Name _____

Date _____

EXPLORATIONS

ACTIVITY 12

What Is the Number "e"?

Exponential growth occurs in many ways; and in investigating exponential growth, you will encounter the number *e*—the base of the natural logarithm.

Why is it "natural," and how does it arise? Answering that question forms the basis of this activity.

Starting with the concept of the integral as calculating the area under the curve leads you to the *Fundamental Theorem of Calculus* (obviously important, or it would not be called "fundamental"). Applying this theorem to the simple function $y = 1/x$ will lead you to the natural logarithm function and the number *e*. Finally, you'll use integration and the natural logarithm function to solve a typical problem in exponential growth.

Part A—Finding the Value of "e"

1. Press [ON] to turn on the Voyage™ 200 PLT. To reset to the default settings, press [2nd] [MEM] [F1], select 1:RAM, select 2:Default, and then press [ENTER] [ENTER].

   Press [♦] [CALC HOME] to access the Home screen. Delete all values of one-character variables by pressing [F6], selecting 1:Clear a–z, and pressing [ENTER].

   On the Home screen, clear the entry line by pressing [CLEAR] once or twice. Clear the History area above the entry line by pressing [F1] **8**.

2. You can use the Voyage 200 PLT to graph and calculate the area under a curve. Think of the definite integral of a function. First of all, you probably think of the area under the curve defined by the function. The graph to the right, for example, displays the area under the curve of

   $y = 1/x$ from $x = 1$ to $x = 4$; that is, $\int_1^4 \frac{1}{x}\,dx = 1.39$.
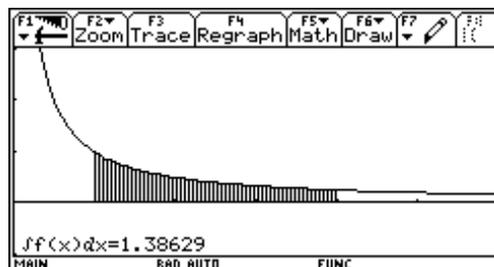
What Is the Number "e"? *(Continued)*

2. (continued)
    Enter the Y= Editor and clear any old functions
    by pressing ◆ [Y=] [F1], select 8:Clear Funtions, and
    press [ENTER].

    Go to the entry line, and enter the function by
   pressing **1** ÷ **X** [ENTER].

    Set the window parameters by pressing
    ◆ [WINDOW]. Move the cursor and enter the
    following values:

xmin = 0	ymin = -1
xmax = 6	ymax = 3
xscl = 1	yscl = 1
	xres = 2

    To display the graph, press ◆ [GRAPH].

    To calculate the area under the curve from 1 to 4,
    press [F5] (Math menu) and select 7:∫f(x)dx. When
    you are prompted for the lower limit, press **1** [ENTER].
    When prompted for the upper limit, press **4** [ENTER].

    You should see the screen to the right.
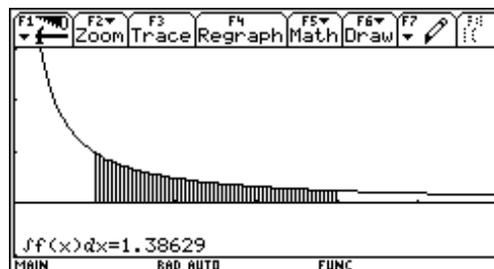
3. Experiment with the upper limit. Can you find a
    value b to the nearest tenth such that

    $$\int_1^b \frac{1}{x}\,dx = 1.0\,?$$

    $b =$ _____

4. As you can see from step 3, changing the value of
    b changes the area under the curve. Thus, if x is
    the variable right endpoint of the domain of
    integration, the function

    $F(x) = \int_1^x \frac{1}{t}\,dt$ gives the area under the curve from

    1 to x. Moreover, $F'(x) = \dfrac{d}{dx}\left(\int_1^x \frac{1}{t}\,dt\right) = \dfrac{1}{x}$.

