Two Investigations of Cubic Functions

By Marlena Herman

Activity Overview

In this activity, two interesting features of cubic functions which have three real roots are explored, namely that:
(i) the root of the equation of the tangent line to a cubic function at the average of two of the function's three roots turns out to be the function's third root, and
(ii) the midpoint between the relative minimum and relative maximum points of a cubic function turns out to be the function's inflection point.

For each of (i) and (ii), an investigation starts with a specific function, $f(x) = x^3 - 3x^2 - 10x + 24$, and then moves to the more general case, g(x)=(x-a)*(x-b)*(x-c). CAS capabilities allow for proofs of the above features to be explored in the more general case.

Note: The ideas presented in this activity were inspired by John F. Mahoney's article entitled "Computer Algebra Systems in Our Schools: Some Axioms and Some Examples," *Mathematics Teacher 95*(8), 2002.

Concepts

- Roots of a Function (i.e., Zeros of the Graph of a Function)
- First and Second Derivatives of a Function
- Tangent Lines in Slope-Intercept Form (using First Derivative for slope)
- Relative Minimum, Relative Maximum, and Inflection Points (using First and Second Derivatives)
- Midpoint between Two Points

Teacher Preparation

The investigations of (i) and (ii) above offer opportunities for students to apply their knowledge of derivatives of a function to interesting properties of cubic functions. Teachers may need to help students review methods of finding roots of a polynomial function, the quadratic formula (optional), and the midpoint formula. Students should also know how to find the first and second derivatives of a function and how to use these derivatives to find slopes/equations of tangents to the function as well as relative minimum, relative maximum, and inflection points of the function.

Screen shots of the student document are provided at the end of this summary. Screenshots of expected results are provided embedded within directions below, as well as at the end of this summary.

Classroom Management

This activity could be teacher-led, or could be used as a self-guided discovery for individual or small groups of students. Teachers might want to go through the first part of each investigation and have students try other parts. Or, teachers might want to use each entire investigation based on $f(x) = x^3 - 3x^2 - 10x + 24$ as an example and have students try other examples using other cubic functions. The function f(x) was created by expanding (x+3)(x-2)(x-4). Teachers can easily create other "nice" examples by expanding (x-a)(x-b)(x-c) where a, b, and c are integers.

The student worksheet Two_Investigations_Of_Cubics_Student is intended to guide students through the main ideas of the two investigations and serve as a place for students to record results.

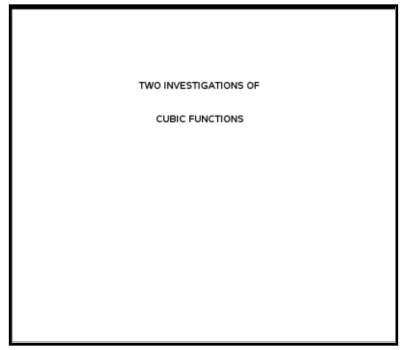
TI-Nspire CAS applications used during this activity

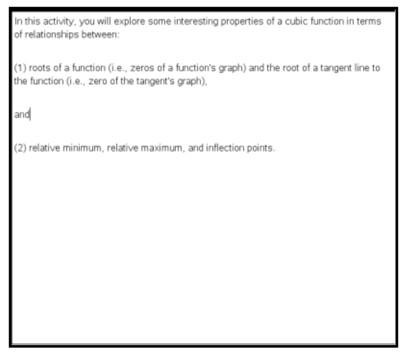
Notes, Calculator, Graphs & Geometry

Getting Started
Students should open the file labeled "CubicInvestigation.tns" and follow instructions provided in the document as well as the student worksheet.

