1(i) The shaded region above represents the region in which \( z \) lies.

Note: Unless required in the question, it would be sufficient to just indicate the centre and radius of the circle in such a locus drawing.

1(ii) Minimum \( |z| \) is given by \( OA \)

\[
OA = OC - AC = \sqrt{2^2 + 5^2} - 3 = \sqrt{29} - 3
\]

Maximum \( |z| \) is given by \( OB \)

\[
OB = OA + AB = \left( \sqrt{29} - 3 \right) + 6 = \sqrt{29} + 3
\]

1(iii) Remark 1: The length \( DP \) in this case, is always the same regardless of which coordinates of \( P \) we are taking. This is because it happens that the line through \( CD \) is perpendicular to the line representing \( \arg(z) = \frac{1}{4} \pi \). This can be easily verified by finding the gradient of the line through \( CD \).
Combining the restriction given by \(0 \leq \arg(z) \leq \frac{1}{4}\pi\), the resulting locus of \(z\) is given as the shaded region above (see diagram).

The point \(D\) represents the complex number \(6+i\).

The maximum value of \(|z-6-i|\) is given by \(DP\) (there are two possible such positions that give this maximum value).

Observe that \((2, 2)\) and \((5, 5)\) both lie on the circle which has Cartesian equation \((x-2)^2 + (y-5)^2 = 3^2\). Thus, these two points are the possible positions of \(P\). This means the maximum value required is \(DP = \sqrt{(6-2)^2 + (1-2)^2} = \sqrt{17}\).

Remark 2: This part can also be solved by changing the equations of the circle and the half-line to Cartesian form and do simultaneous equations solving. The equation of the circle is already given while the equation of the half-line is \(y = x\).

2(i) From the question’s description, the box will have base area \(PS \times SR\) and height \(x\). Thus the volume is given by

\[V = (n-2x)(2n-2x)x\]

\[= 2x(n^2 - nx - 2xn + 2x^2) = 2n^2x - 6nx^2 + 4x^3.\] (Shown!)

2(ii) Find \(\frac{dV}{dx} = 2n^2 - 12nx + 12x^2\) and let \(\frac{dV}{dx} = 0\)

\[\Rightarrow x = \frac{6n \pm \sqrt{(6n)^2 - 4(6)(n^2)}}{12}\]

\[\Rightarrow x = \left(\frac{6 \pm \sqrt{12}}{12}\right)n = \left(\frac{1}{2} \pm \frac{\sqrt{3}}{3}\right)n\]

From the diagram (for example, from the side of \(AD\)), we deduce that \(2x < n\), and thus \(x = \left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right)n\) is the only answer.

Remark 1: Domain of \(f^{-1}\) equals to the range of \(f\) and range of \(f^{-1}\) equals to the domain of \(f\).

Remark 2: Generally, to find the range of a function, we will need a sketch of the graph.

3(i) Write \(f(x) = \ln(2x+1)+3\).

Then, \(2x + 1 = e^{f(x)-3} \Rightarrow x = \frac{1}{2}(e^{f(x)-3} - 1)\).

Thus, \(f^{-1}(x) = \frac{1}{2}(e^{x-3} - 1)\).

The domain of \(f^{-1}\) is \((-\infty, \infty)\) or we can write as \(\mathbb{R}\).

The range of \(f^{-1}\) is \((-\infty, \infty)\).
3(ii) The coordinates where the curves intersect the axes are:

\[ A = (0, 3) \]
\[ B = \left( \frac{1}{2} (e^3 - 1), 0 \right) \]
\[ C = \left( 0, \frac{1}{2} (e^3 - 1) \right) \]
\[ D = (3, 0) \]

3(iii) When the two curves intersect, they also intersect at the line \( y = x \). That is, \( f(x) = f^{-1}(x) \) is equivalent to \( f(x) = x \). Thus we have \( \ln(2x + 1) + 3 = x \)
\[ \Rightarrow \ln(2x + 1) = x - 3 \]

Using the graphic calculator, the values of \( x \) are -0.4847 and 5.482, correct to 4 s.f.