    This is a specific example of the *Fundamental
    Theorem of Calculus*. This theorem states that if
    a function f is continuous on an interval $[a, b]$,
    then for every x in $[a, b]$,

    $F(x) = \int_a^x f(t)\,dt$ such that $F'(x) = f(x)$.

What Is the Number "e"? *(Continued)*

4. (continued)
    Return to the Home screen by pressing
    [♦] [CALC HOME].

    For example, if $F(u) = \int_0^u \sqrt[3]{t^2 + \cos(t)} \cdot dt$, find

    $F(0)$, $F(x)$, $F'(x)$, and $F'(\frac{\pi}{4})$.

    To define the function, press [F4] and select
    1:Define. Now press
    F [(] **U** [)] [=] [2nd] [∫] [(] **T** [^] **2** [+] **COS** [(] **T** [)] [)]
    [^] [(] **1** [÷] **3** [)] [,] **T** [,] **0** [,] **U** [)] [ENTER].

    To find $F(0)$, press **F** [(] **0** [)] [ENTER]. Record your
    answer below.

    $$F(0) = \int_0^0 \sqrt[3]{t^2 + \cos(t)} \cdot dt =$$
    _____

    To find $F(x)$, press **F** [(] **X** [)] [ENTER] and record
    your answer below.

    $F(x) =$
    _____

    To find $F'(x)$, press [2nd] [d] **F** [(] **X** [)] [,] **X** [)]
    [ENTER] and record your answer below.

    $F'(x) =$
    _____

    To find $F'(\frac{\pi}{4})$, press

    [2nd] [d] **F** [(] **X** [)] [,] **X** [)] [2nd] [|] **X** [=] [2nd] [π] [÷] **4**
    [ENTER]. Record your answer below.

    $F'(\frac{\pi}{4}) =$
    _____

5. $\int_1^x \frac{1}{t} dt$ defines a function.
    From step 2 and step 3, you know the function

    $\int_1^x \frac{1}{t} dt$ gives the area under the curve of $y = 1/x$
    from 1 to an arbitrary point x, $x > 0$.

    From the *Fundamental Theorem of Calculus*, you

    know the derivative is $\frac{d}{dx} \left(\int_1^x \frac{1}{t} dt \right) = \frac{1}{x}$.
    Now let's see why this integral is equal to the
    natural logarithm function, $\ln(x)$.

    Define $myst(u) = \int_1^u \frac{1}{t} dt$. Press [F4] and select
    1:Define. Then press **MYST** [(] **U** [)] [=] [2nd] [∫]
    1 [÷] **T** [,] **T** [,] **1** [,] **U** [)] [ENTER].

What Is the Number "e"? *(Continued)*

5. (continued)
   Set the window parameters by pressing
   ◆ [WINDOW] . Move the cursor, and enter the
   following values:

xmin = 0	ymin = -1
xmax = 25	ymax = 4
xscl = 1	yscl = 1
	xres = 2

   Go to the Y= Editor (◆ [Y=]), and use the cursor
   and enter the following functions.

   y1 = myst(x)   y2 = 0   y3 = 1
   y4 = 2   y5 = 3

6. Now press ◆ [GRAPH] to graph the functions as
   shown at the right. (*Be patient!* Functions
   defined by integrals are slow to graph. Zillions
   of rectangles are added for xres = 2; every other
   pixel is plotted—hence, the grumpy message
   "questionable accuracy" at the bottom of the screen.)
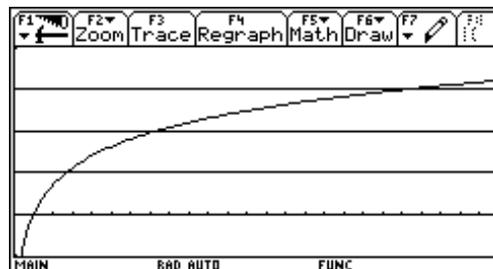
7. Find where $myst(u)$ intersects the horizontal
   lines $y = 0,1,2,3$ to solve the equations

   $$\int_1^u \tfrac{1}{t}dt = 1, \ \int_1^u \tfrac{1}{t}dt = 2, \text{ etc.}$$

   First, find the value of x for which

   $y = \displaystyle\int_1^x \tfrac{1}{t}dt$ intersects the horizontal line $y = 0$
   as follows:

   To go to the Math menu, press [F5] and select
   5:Intersection. The cursor flashes on curve y1.

