

Tessellations and Tile Patterns

Definitions:

- **Tessellation**—covering, or **tiling**, of a plane with a pattern of figures so there are no overlaps or gaps.
- **Monohedral tiling**—tessellation made up of congruent copies of one figure.
- **Dihedral tiling**—tessellation using two different congruent figures.
- **Trihedral tiling**—tessellation using three different congruent figures.

(Grunbaum and Shephard, 1987.)

Construct and Investigate:

1. Using the **Regular Polygon** tool, determine which regular polygons form monohedral, dihedral, or other tessellations of a plane. Draw a regular polygon, and use the **Reflection** tool to reflect the polygon across one of its own sides. Repeat this procedure on some or all sides of the polygon and its reflections to show a tiling pattern (Figure 1). Classify the regular polygons by the types of tiling patterns they make.

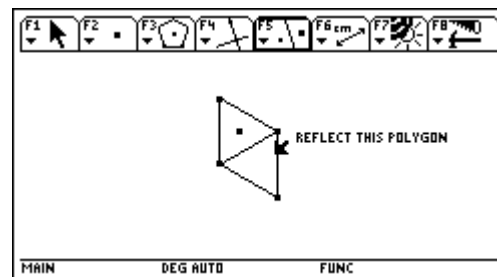


Figure 1

2. Use the transformational geometry tools **Translation**, **Rotation**, **Reflection**, and **Symmetry** to experiment with ways to tessellate the plane with an isosceles triangle. Does an isosceles triangle always tessellate the plane? Explain with examples or counterexamples, or both.
3. Experiment with ways to tessellate the plane with a scalene triangle. Does a scalene triangle always tessellate the plane? Explain by showing examples and counter examples.
4. Find at least two ways to tessellate the plane with a rectangle. Are there special cases that work better than others? Give examples and explain. Does any rectangle always tessellate the plane?
5. Experiment with ways to tessellate the plane with a trapezoid. Some trapezoids tessellate because of their special properties (see Figure 2 and the chapter *Investigating Properties of Trapezoids*). Does a general trapezoid always tessellate the plane? Explain with examples and/or counterexamples.
6. Rectangles and trapezoids are quadrilaterals. Experiment to see whether every quadrilateral tessellates the plane? What about a concave quadrilateral? Explain with examples or counterexamples, or both.

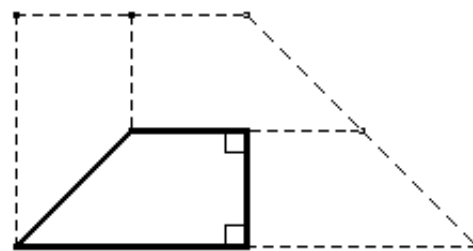


Figure 2

Explore:

1. In part 1 above, you discovered that a regular pentagon will not make a monohedral tiling of the plane. Experiment to see whether any pentagons make a monohedral tiling of the plane. Show examples, and give the conditions under which these figures produce a tessellation.
2. A regular hexagon makes a monohedral tiling of the plane. Find other hexagons that also make a monohedral tiling pattern. Show examples that work, and explain why.

Teacher's Guide: Tessellations and Tile Patterns

Construct and Investigate:

1. The extent of this activity is limited only by students' willingness to experiment. Many more patterns exist than can be shown here. Have students drag the original figure to change its size and show that the pattern remains unchanged. Printing a hard copy of students' work will preserve interesting patterns they discover. Let students be creative. Ask them to explain why certain patterns work.

Three regular polygons, triangles, squares, and hexagons tessellate the plane with a monohedral pattern (Figures 3 and 4).

Ask students to consider the degree measure of the interior angles of each regular polygon and what happens at a common vertex where a number of these figures come together. How many degrees must be accounted for so that no gaps or overlaps occur?

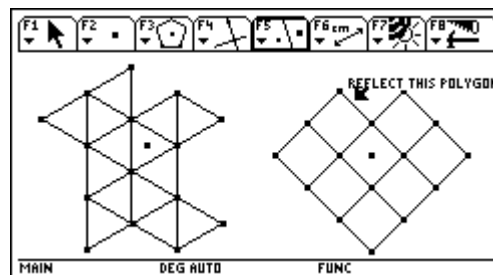


Figure 3

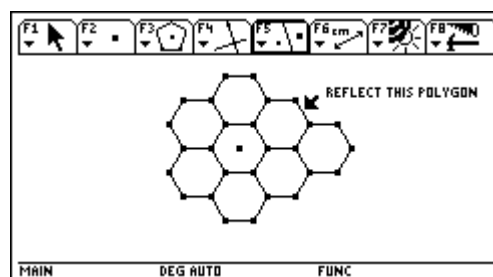


Figure 4

Other regular polygons form dihedral tiling patterns.

