and /or comments.)

Key Topic: Growth and Decay

Abstract:

Newton's Law of Cooling can be used to model the "growth" or "decay" of the temperature of an object over time. In particular, this law states that the rate at which the temperature of an object changes over time is proportional to the difference between the temperature of the object and the temperature of the surroundings. So the temperature of the object is modeled by a differential equation.

In this activity we show how to use this model to solve a murder. In the example provided in the activity, the murder takes place indoors. So the surrounding temperature is a constant room temperature. In the exercise assigned to students, the murder takes place outside. So the students must also devise a model for the surrounding temperature.

Detailed instructions are given on how to use the TI-89 to solve the model.

Prerequisite Skills: Knowledge that the derivative can be used to model rate of change.

Degree of Difficulty: Easy to moderate

Needed Materials: TI-89

NCTM Principles and Standards:

- Content Standards – Algebra
  - Represent and analyze mathematical situations and structures using algebraic symbols
  - Use mathematical models to represent and understand quantitative relationships
  - Draw a reasonable conclusion about situation being modeled

- Process Standards
  - Representation
  - Connections
  - Problem Solving

NEWTON'S LAW OF COOLING

On a cold 26^{o}F December day in Chicago, Detective Daniels went to a large apartment complex to investigate a murder. When he arrived at noon, Sergeant Spencer said that there were many suspects, but they were having trouble narrowing the list since they didn't know the exact time of death. Detective Daniels took out a thermometer and measured the temperature of the body, finding it to be 77.9 ^{o}F. He also noted that the thermostat in the room was set at 72^{o}F. An hour later at 1:00 P.M., he found the body temperature to be 75.6^{o}F. He then left for lunch announcing that when he returned, he would tell them when the murder was committed. How did he plan do this?

At first it looks like Detective Daniels doesn't have enough information to find the time the murder was committed. But Detective Daniels remembered seeing Newton's Law of Cooling when he took calculus. But he didn't remember precisely what the law was. So when he went home for lunch, he borrowed his daughter's calculus book and looked it up. What he found was:

> **Newton's Law of Cooling:** The rate at which the temperature of an object changes is proportional to the difference between the temperature of the object and the temperature of the surroundings.
>
> **Mathematically,** this says:

$$\frac{dy}{dt} = k(y - s)$$

where:

$t =$ time

$y(t) =$ temperature of the object at time t

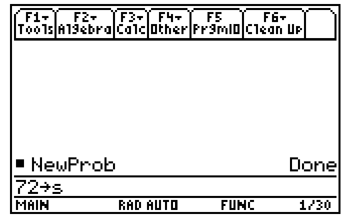
$s(t) =$ temperature of the surrounding area at time t

$k =$ the constant of proportionality

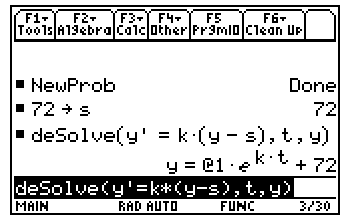
After reading this, Detective Daniels realized that he had forgotten how to solve a differential equation. So he asked his daughter for help. His daughter knew that there was no way she could give her dad a refresher course in differential equations in 15 minutes, so she showed him how to solve the problem using her TI-89 calculator. Here's what she told her dad to do.

***Whenever you start a new problem, clear memory by pressing* [2nd][F6]2.**

1. Set the T-89 to “function” mode by pressing [MODE][▶][6], and while you are there, make sure **Exact/Approx** is set to **AUTO** by pressing [F2][▲][▲][▶][1] [ENTER].



2. Enter the surrounding temperature by storing 72 in **s**. To do this press [7][2]→[alpha] **s**[ENTER].

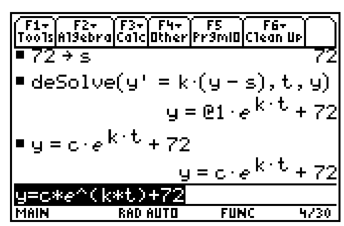


2. First solve Newton’s Law of Cooling by pressing [F3][alpha] **c** [Y][2nd][′][=][alpha] **k** [×][(][Y][−][alpha] **s** [)][,][T][,][Y][)][ENTER].

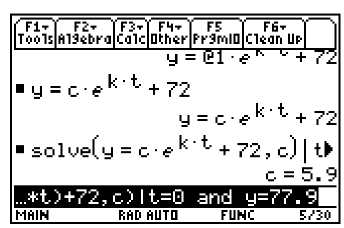
Note: [′] is above the [=] key.

Note: **t** is the independent variable denoting time, and
y is the dependent variable denoting the temperature of the body at time **t**.

3. When integrating, the TI-89 does not tack on the constant of integration for you. But when solving a differential equation, which is done using complicated techniques of integration, that constant of integration is not always as simple as tacking on a $+C$ at the end of the answer. In the TI-89 solution to Newton’s Law of Cooling which we just found, the constant of integration is an integral part of the equation which is denoted by @n, where n is some number. The @ symbol can be accused, as we will soon need to do, by looking in the end of [CATALOG].