   Press [ENTER]. A prompt for y2 ($y = 0$) is displayed.
   Press [ENTER] again. A prompt for a lower bound
   of the point of intersection is displayed. Either
   move the cursor to the left of the point of
   intersection *or* type a left-bound x-coordinate
   value (for example, type **0.1**).Then press [ENTER].
   When prompted for upper bound, either move the
   cursor to the right *or* type a value (for example,
   type **2**). Then press [ENTER].

   Not surprisingly, the point of intersection is (1,0).
   Explain why this is so in the space provided below.

What Is the Number "e"? *(Continued)*

8. To record this point in a data variable called *sysdata*, press ◆ **D**. Notice the message at the bottom of the screen: "DATA PLACED IN VARIABLE SYSDATA." To see the values, go to the Data/Matrix Editor by pressing [APPS], selecting Data/Matrix Editor, and selecting 2:Open.

   If the OPEN dialog box shows Variable:sysdata, press [ENTER]. Otherwise, move the cursor down to the Variable line and press ⓥ. You will see sysdata in a box. Select it by pressing ⊙ until it is highlighted. Then press [ENTER] [ENTER]. You should see the screen to the right.

9. Now find and store the other three points of intersection as follows.

   Press ◆ [GRAPH] [F5] and select 5:Intersection.

   Each time you will be prompted for the two curves. Always press [ENTER] to select the curve y1. To select the next horizontal line, use the up and down cursors (⊙ and ⊙).

   Enter the lower and upper bounds as described in step 7. After you find the point of intersection, press ◆ **D** to store the result in *sysdata*.

   When you have finished, press [APPS], select Data/Matrix Editor, select 2:Open, select variable:sysdata, and press [ENTER] to open the data table. You should see the screen to the right.

10. Notice that the values of y in column c2 increase linearly, while those of x in c1 increase more rapidly. In fact, they form the geometric sequence $\{x_1, x_2, x_3, x_4, \cdots\}$, if $x_2/x_1 = x_3/x_2 = x_4/x_3 = \cdots = constant$. Verify this as follows:

    From the Home screen, enter column 1 of sysdata into a list called xs by pressing **SYSDATA** [2nd] [[ ] **1** [2nd] [ ]] [STO▸] **XS** [ENTER].

    To form the ratio x_2/x_1, press **XS** [2nd] [[ ] **2** [2nd] [ ]] [÷] **XS** [2nd] [[ ] **1** [2nd] [ ]] [ENTER].

    Find more ratios of the sequence and record your results for all three.

    $x_2/x_1 =$ _____ $x_3/x_2 =$ _____ $x_4/x_3 =$ _____

    What is the common ratio?_____

What Is the Number "e"? *(Continued)*

11. The approximate common ratio of 2.7183 defines the number e. To verify that fact, return to the sysdata table, move the cursor to any cell in column c3, and press F4 for Header. Press 2nd [e^x] **C2**) ENTER.

    The values displayed in column c3 are powers of e corresponding to the exponents in column c2. The values are the same as those in column c1.

    You have just demonstrated the equivalence of

    $$y = \int_1^x \tfrac{1}{t}\,dt \iff x = e^y,$$

    thus justifying the definition of

    $$\log_e(x) \equiv \ln(x) \equiv \int_1^x \tfrac{1}{t}\,dt \quad \text{and } \frac{d}{dx}\ln(x) = \frac{1}{x},\text{ and}$$

    by definition of logarithm, $y = \log_b x \leftrightarrow b^y = x$.

