# AIR RESISTANCE

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(Please feel free to email me questions and /or comments.)

**Key Topic:** Motion

**Abstract:**

In this activity the student will use the tools of integral calculus to investigate the motion of a free falling object which is governed both gravity and air resistance. The student is ready for this activity after she/he has had some experience solving simple differential equations.

**Prerequisite Skills:**

- Experience solving simple differential equations
- Experience with the mathematics of free fall motion when air resistance is ignored (preferred, by not required)

**Degree of Difficulty:** Easy to moderate

**Needed Materials:** TI-89

**NCTM Principles and Standards:**

- Content Standards – Algebra
  - Represent and analyze mathematical situations and structures using algebraic symbols
  - Use mathematical models to represent and understand quantitative relationships
  - Draw a reasonable conclusion about situation being modeled

- Process Standards
  - Representation
  - Connections
  - Problem Solving

# AIR RESISTANCE

When calculus students are first introduced to free fall motion and projectile motion, the retarding force caused by air resistance is ignored. The reason for this is not because air resistance is a mathematically difficult force to deal with. Quite the contrary. Air resistance is simply a force which is proportional to the velocity of the object. But when you add the effects of both gravity and air resistance, solving the resulting differential equation is beyond the scope of abilities of most first year calculus students.

Since the TI-89 will solve first order differential equations for us, we won't have any trouble evaluating the resulting differential equation. So let's look at free fall motion when the falling object is subjected to both the force $F_G$ of the earth's gravity and the force $F_R$ caused by air resistance.

We will employ the following notation:

$a = a(t) =$ the acceleration of the object at time $t$.
$v = v(t) =$ the velocity of the object at time $t$.
$s = s(t) =$ the distance of the object from the surface of the earth at time $t$.

$g = 32\text{ ft/s}^2 = 9.8\text{ m/s}^2 =$ the gravitation constant
$v_0 =$ the initial velocity of the object
$s_0 =$ the initial distance of the object from the surface of the earth
$m =$ the mass of the object

Now we have two forces acting on the falling object: the force $F_G$ of the earth's gravity and the force $F_R$ caused by air resistance. $F_G$ is a downward force, so it is negative, and $F_R$ is an upward force which is positive. From physics we know two things: $F_G = -mg$ and $F_R$ is, in many situations, proportional to the velocity $v$ of the falling object. So $F_R = -kv$ for some positive constant $k$. (It should be noted that since the object is falling to earth, $s$ is decreasing. So velocity $v$ is negative. Hence $F_R$ is indeed positive. It should also be noted that $k$ depends on the objects shape and the properties of the air.)

Since the total force acting on the object is $F = F_G + F_R$, we see that

$$F = -mg - kv\,.$$

Now Newton's second law of motion states that $F = ma$ where $F$ is the sum of the forces acting on the object, $m$ is the mass of the object, and $a$ is the acceleration of the object. So this gives us

$$ma = -mg - kv\,,$$

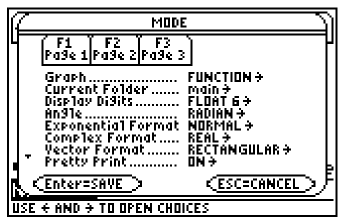
which simplifies to $$a = -g - \frac{k}{m}v$$

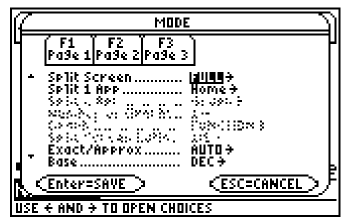
where $a$ is the acceleration of the object, $v$ is its velocity, and $g$ and $\frac{k}{m}$ are constants.

Since $a(t) = v'(t)$, $s(t) = v'(t)$, $v(0) = v_0$, and $s(0) = s_0$, we can use the above equation to find the velocity and distance functions of the free falling object. In the following we show you how to use the TI-89 to do this.

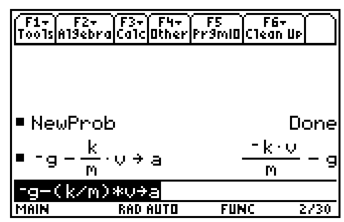
***Whenever you start a new problem, clear memory by pressing* [2nd][F6]2.**

Set the first two pages of the **MODE** as shown at the right.