Directions

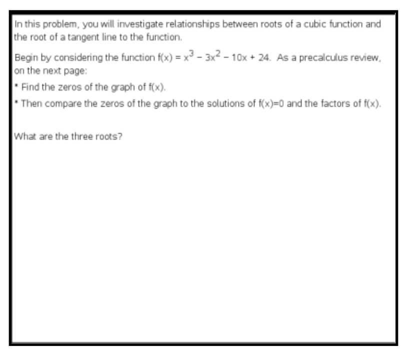
Problem 1 (Introduction)
The first two pages are simply introductory.





Problem 2 (Investigation 1)

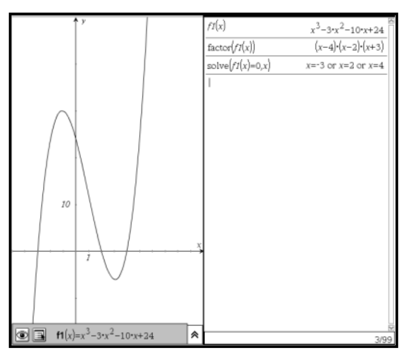
The first investigation begins as a problem in page 2.1 with a consideration of the function $f(x) = x^3 - 3x^2 - 10x + 24$.



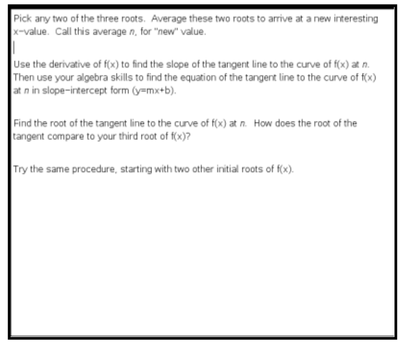
Students start by finding the three roots of f(x) by examining zeros on the graph of f(x). They should notice that the given graph on page 2.2 appears to cross the x-axis at -3, 2, and 4.

With the calculator in a split-screen beside the graph, factoring f(x) yields (x-4)(x-2)(x+3) and solving f(x)=0 yields x = -3, x = 2, and x = 4, thereby confirming that the roots are indeed -3, 2, and 4.

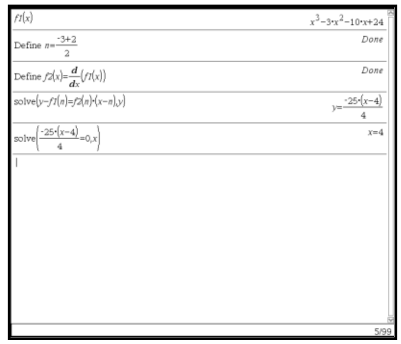
The FACTOR and SOLVE commands are available in the ALGEBRA menu.



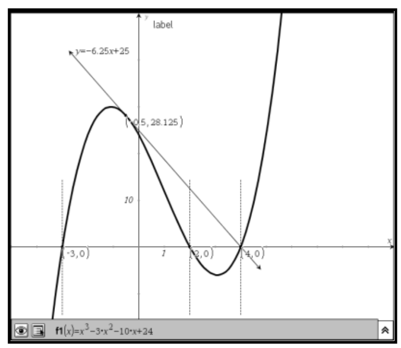
Instructions on page 2.3 guide the students to average any two of the roots, call this average "***n***," and find the equation of the tangent line to f(x) at ***n***. Then students find the root of the tangent.



An example (Case 1) is given on page 2.4 in the document.
Case 1: Beginning with roots -3 and 2, which average to ***n*** = -0.5, the slope of the tangent at ***n*** is f'(-0.5) = -25/4 and the equation of the tangent at ***n*** is y = (-25/4)x + 25. Finding the tangent's root by solving (-25/4)x+25=0 algebraically yields x = 4.



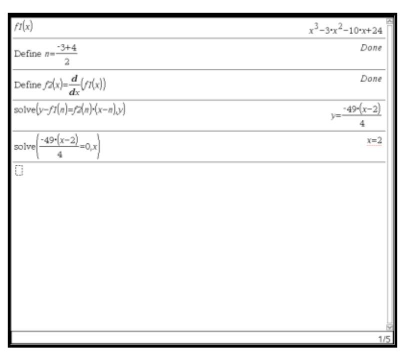
As given on page 2.5, a graphical inspection shows that the x-intercept of the tangent at ***n*** is the same as the remaining x-intercept of f(x).