    On the Home screen (● [HOME]), evaluate $\frac{d}{dx}\ln(x)$.

    Then evaluate $\frac{d}{dx}\operatorname{myst}(x)$.

    $$\frac{d}{dx}\ln(x) = \underline{\hspace{4cm}}$$

    $$\frac{d}{dx}\operatorname{myst}(x) = \underline{\hspace{4cm}}$$

    Type **MYST** and press (**X**) ENTER.

    $$\operatorname{myst}(x) = \underline{\hspace{4cm}}$$

What Is the Number "e"? *(Continued)*

Part B—A Problem in Exponential Growth

The Klingon® empire is expanding at the rate of 23% per year. If the Klingons now control 1864 star systems, how many years will elapse before their empire exceeds the 23,200 systems of the Galactic Federation?

1. Let y = the number of Klingon star systems in year t from the present. Then $\frac{dy}{dt} = .23y$.

    On the Home screen, delete all values of one-character variables by pressing F6, selecting 1:Clear a–z, and pressing ENTER. Clear the entry line by pressing CLEAR once or twice. Clear the History area above the entry line by pressing F1 **8**. Clear values of dy and dt by entering **DELVAR DY** ⎡,⎤ **DT** ENTER.

2. Enter the differential equation, and name it eq by pressing **DY** ⎡÷⎤ **DT** ⎡=⎤ ⎡,⎤ **23** ⎡×⎤ **Y** STO▶ **EQ** ENTER.

    Separate the variables by entering **EQ** ⎡×⎤ **DT** ⎡÷⎤ **Y** STO▶ **EQ** ENTER.

    The result should be $\frac{dy}{y} = .23 \cdot dt$.

3. On the Voyage™ 200 PLT, dy and dt are not necessary. To integrate each side of the equation separately with a constant k of integration on the right, divide dy and dt out by using the **left(** and **right(** extraction commands.

    Press 2nd [∫] **LEFT** ⎡(⎤ **EQ** ⎡)⎤ ⎡÷⎤ **DY** ⎡,⎤ **Y** ⎡)⎤ ⎡=⎤ 2nd [∫] **RIGHT** ⎡(⎤ **EQ** ⎡)⎤ ⎡÷⎤ **DT** ⎡,⎤ **T** ⎡,⎤ **K** ⎡)⎤ STO▶ **EQ** ENTER.
    The result should be $\ln(|y|) = .23 \cdot t + k$.

    Solve the resulting equation for y, given $y > 0$. Press **SOLVE** ⎡(⎤ **EQ** ⎡,⎤ **Y** ⎡)⎤ 2nd [|] **Y** 2nd [>] **0** STO▶ **EQ** ENTER.
    The result should be $y = e^{.23\,t\,+\,k}$.

    Solve the resulting equation for k, given $y = 1864$ when $t = 0$. Press **SOLVE** ⎡(⎤ **EQ** ⎡,⎤ **K** ⎡)⎤ 2nd [|] **Y** ⎡=⎤ **1864 AND T** ⎡=⎤ **0** ⎡♦⎤ ENTER.
    The result should be $k = 7.53048$.

*Note: You can find the **left(** and **right(** extraction commands under the Extract submenu of the Algebra menu, F2 B3 and F2 B4, respectively.*

*Note: A short function sepvar to handle this is given on the next page. It can be defined in the Matrix/Function Editor. If you have defined it, simply enter **sepvar(eq,y,t)** ENTER.*

What Is the Number "e"? *(Continued)*

4. Define k as the result as follows:

    Press F4 and select 1:Define. To retrieve the last answer, press ⊙ ENTER ENTER. Then display the equation by pressing **EQ** ENTER.

    The result should be $y = 1864 \cdot (1.2586)^t$. This equation expresses the number y of star systems in the Klingon® empire after t years.

