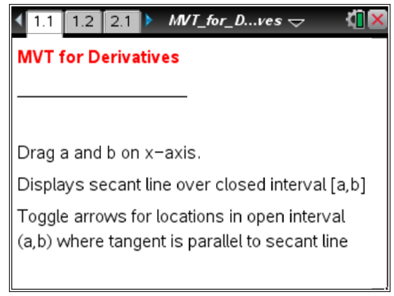
MVT for Derivatives



Math Objectives

- Students will be able to interpret the statement of the Mean Value Theorem in terms of its graphical representation.
- Students will be able to use slopes of secant and tangent lines to explain the relationship between average and instantaneous rates of change demonstrated by the MVT.
- Students will be able to identify functions and/or intervals for which the MVT cannot be applied.

Vocabulary

- secant line
- tangent line
- average rate of change
- instantaneous rate of change

About the Lesson

- This lesson uses a graphical representation of the Mean Value Theorem (MVT) to demonstrate how the theorem relates information about the average rate of change of a function to an instantaneous rate of change.
- As a result, students will:
    - Change the endpoints of intervals and relate the changes in the slopes of secant lines to the average rate of change.
    - Observe that for the first two functions provided there is always a point in the intervals where the tangent line is parallel to the secant line connecting the endpoints, supporting the conclusion of the MVT.
    - Observe a function that is not everywhere differentiable to note when the conclusion of the MVT will not hold.

TI-Nspire™ Navigator™ System

- Use Class Capture to check students' reasoning about a range of intervals and to explore multiple solutions.
- Use Quick Poll to assess student understanding of the Mean Value Theorem.

Activity Materials

Compatible TI Technologies: TI-Nspire™ CX Handhelds, TI-Nspire™ Apps for iPad®, TI-Nspire™ Software

Tech Tips:

- This activity runs best on a computer. It is runs slowly on a handheld device. Dragging points *a* and/or *b* will be a little faster on a handheld device if the tangent line feature is turned off.
- Watch for additional Tech Tips throughout the activity for the specific technology you are using.
- Access free tutorials at http://education.ti.com/calculators/pd/US/Online-Learning/Tutorials

Lesson Files:
Student Activity

- MVT_for_Derivatives_Student.pdf
- MVT_for_Derivatives_Student.doc

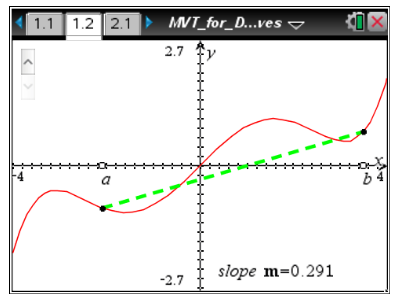
TI-Nspire document

- MVT_for_Derivatives.tns

Discussion Points and Possible Answers

> **Tech Tip:** If students experience difficulty dragging a point, check to make sure that they have moved the cursor until it becomes a hand ( ) getting ready to grab the point. If tab appears next to the object name, then there are multiple objects near the cursor and you may have to press ctrl to cycle through to the object you desire to grab. Also, be sure that the word *point* appears, not the word *text.* Then press ctrl to grab the point and close the hand ( ).

The Mean Value Theorem states: If ***f*** is continuous on the interval [*a, b*] and differentiable on the open interval (*a, b*), then there must exist at least one number *c* in the interval (*a, b*) such that $\mathbf{f}\,'(c)=\dfrac{\mathbf{f}(b)-\mathbf{f}(a)}{b-a}$. In this activity, you will explore a visual representation of the theorem and consider some of its applications.



Move to page 1.2.

1. The screen displays a graph of the function **f** with the secant line drawn on the closed interval [*a, b*]. Drag the point *a* and/or the point *b* along the *x*-axis to change the interval and note changes in the secant line.

   a. The slope of the secant line is displayed. Determine the slope of the secant line on the interval
   [–3, 2] _______________ [–1, 3]________________

   <u>Answers:</u> The slope is approximately .29 on the interval [–3, 2] and .34 on [–1, 3].

> **Tech Tip:** Students can use the Scratchpad feature to verify the slope that is displayed on the graph. For example, since the function graphed has been defined as $\mathbf{f}1(x)$, students can enter $\dfrac{\mathbf{f}1(2)-\mathbf{f}1(-3)}{2-(-3)}$ to find the slope on the interval [–3, 2].

b. This slope can be interpreted as the average rate of change of the function values **f** over the interval [*a, b*]. Explain how this is related to the conclusion of the Mean Value Theorem as written above.