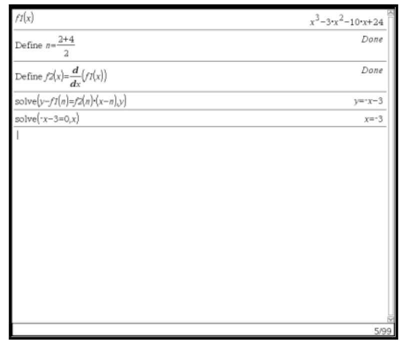
Students should try other cases of initial pairs of roots, altering pages 2.4 and 2.5 as needed.

Case 2: Beginning with roots -3 and 4, which average to ***n*** = 0.5, the equation of the tangent at ***n*** is y =(-49/4)x + (49/2).

Finding the tangent's root with algebra yields x = 2.

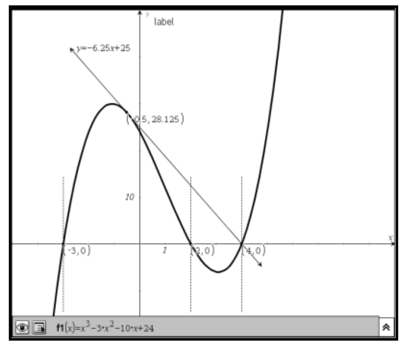


Case 3: Beginning with roots 2 and 4, which average to ***n*** = 3, the equation of the tangent at ***n*** is y = - x - 3. Finding the tangent's root with algebra yields x = -3.

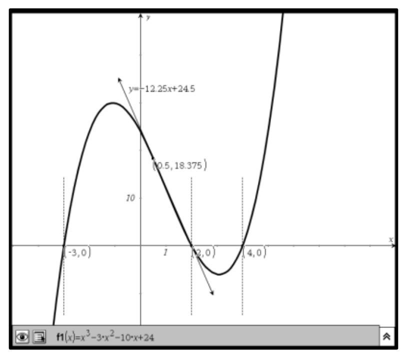


In each possible case involving f(x) = x^3 - $3x^2$ - 10x + 24, the x-intercept of the tangent at ***n*** is the same as the remaining x-intercept of f(x). In other words, the root of the equation of the tangent line to f(x) at the average of two of the three roots of f(x) turns out to be the remaining third root of f(x).

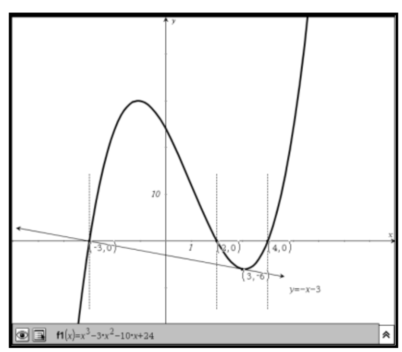
Case 1



Case 2

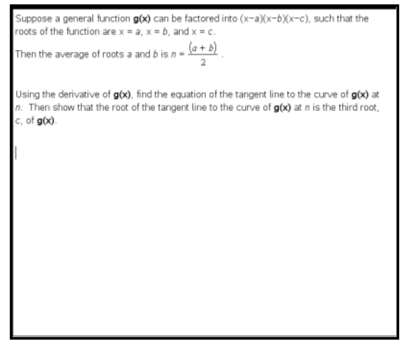


Case 3

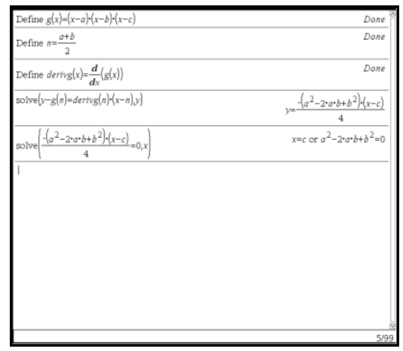


Is this always true? Students could try other sample cubic functions by redefining f(x).