4(a)(i) Use integration by parts twice:
\[
\int_0^n xe^{-2x} \, dx = \left[ x^2 \left( -\frac{1}{2} e^{-2x} \right) \right]_0^n - \int_0^n 2x \left( -\frac{1}{2} e^{-2x} \right) \, dx \\
= -\frac{n^2}{2} e^{-2n} + \int_0^n xe^{-2x} \, dx \\
= -\frac{n^2}{2} e^{-2n} + \left[ x \left( -\frac{1}{2} e^{-2x} \right) \right]_0^n - \int_0^n -\frac{1}{2} e^{-2x} \, dx \\
= -\frac{n^2}{2} e^{-2n} - \frac{n}{2} e^{-2n} - \frac{1}{4} e^{-2n} \\
= -\frac{n^2}{2} e^{-2n} - \frac{n}{2} e^{-2n} - \frac{1}{4} e^{-2n} + \frac{1}{4} = \frac{-e^{-2n}}{4} (2n^2 + 2n + 1) + \frac{1}{4}.
\]

Note: Usually when the integrand consists of two different families of terms, we should be prepared to apply integration by parts. This is only a guideline, as there are exceptions such as \( \int \frac{\ln x}{x} \, dx \) where it is not necessary to employ the technique of integration by parts. Another familiar exception is \( \int \sec^3 x \, dx \) where the usual method is to use integration by parts to solve it.
4(a)(ii) 

\[ \int_0^\infty x^2 e^{-2x} \, dx = \lim_{n \to \infty} \left( \frac{-e^{-2n}}{4} \left( 2n^2 + 2n + 1 \right) + \frac{1}{4} \right) = \frac{1}{4}. \]

Note that \( \frac{-e^{-2n}}{4} \left( 2n^2 + 2n + 1 \right) \to 0 \) as \( n \to \infty \).

Note 1: Integrals which have limits ending with infinity such as \( \int_0^\infty x^2 e^{-2x} \, dx \) are called improper integrals.

Note 2: A way to understand why \( \frac{-e^{-2n}}{4} \left( 2n^2 + 2n + 1 \right) \to 0 \) as \( n \to \infty \) is by manipulating the expression

\[ \frac{-e^{-2n}}{4} \left( 2n^2 + 2n + 1 \right) = \frac{-2n^2 + 2n + 1}{4e^{2n}} \]

and thinking of

\[ e^{2n} = 1 + \left( 2n \right)^2 + \left( 2n \right)^3 + \cdots \]

(according to MF15) which can be considered an infinite sum consisting of terms in \( n \) of much higher degrees.

4(b) 

From the diagram, the shaded region will undergo a rotation of one full round about the \( -y \)-axis.

Volume of solid obtained is given by .

\[ V = \int_0^1 \pi y^2 \, dx = \pi \int_0^1 \frac{16x^2}{(x^2 + 1)^2} \, dx. \]

Using the substitution \( x = \tan \theta \):

- \( \frac{dx}{d\theta} = \sec^2 \theta \).

- When \( x = 0 \Rightarrow \theta = 0 \) and when \( x = 1 \Rightarrow \theta = \frac{\pi}{4} \).

Thus, \( V = 16\pi \int_0^{\pi/4} \frac{\tan^2 \theta}{\left( \tan^2 \theta + 1 \right)^2} \sec^2 \theta \, d\theta \)
\[ = 16\pi \int_{0}^{\frac{\pi}{4}} \frac{\tan^2 \theta}{\tan^2 \theta + 1} \, d\theta \text{ (since } \tan^2 \theta + 1 = \sec^2 \theta) \]
\[ = 16\pi \int_{0}^{\frac{\pi}{4}} \tan \theta \, d\theta. \]
\[ = 16\pi \int_{0}^{\frac{\pi}{4}} \frac{\sin^2 \theta}{\cos^2 \theta \sec^2 \theta} \, d\theta = 16\pi \int_{0}^{\frac{\pi}{4}} \sin^2 \theta \, d\theta. \text{ (Shown!)} \]

Thus, \( V = 16\pi \int_{0}^{\frac{\pi}{4}} \sin^2 \theta \, d\theta \)
\[ = 16\pi \int_{0}^{\frac{\pi}{4}} \frac{1 - \cos 2\theta}{2} \, d\theta \text{ (by cosine double angle formula)} \]
\[ = 8\pi \left[ \theta - \frac{1}{2} \sin 2\theta \right]_{0}^{\frac{\pi}{4}} = 8\pi \left( \frac{\pi}{4} - \frac{1}{2} \right) = 2\pi \left( \pi - 2 \right). \]