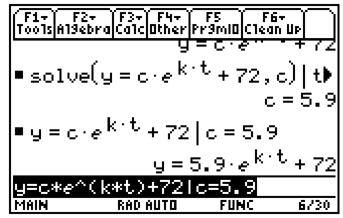
But this is not always as easy as it sounds. It’s easier to edit our TI-89 solution by replacing the @n symbol with the letter **c**. To do this, press [▲][ENTER] to place the solution on the command line, then left arrow to the right of the @n symbol, press [←] twice to delete it, and replace it with the letter **c** by pressing [alpha] **c**. Then press [ENTER].



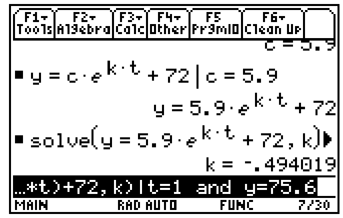
4. Time $t=0$ occurred at noon when the body temperature was found to be 77.9 ^{o}F. This will allow us to solve for **c** by pressing [F2][ENTER][▲][ENTER][,][alpha] **c** [)][|][T][=][0] [2nd][MATH][8][8][Y][=][7][7][.][9][ENTER].

Note: The word “and” could have been entered using the [alpha] provided you put a space before and after this word. The space symbol [␣] is above the negation key [(-)].

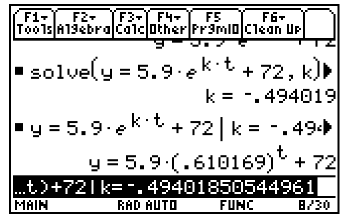
5. Let's put this value for **c** back into our equation before solving for **k**. To do this press ▲▲▲[ENTER][|]▲[ENTER][ENTER].



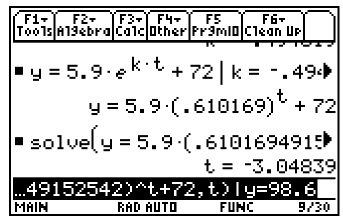
6. Now solve for **k** by using the fact that at 1 p.m., the temperature of the body was 75.6^{o}F. By now you should be able to do this on your own.



7. And put this value for **k** back into the equation for **y**.



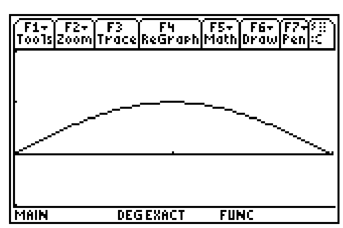
8. We are now ready to find out when the murder was committed. To do this, we have to find the value of **t** when the body was at normal body temperature, which is 98.6^{o}F.



$t \approx -3.05$ indicates that the murder occurred at 9 a.m., approximately 3 hours before noon.

EXERCISE: At noon next day, Detective Daniels arrived on the scene of another murder. But this one took place in a vacant lot. He took the temperature of the body and found that it was 60^{o}F. He also noted that the outdoor temperature was 26^{o}F. An hour later he found that the temperature of the body had dropped to 40^{o}F. He then called the weather bureau to find out what the temperature was at 9 a.m., being informed that it was 18^{o}F. He then used his daughter's TI-89 to find the time of the murder. What time did he find?

Note that in the great outdoors, the surrounding temperature is *not* constant. It is a function of time. One half the period of *some* sine function is usually used to model temperature during a 24 hour period of time. But $x = 0$ is not midnight since the coldest temperature of the night usually occurs between 2 and 5 a.m.

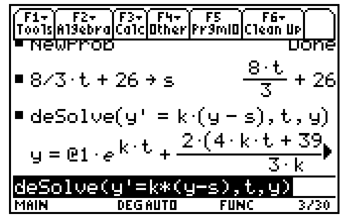


However, when looking at the change in temperature over a period of only a few hours, as you are in this exercise, it is safe to assume a linear model for time.

SOLUTION:

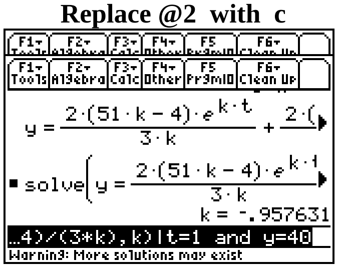
Whenever you start a new problem, clear memory by pressing [2nd][F6]**2.**

Since $t = 0$ occurs at noon, the linear model for the outside temperature s passes through the points $(-3, 18)$ and $(0, 26)$. So the equation for s is $s = \frac{8}{3}t + 26$. Store this value in **s** and then solve Newton's Law of Cooling.



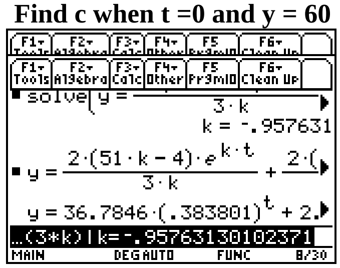
From here, all you have to do is follow steps 3 through 8 in the above example to solve the problem. The screen shots for these steps are:

Replace @2 with c



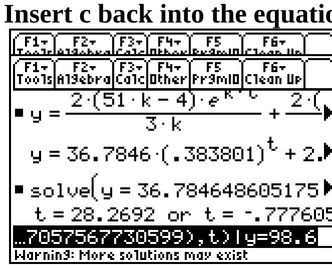
Find k when t =1 and y = 40

Find c when t =0 and y = 60



Insert k back into the equation

Insert c back into the equation



Find t when y =98.6

Since $t \approx -0.8$ hours, the murder took place approximately 48 minutes before noon. That is, at 11:12 a.m.