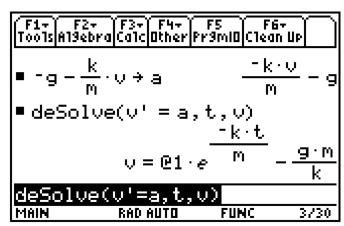




Define the acceleration function $a = -g - \frac{k}{m}v$ by storing the right side of the equation in the variable $a$.
[(-)][alpha] **g** [−][(][alpha] **k** [÷][alpha] **m** [)][×][alpha] **v**
[STO▸][alpha] **a** [ENTER].



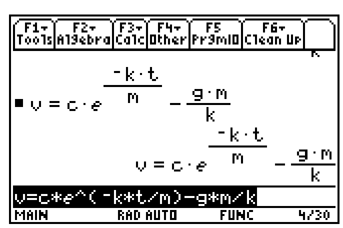

Solve the differential equation $v' = a$ (which is a function of $t$) for the variable $v$.
[F3][alpha] **c** [alpha] **v** [2nd][′][=][alpha] **a** [,][T][,][alpha] **v**
[)][ENTER] .

**Note:** [′] is above the [=] key. Also, **deSolve** can also be accessed by pressing [F3] followed by pressing ⊖ eleven times, and then pressing [ENTER].



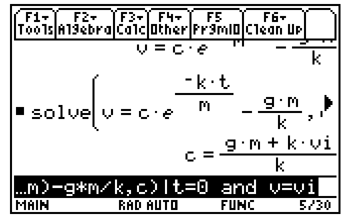

The constant of integration appears in the solution as @1. Although the @ symbol can be found in [CATALOG], doing such is not always that easy.

It's easier to edit our TI-89 solution by replacing the @1 symbol with the letter **c**. To do this, press ⊖[ENTER] to place the solution on the command line, then left arrow to the right of the @1 symbol, press [←] twice to delete it, and replace it with the letter **c** by pressing [alpha] **c**. Then press [ENTER].

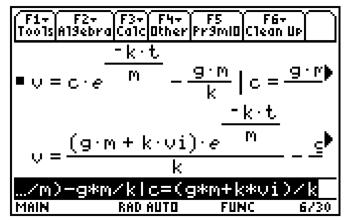


We now want to solve for $c$ when $t=0$ and $v=v_0$. We need a TI-89 compatible symbol for $v_0$. Since the TI-89 will interpret the symbol $v0$ as $v \cdot 0 = 0$, let's use $vi$ to denote initial velocity.

Solve the last equation for $c$ when $t=0$ and $v=vi$.
[F2][ENTER][▲][ENTER][,][alpha] **c** [)][|][T][=][0]
[2nd][MATH][alpha] **v** [=][alpha] **v** [alpha] **i** [ENTER].




**Note:** The word "and" could have been entered using [alpha] provided you put a space before and after this word. The space symbol [␣] is above the negation key [(-)].




Let's put this value for **c** back into our equation for velocity. To do this press [▲][▲][▲][ENTER][|][▲][ENTER][ENTER].

So the velocity function for this free falling object is

$$v(t) = \left( \frac{mg}{k} + v_0 \right) e^{-kt/m} - \frac{mg}{k}.$$

Note that since $k$ is positive, $\lim_{t \to \infty} |v(t)| = \frac{mg}{k}$. This says that the speed of the object will eventually reach a *terminal speed* equal to $\frac{mg}{k}$.

This is not the case when air resistance is ignored. In this case $v(t) = -gt + v_0$, and thus $\lim_{t \to \infty} |v(t)| = \infty$.

The techniques outlined above can be used to get the TI-89 to find the distance function $s(t)$. You may want to do such. If you do, the answer is equivalent to

$$s(t) = -\frac{m}{k}\left( \frac{mg}{k} + v_0 \right) e^{-kt/m} - \frac{mg}{k} t + \frac{m}{k}\left( \frac{mg}{k} + v_0 \right) + s_0.$$