### 2011 GCE ‘A’ Level H2 Maths Solution Paper 2
**SECTION B (STATISTICS)**

5 From \( P( X < 40.0 ) = 0.05 \), we standardise \( X \):
\[ P\left( \frac{X - \mu}{\sigma} < \frac{40.0 - \mu}{\sigma} \right) = 0.05 \]
\[ \Rightarrow P\left( Z < \frac{40.0 - \mu}{\sigma} \right) = 0.05 \]
\[ \Rightarrow \frac{40.0 - \mu}{\sigma} = -1.64485 \text{ ----- (1)} \]

Similarly, for \( P( X < 70.0 ) = 0.975 \), we have
\[ \Rightarrow P\left( Z < \frac{70.0 - \mu}{\sigma} \right) = 0.975 \]
\[ \Rightarrow \frac{70.0 - \mu}{\sigma} = 1.95996 \text{ ----- (2)} \]

Solving (1) and (2): \( \mu = 53.7 \), \( \sigma = 8.32 \), correct to 3 s.f.

Note 1: To use the InvNorm function in the graphic calculator, we must ensure the probability is of the form \( P( X < ...) = ... \) or \( P( X \leq ...) = ... \), with the inequality sign being “less than” or “less than or equal”.

Note 2: The method of standardisation is very efficient in solving normal distribution problems involving unknown mean and/or unknown variance (or standard deviation).

6(i) Quota sampling might be carried out by

- First divide the range of ages into several groups (12 – 16, 17 – 21, 22 – 26 etc).
- Next, assign a quota (required number of residents to be interviewed) for each age group.
- Interviewer will then pick residents at his/her own discretion for interviews, until all the quotas are fulfilled.
6(ii) One disadvantage is that the interviewer may likely have collected a biased sample of residents due to his/her non-random selection process of residents (selection is based solely on his/her judgement).

**Remark:** Despite the obvious disadvantage of quota sampling, it is still used widely nowadays mainly due to its big advantage of not needing to have a sampling frame to carry out the sampling process.

6(iii) **Stratified sampling.**

This method is not realistic in this context because it will be extremely difficult to gain access to information about our sampling frame. For example, it will be difficult to get a whole list of residents with information such as their ages from a government or public agency.

**Note:** Other answers are possible such as randomly selected residents may not be willing to participate in the interview.

7(i) **Assumption 1:** All the friends should not have any knowledge of this experiment to be carried out, before or during the experiment, to ensure that each trial of contacting a friend is independent of one another.

**Assumption 2:** The experiment should be conducted in a realistic span of time in the evening to ensure that time does not become a factor to affect the constant success probability of contacting a friend.

7(ii) No matter how quickly the experiment can be carried out, all the calls are bound to be done at different times in the evening, and in turn will make it difficult for the probability of success in contacting a friend to stay constant.

7(iii) For $n = 8, R \sim B(8, 0.7)$ and $P(R \geq 6)$

$= 1 - P(R \leq 5)$

$= 0.552$ correct to 3 s.f.

TI84 Plus screenshots:
7(iv) For \( n = 40 \), \( R \sim \text{B}(40, 0.7) \).

Since \( n = 40 \) is considered large, \( np = 28 > 5 \) and \( n(1-p) = 12 > 5 \), thus, \( R \sim \text{N}(28, 8.4) \) approximately.

The parameters in this new distribution are the mean (28) and variance (8.4).

Therefore \( P(R < 25) \xrightarrow{c.c} P(R < 24.5) \)

\[ = 0.114 \text{ correct to 3 s.f.} \]

Note: \( c.c \) stands for continuity correction.

8(i) From the graphic calculator, the value of the product moment correlation coefficient, \( r \) is \(-0.992\) correct to 3 s.f.

If we observe the differences of consecutive \( y \) values such as \( 18.8 - 16.9 = 1.9 \) etc:

\[
\begin{array}{c|cccccc}
D & 1.9 & 2.4 & 2.8 & 3.1 & 3.7 & 4.1 \\
\hline
\end{array}
\]

We would notice that differences follow an increasing trend, which would not be the case if the data have followed a linear model (the differences would be more or less kept constant).
Though the linear model is not the best, it is not obvious from the diagram to conclude that the points necessary follow a non-linear model.