A regular pentagon together with a rhombus with sides equal to the length of the side of the pentagon tessellates the plane (Figure 5).

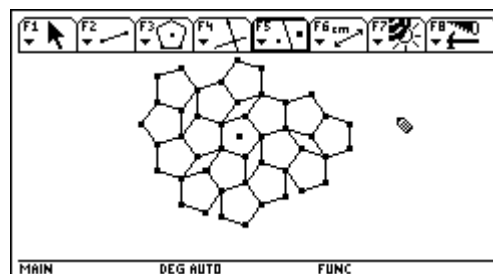


Figure 5

A regular heptagon together with a pentagon tessellates the plane. In this tiling pattern, is the pentagon regular (Figure 6)?

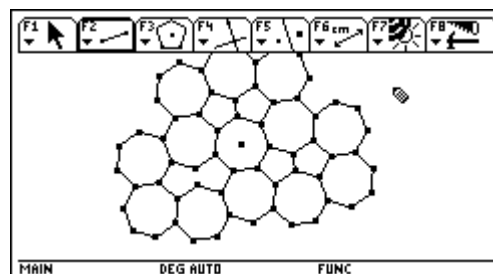


Figure 6

Teacher's Guide: Tessellations and Tile Patterns (Cont.)

A regular octagon and a square tessellate the plane (Figure 7).

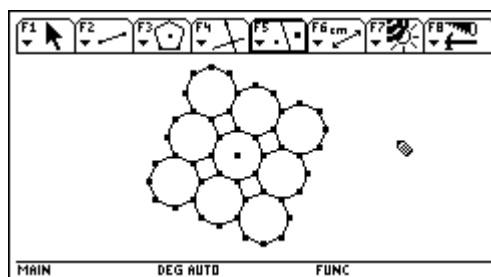


Figure 7

A regular dodecagon and an equilateral triangle tessellate the plane (Figure 8).

- Reflecting an isosceles triangle over its own sides does not necessarily produce a monohedral tessellation unless the triangle is an equilateral or an isosceles right triangle. Place a vector on each side of the original isosceles triangle. Use the **Translate** tool to translate the original triangle, or any copies, by these vectors to produce a monohedral tiling pattern (Figure 9).

Another way to produce the same tile pattern is to use **symmetry** (a half-turn or point reflection). Symmetry reflects the triangle through the midpoint of any side or through the midpoint of any side of a reflected triangle (Figure 10).

The same construction can be done using the **Rotation** tool with a 180° angle of rotation. Once the students establish a pattern, it is very important that they drag the original figure to confirm that the pattern does not depend on only one shape but is true for all isosceles triangles.

- All of the methods that work for a general isosceles triangle also work for a scalene triangle.

Figure 11 shows the original triangle and midpoint reflections of this triangle. Drag the vertices of the original triangle to show that these techniques work for any scalene triangle.