Eventually, as described on page 2.6, students should consider the general case g(x)=(x-a)(x-b)(x-c), where g(x) is a cubic function with roots a, b, and c.



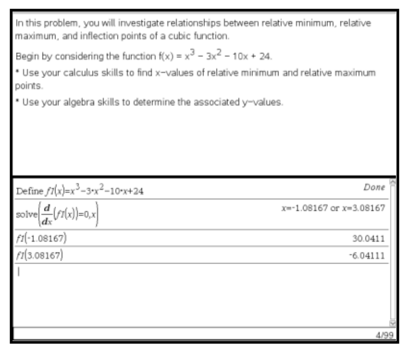
Beginning with roots a and b, which average to ***n***=(a+b)/2, CAS capabilities can be used to determine that the equation of the tangent at ***n*** is y = −(a^2-2ab+b^2)*(x-c)/4. Finding the tangent's root with algebra yields x = c, which is the third root of g(x).



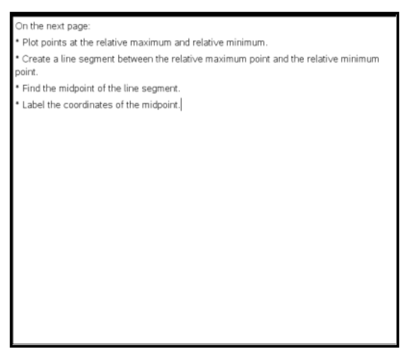
Voila!

<u>Problem 3 (Investigation 2)</u>

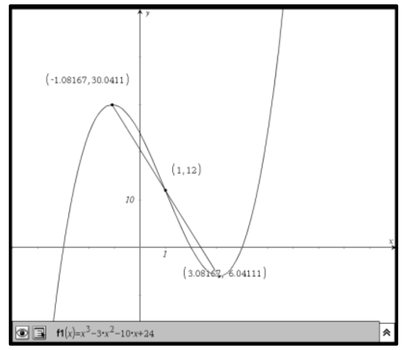
The second investigation begins as a problem in page 3.1 with a consideration of the function f(x) = x^3 - $3x^2$ - 10x + 24. Students start by finding the relative minimum and relative maximum points of f(x). The x-values of these points can be found by setting the first derivative equal to zero. When plotted, the points should fall in appropriate positions on the graph of f(x). That is, it should be obvious to students that the relative minimum occurs at the lowest point on a concave upward region of the graph and that the relative maximum occurs at the highest point on a concave downward region of the graph.



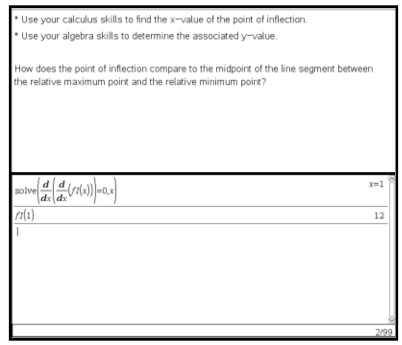
Instructions on page 3.2 guide the students to plot the relative minimum and relative maximum points, create a line segment between these two points, find the midpoint of the line segment, and label the coordinates of the midpoint.



The graph of the function is provided on page 3.3. Students can plot the points with the POINT ON command from the POINTS & LINES menu, create the line segment with the SEGMENT command from the POINTS & LINES menu, find the midpoint with the MIDPOINT command from the CONSTRUCTION menu, and label the midpoint with the COORDINATES AND EQUATIONS command from the ACTIONS menu.