8(iii) Let $r_1$ and $r_2$ denote the product moment correlation coefficients based on the linear model and the quadratic model respectively. If $|r_2| > |r_1|$, then we would deem the quadratic model to be the better model. If $|r_2| < |r_1|$, then the linear model would be better.

8(iv) From the graphic calculator, the equation of the least-squares regression line of $y$ on $x^2$ is

$$y = -0.85621x^2 + 22.23049.$$  

When $x = 3.2$, the estimate value of $y$ is

$$y = -0.85621(3.2)^2 + 22.23049 = 13.5$$ correct to 3 s.f.

**Note:** We can of course find the value of $r$ for the quadratic model and do a comparison with that obtained for the linear model. However, since the question asked us to “explain how to use”, it will be more ideal to just propose a general way to help decide which model is better.

Some TI84 Plus screenshots:

<table>
<thead>
<tr>
<th>L1</th>
<th>L2</th>
<th>L3 * 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>10.8</td>
<td>4.25</td>
</tr>
<tr>
<td>3.5</td>
<td>19.5</td>
<td>9.25</td>
</tr>
<tr>
<td>4.5</td>
<td>11.2</td>
<td>12.25</td>
</tr>
<tr>
<td>4.5</td>
<td>8.6</td>
<td>16.25</td>
</tr>
<tr>
<td>5</td>
<td>4.9</td>
<td>20.25</td>
</tr>
<tr>
<td>L3 = &quot;L1^2&quot;</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**LinReg(ax+b)**

Xlist:L3
Ylist:L2
FreqList:
Store RegEQ:
Calculate

```
y=a+b
a=-.8562096195
b=22.23049174
r^2=.9999676152
r=+.9999838075
```
A probability tree is drawn for the situation:

```
   A
  0.05  Faulty
/   \
0.6  0.95  Not faulty
   /   \
0.4  0.07  Faulty
  /   \    
B  0.93  Not faulty
```

**Remark:** A probability tree is for the purpose of clearer illustration. If the question did not ask, it is not necessary to draw a probability tree and use it to solve probability problems.

9(i)(a) \( P(\text{a lens is faulty}) = 0.6 \times 0.05 + 0.4 \times 0.07 = 0.058. \)

9(i)(b) \( P(\text{a lens is made by } A \text{ given that the lens is faulty}) = P(\text{lens is made by } A | \text{lens is faulty}) = \frac{P(\text{lens is made by } A \text{ and lens is faulty})}{P(\text{lens is faulty})} = \frac{0.6 \times 0.05}{0.058} = 0.517 \text{ correct to 3 s.f.} \)

9(ii)(a) \( P(\text{exactly one of the lenses is faulty}) = P(\text{1st lens faulty and 2nd lens not faulty}) + P(\text{1st lens not faulty and 2nd lens faulty}) = 0.058 \times (1 - 0.058) + (1 - 0.058) \times 0.058 = 0.109272 \)

9(ii)(b) \( P(\text{both were made by } A, \text{ given that exactly one is faulty}) = P(\text{both were made by } A | \text{exactly one is faulty}) = \frac{P(\text{both were made by } A \text{ and exactly one is faulty})}{P(\text{exactly one is faulty})} = \frac{(0.6 \times 0.05)(0.6 \times 0.95) \times 2}{0.109272} = 0.313 \text{ correct to 3 s.f.} \)

9(ii) We can also use the complementary method to obtain the probability:
\[ 1 - P(\text{both faulty}) - P(\text{both not faulty}) \]

**Note 2:** The answer in this case is exact, thus it is not necessary to round it off to 3 significant figures.

10(i) Let \( \mu \) denote the mean time taken to install a component.

Null hypothesis \( H_0 \) is \( \mu = 38.0 \).
Alternative hypothesis \( H_1 \) is \( \mu < 38.0 \).
### 10(ii)

To reject null hypothesis, $p$-value ≤ level of significance. That is, $P(\bar{T} \leq T) \leq 0.05$

\[
\Rightarrow P \left( Z \leq \frac{\bar{T} - 38.0}{5.0} \right) \leq 0.05
\]

\[
\Rightarrow \left( \bar{T} - 38.0 \right) \frac{50}{5} \leq -1.64485
\]

\[
\Rightarrow \bar{T} \leq 36.837.
\]

Since $\bar{T}$ denotes the sample mean time, therefore the set of values required is \( \{ \bar{T} \in \mathbb{R} : 0 < \bar{T} < 36.8 \} \).