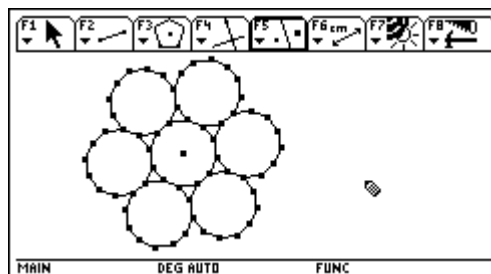


Figure 8

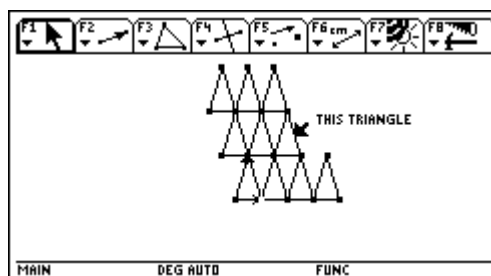


Figure 9

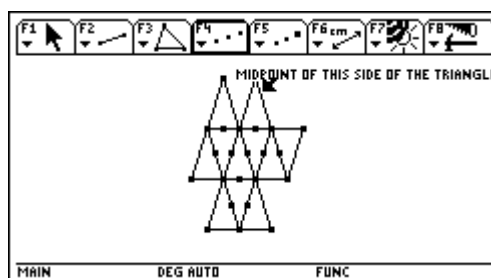


Figure 10

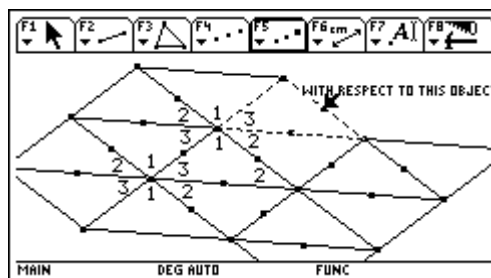


Figure 11

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To understand why reflecting through a midpoint produces a tessellation, number the original angles and follow the position of these angles through the point reflections. At each common vertex, the three angles of the original triangle are each represented twice, resulting in a total of 360° (Figure 11).

Figure 12 shows the vector translation of the original triangle producing the same tiling pattern.

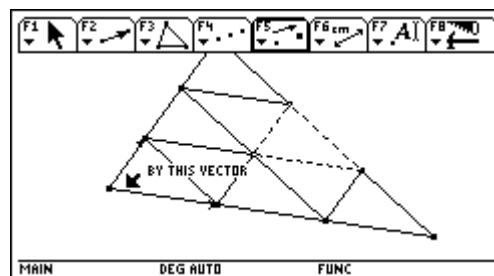


Figure 12

4. You can tessellate the plane with a rectangle in the following three ways:
 - Reflect the rectangle with respect to its sides.
 - Translate the rectangle through vectors equal to the side lengths of the figure.
 - Rotate the rectangle through 180° about the midpoint of any side using the **Rotate** tool, or do a half-turn (or point reflection) using the **Symmetry** tool.
5. In general, any trapezoid tessellates the plane. Reflect the original trapezoid (or any images of it) through the midpoints of its sides to produce the monohedral tiling pattern shown in Figure 13.

A reflection through a point is the same as a 180° rotation about that point. This rotation is also known as a half-turn. Drag the vertices of the original trapezoid to show that the pattern is the same for any trapezoid.

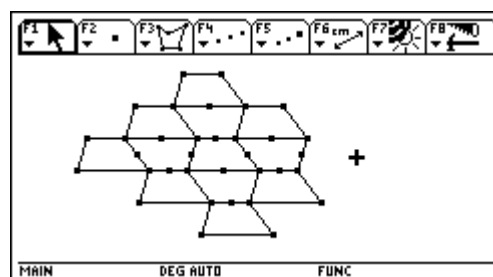


Figure 13

The trapezoid in Figure 2 tessellates the plane using two reflections and a rotation about a vertex or two reflections and a translation.

Figure 14 shows both methods.

- The first-level tessellation is two reflections and a translation. (Notice the vector on the slanted side of the trapezoid in the lower right-hand corner.)
- The second-level tessellation is two reflections and a rotation of 90° about the midpoint of the shorter base. This trapezoid is a combination of a square and half of the same square.

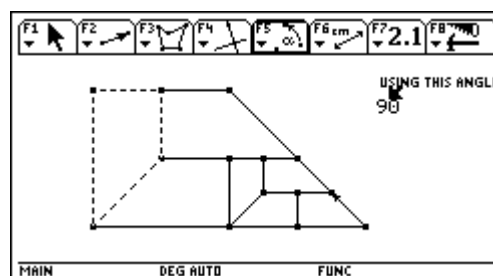


Figure 14

Teacher's Guide: Tessellations and Tile Patterns (cont.)

Other special trapezoids tessellate the plane by using some or all of the transformations available on the Voyage™ 200.

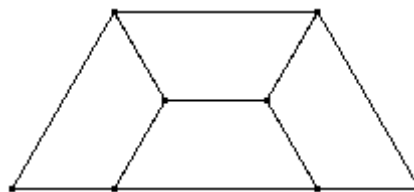


Figure 15

The trapezoids in Figure 15 and Figure 16 are made of three congruent equilateral triangles with all angles either 60° or 120° .