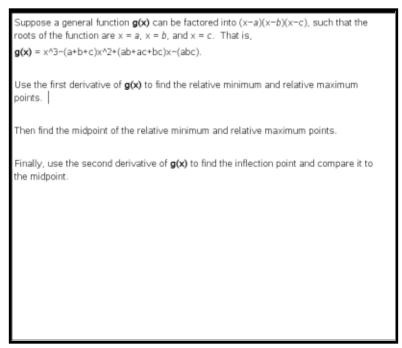


On page 3.4, students find the inflection point of f(x). The x-value of this point can be found by setting the second derivative equal to zero. In the problem involving $f(x) = x^3 - 3x^2 - 10x + 24$, both the midpoint and the inflection point turn out to be (1, 12).



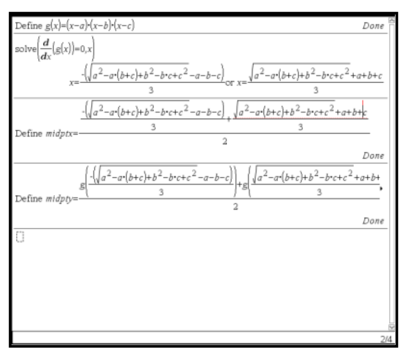
Is it always the case that both the midpoint and the inflection point turn out to be the same? Students could try other sample cubic functions by redefining f(x).

Eventually, as described on page 3.5, students should consider the general case g(x)=(x-a)(x-b)(x-c), where g(x) is a cubic function with roots a, b, and c.

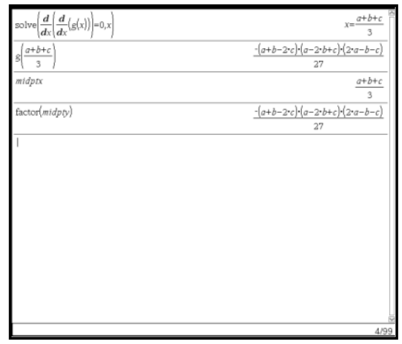


Note:
$g(x) = x^3 - (a+b+c)x^2 + (ab+ac+bc)x - (abc)$
$g'(x) = 3x^2 - 2(a+b+c)x + (ab+ac+bc)$
$g''(x) = 6x – 2(a+b+c)$

Setting the first derivative equal to zero and solving for x with the quadratic formula or CAS capabilities determines that the x-coordinates of the relative minimum and relative maximum points are
$x = [(a+b+c) \pm \sqrt{(a^2+b^2+c^2-ab-ac-bc)}] / 3$.



Once y-coordinates are found with
$g ([(a+b+c) \pm \sqrt{(a^2+b^2+c^2-ab-ac-bc)}] / 3)$, the midpoint formula can be used to find the x- and y-coordinates of the midpoint between the relative minimum and relative maximum points.



Setting the second derivative equal to zero and solving for x determines that the x-value of the inflection point is x = (a+b+c)/3. The y-coordinate can be found with g ((a+b+c)/3).

Using CAS capabilities, the coordinates of the midpoint and the coordinates of the inflection point are the same. Hence, the midpoint between the relative minimum and relative maximum points is the inflection point of the cubic function g(x).

Voila!

Ideas for Extension

Students could make conjectures and/or read about other properties of cubic functions and test ideas with TI-Nspire CAS.

Students could investigate whether similar properties hold for other types of polynomial functions. For example, if “cubic” is changed to “quartic” in the above explorations, is there any relationship(s) between the roots of a quartic g(x)=(x-a)(x-b)(x-c)(x-d) and roots of tangent lines to the quartic? Is there any relationship(s) between relative minimum, relative maximum, and inflection points?

Screen Captures of "CubicInvestigation.tns" Document

Problem 1 (Introduction)

TWO INVESTIGATIONS OF

CUBIC FUNCTIONS

In this activity, you will explore some interesting properties of a cubic function in terms of relationships between:

(1) roots of a function (i.e., zeros of a function's graph) and the root of a tangent line to the function (i.e., zero of the tangent's graph),

and

(2) relative minimum, relative maximum, and inflection points.