### 10(iii)

If the null hypothesis is not rejected, $p$-value > level of significance. That is, $P(\bar{T} \leq T) > 0.05$

\[
\Rightarrow P \left( Z \leq \frac{37.1 - 38.0}{5.0} \right) > 0.05
\]

\[
\Rightarrow -0.9\sqrt{n} > -1.64485
\]

\[
\Rightarrow n < 83.5
\]

Since $n$ is a positive integer, the set of values required can be written as \( \{ n \in \mathbb{Z} : 1 \leq n \leq 83 \} \).

### 11(i)

\[
P(R = 4) = \binom{18}{4} \binom{12}{6} \binom{30}{10} = 0.0941 \text{ correct to 3 s.f.}
\]

### 11(ii)

Starting with $P(R = r) > P(R = r + 1)$, we have

\[
\binom{18}{r} \binom{12}{10-r} > \binom{18}{r+1} \binom{12}{9-r}
\]

\[
\Rightarrow \binom{18}{r} \binom{12}{10-r} > \binom{18}{r+1} \binom{12}{10-r}
\]

\[
\Rightarrow \frac{18!}{r!(18-r)!} > \frac{18!}{(r+1)!(17-r)!}
\]

(Shown!)

---

**Note 1:** \( \binom{n}{r} = \frac{n!}{r!(n-r)!} \)

**Note 2:** Most probable number here refers to the mode of $R$.

**Note 3:** We can use TI84 Plus to verify the correctness of $r$. 

![TI84 Plus screenshot](image-url)
\[
\begin{align*}
\Rightarrow r+1 & > \frac{10-r}{3+r} \\
18-r & > 3r \\
(18-r)(10-r) & > (r+1)(3+r) \\
r & > 5.53 \\
\end{align*}
\]
Thus, the required value of \( r \) is 6.

(iii) Let \( X \) denote the number of people joining the queue in a period of 4 minutes. Then \( X \sim \text{Po}(4.8) \).

Required probability
\[
P(X \geq 8) = 1 - P(X \leq 7) = 0.113 \text{ correct to 3 s.f.}
\]

(ii) Let \( Y \) denote the number of people joining the queue in a period of \( t \) seconds.

Then \( Y \sim \text{Po} \left( \frac{1.2}{60}t \right) \Rightarrow Y \sim \text{Po}(0.02t) \)

Given \( P(Y \leq 1) = 0.7 \)
\[
P(Y = 0) + P(Y = 1) = 0.7.
\]
Thus \( e^{-0.02t} \left( 0.02t \right)^0 + e^{-0.02t} \left( 0.02t \right)^1 / 1! = 0.7 \).
\[
e^{-0.02t} + 0.02re^{-0.02t} = 0.7.
\]

From the graphic calculator, \( t = 54.87 \).
Thus \( t = 55 \) seconds (correct to the nearest whole number).

(iii) Let \( J \) denote the number of people joining the queue in a period of 15 minutes and let \( L \) denote the number of people joining the queue in a period of 15 minutes. Then \( J \sim \text{Po}(15 \times 1.2) \) and \( L \sim \text{Po}(15 \times 1.8) \)
\[
\Rightarrow J \sim \text{Po}(18) \text{ and } L \sim \text{Po}(27).
\]
Since in both cases, the mean number is larger than 10. Thus, \( J \sim \text{N}(18, 18) \) and \( L \sim \text{N}(27, 27) \) approximately.

This means \( J - L \sim \text{N}(18 - 27, 18 + 27) \) approximately.
\[
\Rightarrow J - L \sim \text{N}(-9, 45).
\]
By graphic calculator, the required probability is \( P(J - L \geq -11) = 0.617 \) correct to 3 s.f.

(iv) If the time period is in terms of several hours, the rate of people joining the check-in queue would probably not be constant throughout, which violates one of the conditions for a Poisson model to be valid.

Remark: In the process of computing \( P(J - L \geq -11) \), we did not do a continuity correction even though \( J - L \) is a discrete random variable (it follows a Skellam distribution).

Remark: The rate of people joining the queue would certainly be higher at a certain period of time just before the departure of a flight.