

Teacher Notes



Activity 17

Fundamental Theorem of Calculus

Objectives

- Explore the connections between an accumulation function, one defined by a definite integral, and the integrand
- Discover that the derivative of the accumulator is the integrand

Materials

- TI-84 Plus / TI-83 Plus

Teaching Time

- 50 minutes

Abstract

This activity follows the Accumulation Function activity and explores graphic and numeric manifestations of the Fundamental Theorem of Calculus. The version covered here states that if f is a function continuous on the closed interval $[a, b]$, and x is any number in $[a, b]$, then the function

$$F(x) = \int_a^x f(t) dt$$

is an antiderivative for f . Another way of expressing this result is

$$\frac{d}{dx} \int_a^x f(t) dt = f(x)$$

Students evaluate an accumulation function using list variables in the graphing handheld. They note the location of relative maxima in the table of values for the accumulation function and see that each corresponds to a sign change from positive to negative in the values of the integrand. Similarly, they see that a minimum on the accumulator occurs where the integrand changes sign from negative to positive.

With the values of the accumulator in a list variable, it is possible to verify the Fundamental Theorem of Calculus by forming a difference quotient. This is suggested as a follow-up activity.

Evidence of Learning

Students will:

- be able to predict the location of local extrema on the graph of functions of the form $g(x) = \int_a^x f(t) dt$ by inspecting the graph of $y = f(x)$.
- recognize that changing the lower limit of integration in the function $g(x) = \int_a^x f(t) dt$ results in a vertical translation of the graph of $y = g(x)$.

- be able to differentiate functions defined by definite integrals, such as

$$\frac{d}{dx} = \int_a^x f(t)dt.$$

Management Tips and Hints

Prerequisites

Students should:

- be able to graph functions and navigate the **CALC Menu** to find zeroes of a function.
- be able to produce a scatter plot. The activity introduces them to the **seq** command.
- know the notation for defining a definite integral; an integral's value is positive when the integrand is positive over the interval of integration, and an integral's value is negative when the integrand is negative.

The accumulation function activity or some other experience with functions defined by a definite integral is necessary preparation for students.

Common Student Errors/Misconceptions

- Students may have difficulty with the idea of a "dummy" variable (the variable of integration) and distinguishing it from the independent variable of the accumulation function (usually an upper limit of integration).
- Students can lose track of where the values in the list **L2** actually come from. Each results from the evaluation of a definite integral.

Extension #1

After students have seen that the derivative of **Y2(X)** is **Y1(X)**, it is sensible to confirm this numerically. Follow the steps outlined below.

1. Have half of the students copy **L2** into **L4**, and have the other half copy **L3** into **L4**. Shown are the screenshots that came from using **L2**.

L2	L3	L4
0	-1.478	-----
.15957	-1.318	
.31914	-1.159	
.47865	-.9992	
.63799	-.8399	
.79693	-.681	
.95509	-.5228	
L4 = L2		

2. Position the cursor on the first value in **L4**, and press **DEL** to delete it. Do the same for the last value in **L2**. Each list will then contain a total of 47 values, with the values of **L2** (or **L3**) and **L4** offset by one.

L2	L3	L4	4
0	-1.478	1.3298	
.15957	-1.318	.31914	
.31914	-1.159	.47865	
.47865	-.9992	.63799	
.63799	-.8399	.79693	
.79693	-.681	.95509	
.95509	-.5228	1.1119	
L4(1) = .1595741652...			

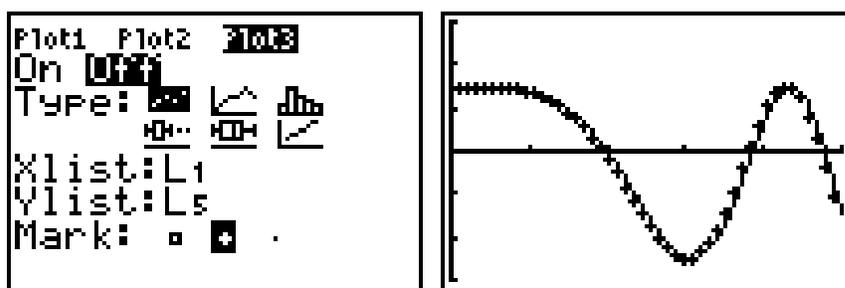
L2	L3	L4	2
-.0209	-1.499	.13298	
.13298	-1.345	.26246	
.26246	-1.215	.34818	
.34818	-1.13	.37418	
.37418	-1.104	.33025	
.33025	-1.148	.21384	
-----	-1.264	-----	
L2(47) = .330254116...			