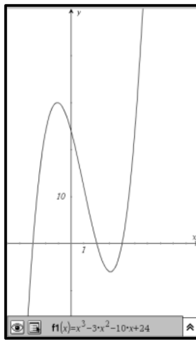
Problem 2 (Investigation 1)

In this problem, you will investigate relationships between roots of a cubic function and the root of a tangent line to the function.

Begin by considering the function $f(x) = x^3 - 3x^2 - 10x + 24$. As a precalculus review, on the next page:

* Find the zeros of the graph of f(x).
* Then compare the zeros of the graph to the solutions of f(x)=0 and the factors of f(x).

What are the three roots?



$f1(x)$ — $x^3-3\cdot x^2-10\cdot x+24$

1/99

Pick any two of the three roots. Average these two roots to arrive at a new interesting x–value. Call this average *n*, for "new" value.

Use the derivative of f(x) to find the slope of the tangent line to the curve of f(x) at *n*. Then use your algebra skills to find the equation of the tangent line to the curve of f(x) at *n* in slope–intercept form (y=mx+b).

Find the root of the tangent line to the curve of f(x) at *n*. How does the root of the tangent compare to your third root of f(x)?

Try the same procedure, starting with two other initial roots of f(x).

$f1(x)$	$x^3-3\cdot x^2-10\cdot x+24$
Define $n=\frac{-3+2}{2}$	*Done*
Define $f2(x)=\frac{d}{dx}(f1(x))$	*Done*
$\text{solve}(y-f1(n)=f2(n)\cdot(x-n),y)$	$y=\frac{-25\cdot(x-4)}{4}$
$\text{solve}\left(\frac{-25\cdot(x-4)}{4}=0,x\right)$	$x=4$

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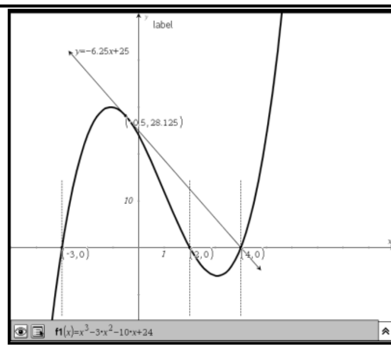

Suppose a general function **g(x)** can be factored into $(x-a)(x-b)(x-c)$, such that the roots of the function are $x = a$, $x = b$, and $x = c$.

Then the average of roots *a* and *b* is $n = \frac{(a+b)}{2}$.

Using the derivative of **g(x)**, find the equation of the tangent line to the curve of **g(x)** at *n*. Then show that the root of the tangent line to the curve of **g(x)** at *n* is the third root, *c*, of **g(x)**.