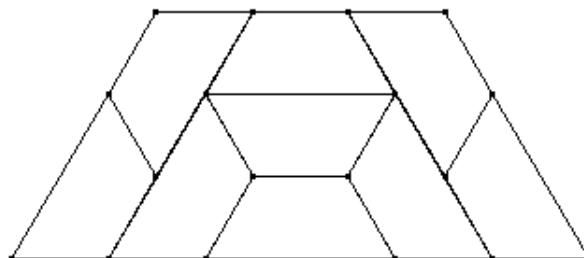


Figure 16

Figure 17 shows trapezoids formed by two congruent squares combined with a 30° 60° 90° triangle.

Because these examples can be dissected or expanded into infinitely many identical parts, all represent monohedral tilings.

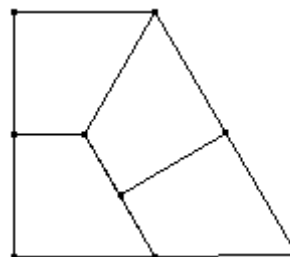


Figure 17

6. Any quadrilateral tessellates the plane by a 180° rotation about the midpoint of its sides (Figure 18). This is true for convex as well as concave quadrilaterals (Figure 19).

When the original quadrilateral and its images are rotated 180° about the midpoint of its sides, the four angles share a common vertex. Common sides align with each other forming a monohedral tessellation.

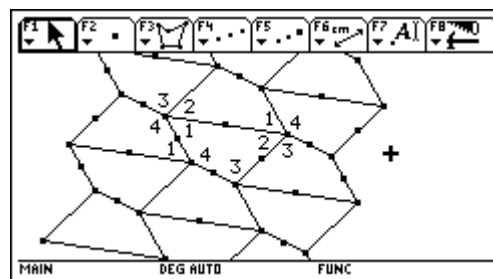


Figure 18

Figure 18 shows the angles of this tessellation numbered so that the pattern is clear. Because the sum of the interior angles of any quadrilateral is 360° , when these four angles share a common vertex, the rotated figures cover the plane with no gaps or overlaps.

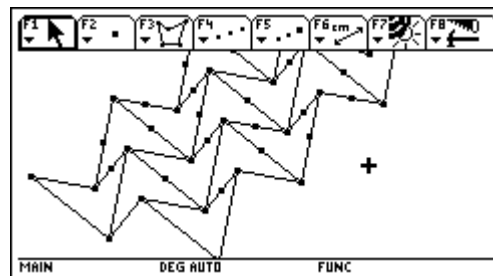


Figure 19

Teacher's Guide: Tessellations and Tile Patterns (Cont.)

Explore:

1. According to David Wells in *The Penguin Dictionary of Curious and Interesting Geometry* (Wells, 1991), there are currently 14 known monohedral pentagonal tilings.

Probably the most famous of these pentagonal patterns is the “Cairo Tessellation” named after the Islamic decorations found on the streets of Cairo (Figure 20).

While this pattern is made up of congruent pentagons, it can also be viewed as overlapping congruent hexagons that, in turn, can be seen as pairs of congruent isosceles trapezoids. This tessellation is constructed from an isosceles trapezoid with adjustable dimensions that is reflected, translated, and rotated to complete the pattern.

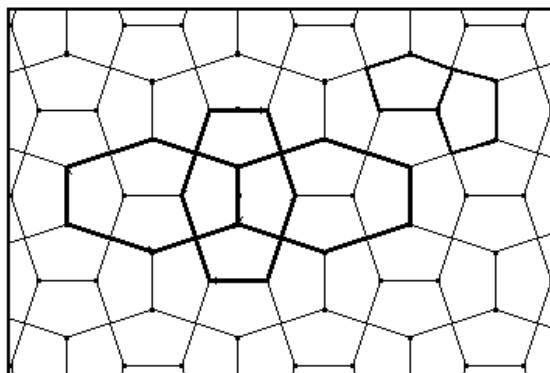


Figure 20: The Cairo Pentagonal Tessellation

K. Reinhardt described what was thought to be the complete list of the five possible types of monohedral pentagonal patterns in 1918. In 1967, Richard Kershner found three new patterns. Then in 1975, Richard E. James III discovered another new pattern.

In 1976, Marjorie Rice, a nature artist from San Diego, found a new type of pentagonal tessellation shown in Figure 21.

Four of the five sides of the basic pentagon are equal. This figure can be duplicated through a series of reflections, half-turns, and rotations.