3. Define **L5** as $(L4 - L2)/(2 * DX)$ as the difference quotient for the accumulation function, **Y2** or **Y3**. Replace **L2** with **L3** for half of the students. This will approximate its derivative.

L3	L4	L5	5
-1.478	.15957	-----	
-1.318	.31914		
-1.159	.47865		
-.9992	.63799		
-.8399	.79693		
-.681	.95509		
-.5228	1.1119		
L5 = (L4 - L2) / (2 * Δ...			

L3	L4	L5	5
-1.478	.15957	1.3298	
-1.318	.31914	1.4999	
-1.159	.47865	1.4994	
-.9992	.63799	1.4978	
-.8399	.79693	1.494	
-.681	.95509	1.4868	
-.5228	1.1119	1.4744	
L5(1) = 1.499997153...			

4. Delete the first element of **L1**. Define **PLOT3** as a scatter plot of **L5** versus **L1**. Select only **Y1** and **PLOT3** for graphing. Draw the graph. If you get an error message about a DIM MISMATCH, check that the lengths of **L1** and **L5** are the same.



The scatter plot and the graph of **Y1** coincide.

Extension #2

Ask students to examine the locations of points of inflection on the graphs of **PLOT1** and **PLOT2**. They should see that these points occur at extrema on the graph of **Y1**. Points of inflection on the graph of a function always occur at extrema on the graph of the derivative. Moreover, a point of inflection is a place where outputs are locally changing the fastest. Because the function is accumulating area, it should change the fastest where the integrand is farthest from the x-axis.

Activity Solutions

- There is no area to calculate.
- The integrand, $Y_1(X)$, is positive over the interval $[0, 0.10638]$.
- $\int_0^{0.21277} 2 \cos\left(\frac{x^2}{3}\right) - 0.5 dt > \int_0^{0.10638} 2 \cos\left(\frac{x^2}{3}\right) - 0.5 dt$
because the integrand is positive over the interval $[0.10638, 0.21277]$. More positive area has been accumulated.
- 2.0213
- 1.9886
- $\int_0^{2.1277} 2 \cos\left(\frac{x^2}{3}\right) - 0.5 dt < \int_0^{2.0213} 2 \cos\left(\frac{x^2}{3}\right) - 0.5 dt$
because the integrand, $Y_1(X)$, is negative on the interval $[2.0213, 2.1277]$. Negative area is accumulating, so the accumulation function is decreasing.
- The answers to Questions 4 and 5 are approximately the same because the maximum value of the accumulation function, $Y_2(X)$, will occur when accumulation of the positive area stops and accumulation of the negative area starts. This occurs where the integrand changes sign from positive to negative (the answer to Question 5). The answer to Question 4 is the approximate location where the values of $Y_2(X)$ reach a maximum.
- 3.8298
- 3.8594
- The answers to Questions 8 and 9 are similar because the accumulation function reaches a minimum when negative area stops accumulating and positive area starts accumulating. As in Question 7, this occurs where the integrand changes sign from negative to positive.
- 4.7753
- $Y_2(X)$ reaches another local maximum at that point. Looking at the table, the maximum occurs near 4.7872.
- Answers will vary. Focusing on $x = 0$,

$$Y_2(0) = \int_0^0 2 \cos\left(\frac{x^2}{3}\right) - 0.5 dt$$

is zero because the upper and lower limits of integration are equal. However,

$$Y_3(0) = \int_1^0 2 \cos\left(\frac{x^2}{3}\right) - 0.5 dt < 0$$

because the integrand is positive, but the integration occurs from right to left. This places the graph of **L3** below the graph of **L2**.

14. PLOT1 and **PLOT2** have the same shape because each increases where the integrand is positive, and each decreases where the integrand is negative. In fact, at each value of x , the two plots are changing at exactly the same rate because each accumulates area under the graph of **Y1**. The Fundamental Theorem says that the two accumulation functions, **Y2** and **Y3**, have the same derivative, **Y1**.

15. $\frac{d}{dX}(Y_2(X)) = Y_1(X)$