Define $g(x)=(x-a)\cdot(x-b)\cdot(x-c)$	*Done*

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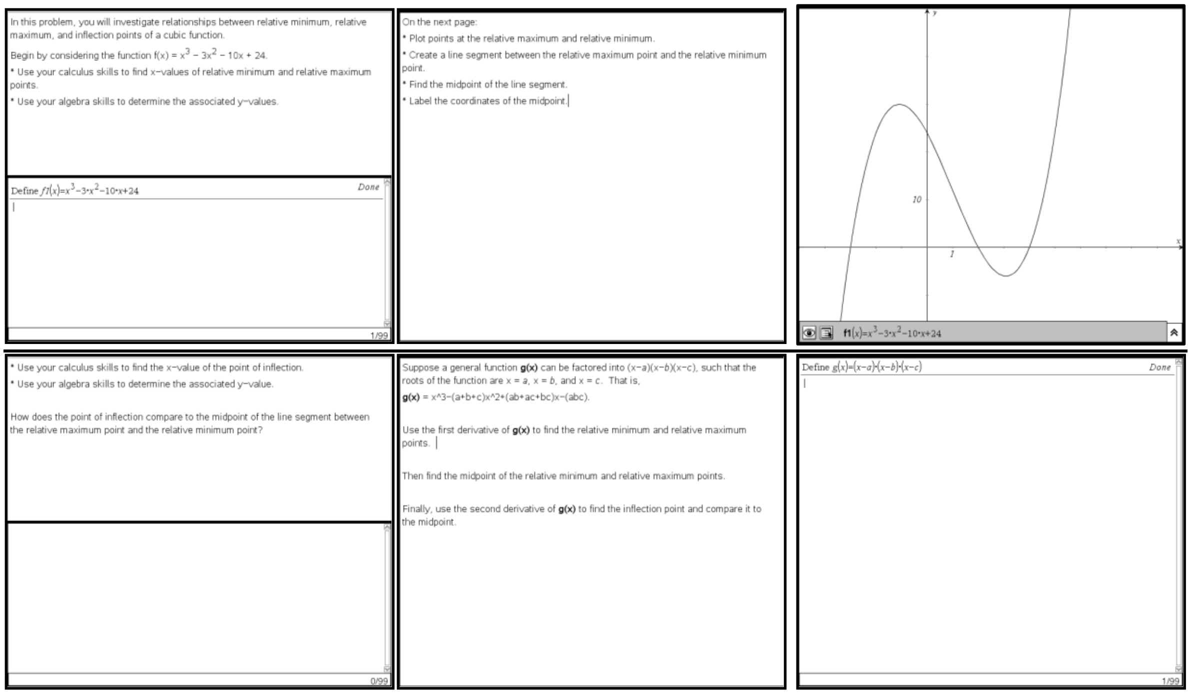
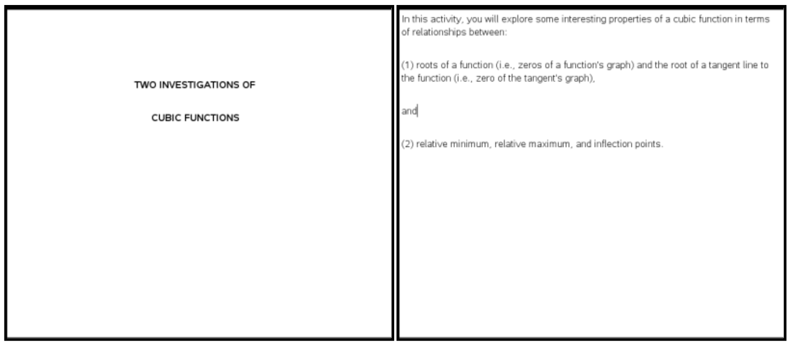
Problem 3 (Investigation 2)