    To calculate when the Klingons will surpass the Federation's 23,200 systems, enter **SOLVE** ⸨ **EQ** ⸴ **T** ⸩ 2nd [|] **Y** = **23200** ENTER. The result should be $t = 10.9627 \approx 11$ years.

Creating the Sepvar Function

To solve a differential equation with separated variables such as $\dfrac{dy}{y} = r \cdot dt$, create the function sepvar.

The result returned is $\ln(|y|) = r \cdot t + k$.

Sepvar takes the following three arguments:

*   A differential equation with separated variables
*   The variable of the left member of the equation
*   The variable of the right member of the equation

1. Before you begin typing, go to the Home screen and delete any value from k by pressing **DELVAR K** ENTER.

2. To create a *new function* in the Program Editor, type APPS, select Program Editor, and select 3:New. Then press ⊙, and select 2:Function.

3. Press ⊙ ⊙ and type **SEPVAR** in the Variable box and press ENTER ENTER.

    You are now in the Program Editor.

4. Enter the commands as shown on the screen to the right.

*Note: Pressing 2nd **h** enters the **&**.*

What is the Number "e"? *(Continued)*

Part A—Extra Practice

Let $f(u) = \int_0^u \sqrt{t^3 + 1} \cdot dt$ and $g(u) = \int_1^u \sqrt{t^3 + 1} \cdot dt$.

1. Evaluate $f(0), f(1), g(0), g(1)$ _____.

2. $f'(x) =$ _____ and $g'(x) =$ _____.

3. As a result of step 2, what can you conclude about the relationship between f and g?

   _____

4. $f''(x) =$ _____.

5. The *Chain Rule:*

   Define u to be functions of x. For example, if $u = 2x$, press **2** ⊠ **X** STO▶ **U** ENTER, and then

   $f(u) =$ _____ and $\dfrac{d(f(u))}{dx} =$ _____.

   Fill in the following table for various functions $u(x)$.

$u(x)$	$f(u)$	$\dfrac{df(u)}{dx}$
$u = x^2$		
$u = x^3$		
$u = e^x$		
$u = \sin(x)$		

   What inference do you draw for

   $$\frac{d}{dx}\left(\int_0^{u(x)} \sqrt{t^3 + 1} \cdot dt\right)_?$$

   _____

6. Repeat the exercise for various functions $u(x)$ and another function defined by an integral.

   Try to generalize your conclusion for

   $$\frac{d}{dx}\left(\int_a^{u(x)} g(t) \cdot dt\right)_.$$

   _____

Part B—Extra Practice

Use the Klingon® problem in *Part B* as a model to solve the following problems in exponential growth. You may wish to define the function sepvar in the Program Editor. It is given in the section "Creating the Sepvar Function."

1. A certain bacteria culture grows at an exponential rate; that is, its rate of growth is directly proportional to its population. If it grows at a rate of 8.2% per hour, how long will it take 100 bacteria to increase to 10,000?

2. In decaying organic matter such as wood, Carbon 14 decays at the rate of 50% every 5700 years. If the remains of a wood fire at a camp site found in an archaeological dig show 4.7% of the original Carbon 14 remaining, how old is the camp site?

***Hint:** Let r = the rate of growth, and proceed as above. Use as amounts 100 for 100%, 50 for 50% and 4.7 for 4.7%, and solve for k and r. After you have the equation expressing y as a function of t, take the natural logarithm of the equation and solve the result for t.*

What Is the Number "e"? *(Continued)*

Extensions

1. Using your Voyage™ 200 PLT, find a property of ln*(x)*.

   $$\frac{d\ln(x)}{dx} = \underline{\hspace{3cm}} \text{ and}$$

   $$\frac{d\ln(a \cdot x)}{dx} = \underline{\hspace{3cm}}.$$

2. Based on these results,
   ln*(a · x)* − ln*(x)* = _____.

   Why?

   _____

3. To find the value of the preceding difference, evaluate the expression when $x = 1$. Write the resulting equation below.

   _____

   What property of logarithms does this illustrate?

   _____