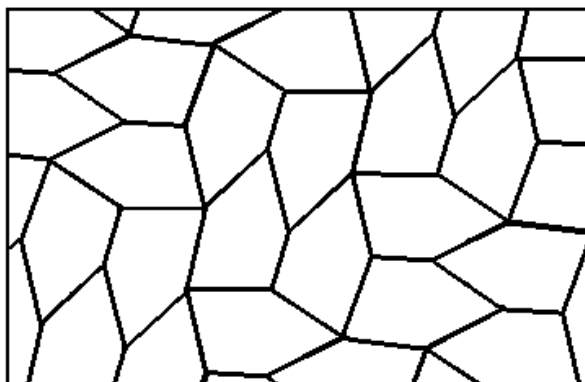


Figure 21: The Rice Pentagonal Tessellation

Soon after the 1976 Rice pattern was found, three more new pentagonal tessellations were found.

Rolf Stein found the fourteenth known pentagonal tessellation in 1985 (Figure 22).

No one knows whether there are other pentagonals. See whether you can find a new one!

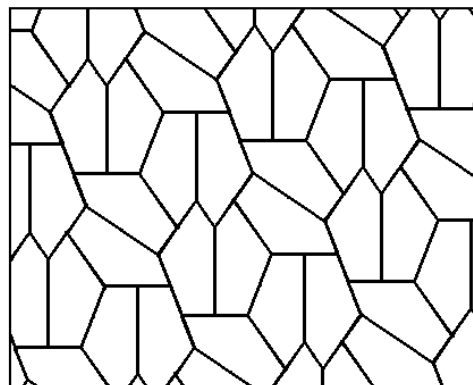


Figure 22: The Stein Tessellation

Teacher's Guide: Tessellations and Tile Patterns (cont.)

2. One example of a nonregular hexagon that tessellates the plane is one with opposite sides parallel and two pairs of opposite sides of equal length. Using the **Symmetry** tool, you can form the tile pattern by rotating the basic hexagon through a series of half-turns about the midpoints of the sides of the figures.

Figure 23 shows this tile pattern. Opposite interior angles of this hexagon are equal in pairs, and the sum of the measure of three different angles is 360° . This makes the tessellation possible because these three different angles meet at the common vertices of the tessellation, ensuring that there are no gaps or overlaps.

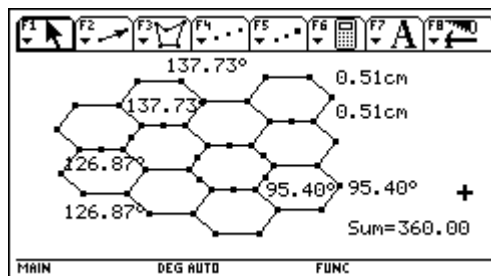


Figure 23

Another interesting hexagonal tessellation is a “star burst” of semiregular hexagons with six equal sides that are parallel in opposite pairs (Figure 24).

Two of the opposite interior angles are 60° . The other four interior angles measure 150° , for a total of 720° , the sum of the measures of the interior angles of any hexagon.

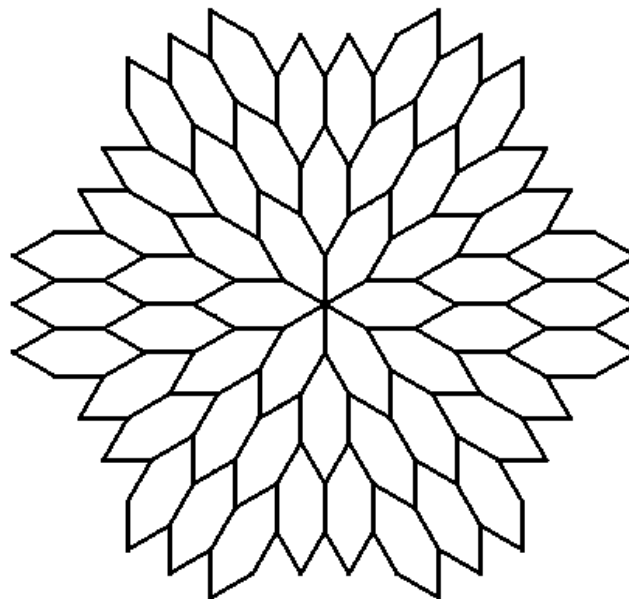


Figure 24