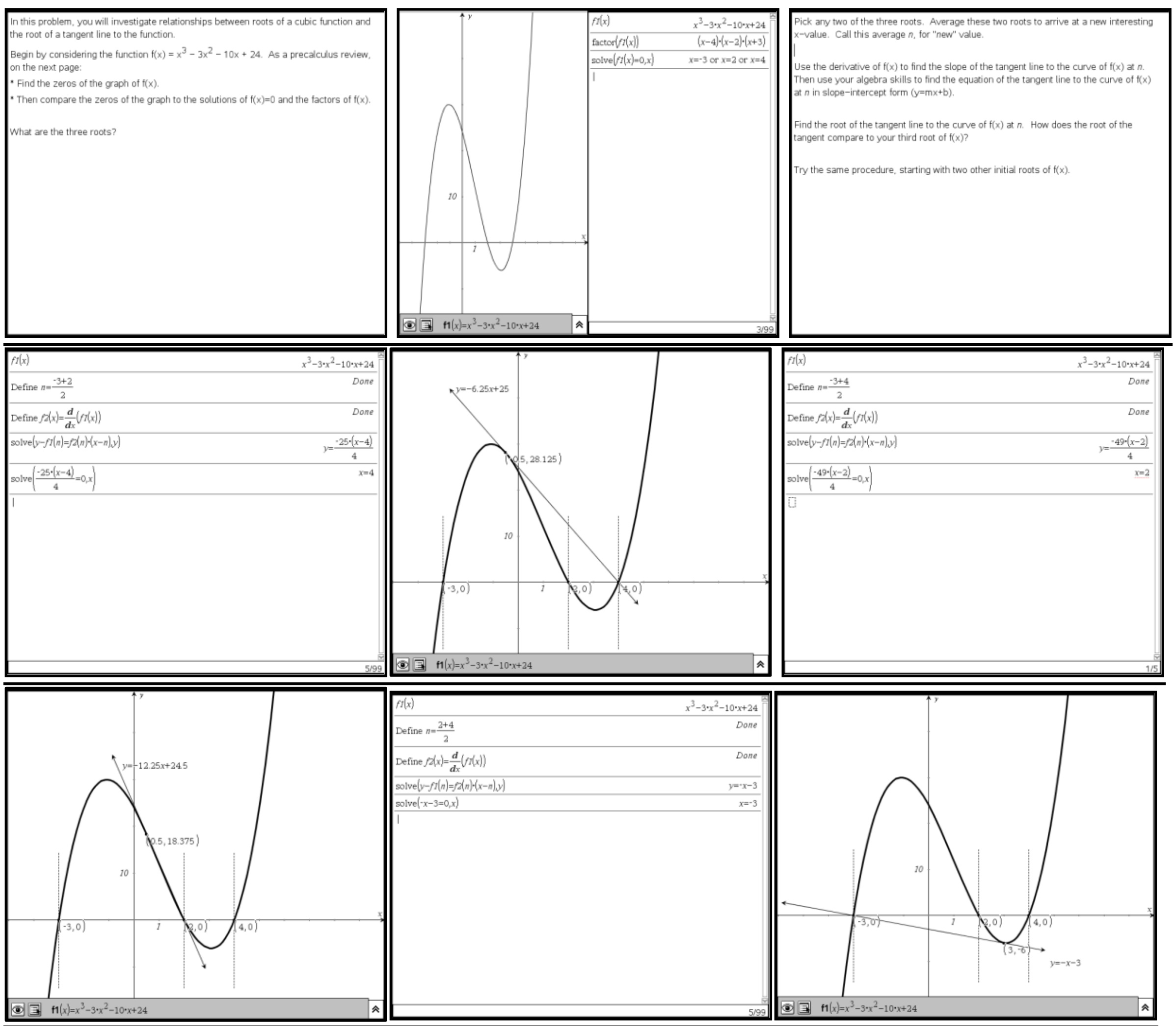


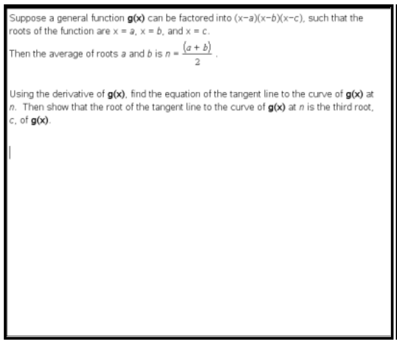
Screen Captures of "CubicInvestigationSampleSoln.tns" Document

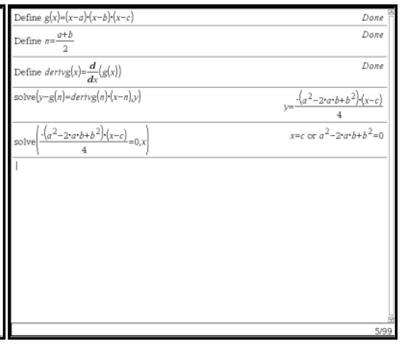
Problem 1 (Introduction)



Problem 2 (Investigation 1)







Problem 3 (Investigation 2)