Sample answer: The conclusion of the MVT states that $\mathbf{f}'(c) = \dfrac{\mathbf{f}(b) - \mathbf{f}(a)}{b - a}$. The right-hand side of this equation is the slope of the line through the points (*a*,f(a)) and (*b*,f(b)) and can be interpreted as the average rate of change of the function *f* over the interval [*a*,*b*]. The total change in function values *f*(*b*) – *f*(*a*), divided by the change in *x* (the length of the interval *b*-*a*), is the average rate of change.

2. Press the up arrow to see locations in the open interval (*a, b*) where the tangent line is parallel to the secant line displayed for the interval [*a, b*].

a. What does the slope of the tangent line represent?

Sample answer: The slope represents the rate of change of the function at a specific value, the instantaneous rate of change, or the derivative of the function $\mathbf{f}'(c)$ at a point.

Since the tangent line is parallel to the secant line and has the same slope, students might note that this is also the average rate of change.

> **Teacher Tip:** This is a good time to review definitions and discuss the relationships among the terms students have used to answer the question**.** It is important to connect the right-hand side of the conclusion of the MVT described in question 1 to the left-hand side of the equation, which is written symbolically as $\mathbf{f}'(c)$, the derivative of the function at a value.

b. Continue dragging the point *a* and/or the point *b* to explore different intervals. Is it always possible to find a point where the tangent line is parallel to the secant line?

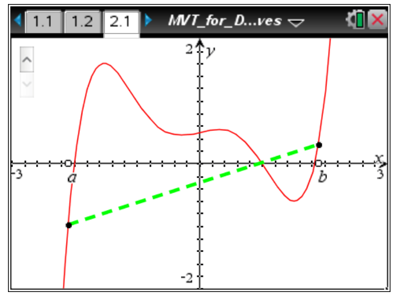
Answer: Yes

TI-Nspire Navigator Opportunity: *Class Capture*
See Note 1 at the end of this lesson.

c. Explain how this is related to the conclusion of the Mean Value Theorem as written above.

Answer: This means it is always possible to find a point *c* in the interval (*a, b*) where the instantaneous rate of change is equal to the average rate of change of the function over the interval [*a, b*].

Move to page 2.1.



3. Drag the point *a* and/or the point *b* to change intervals for this continuous and differentiable function. Press the up arrow to see locations in the open interval (*a, b*) where the tangent line is parallel to the secant line.

   a. Can you find an interval [*a, b*] where there is more than one value in (*a, b*) such that the instantaneous rate of change is equal to the average rate of change? If so, give an example.

   Answer: Yes. (There are many such intervals, some having as many as three points where the tangent lines displayed are parallel to the secant line.)

   b. Can you find an interval [*a, b*] where there are no points in (*a, b*) such that the instantaneous rate of change is equal to the average rate of change? If so, give an example.

   Answer: No. That is the whole point of the Mean Value Theorem!

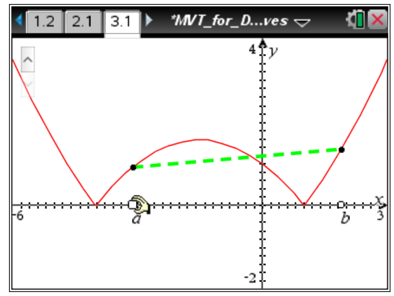
4. Cal says that according to the Mean Value Theorem, it is not possible to find a polynomial function such that: $\mathbf{f}(0) = -1$, $\mathbf{f}(2) = 4$, and $\mathbf{f}'(x) \leq 2$ for all *x* in the interval [0, 2].

   Explain how Cal might support his argument both numerically and graphically.

   Sample answer: Numerically, you can use the first two conditions to calculate the average rate of change $\frac{4-(-1)}{2-0} = \frac{5}{2}$. According to the MVT, there must be a point in the interval where $\mathbf{f}'(x) = \frac{5}{2}$, which contradicts the final condition that $\mathbf{f}'(x) \leq 2$ for all *x* in the interval [0, 2]. Graphically, you could plot the points (0, –1) and (2, 4) and imagine tracing possible curves between these points. The tangent line at some point along the curve has to be at least as steep as the straight line joining these two points.

> **Teacher Tip:** You might want to come back to this question after exploring question 5 to discuss what other conditions are assumed in Cal's argument. For example, you were restricted to polynomial functions, which all meet the conditions of continuity and differentiability necessary to apply the MVT.

Move to page 3.1.



5. Press the up arrow to display locations in the open interval (a, b) where the tangent line is parallel to the secant line for this new function.

   a. Move a and b to display a secant line for the interval [–5, –3]. Is there a tangent line shown?

   Answer: The tangent line disappears.

   b. Explain what this answer means in terms of rates of change.

   Sample answer: There is no point in the interval [–5, –3] where the instantaneous rate of change (the derivative of the function) is equal to the average rate of change of the function over this interval. Alternatively, there is no value c in the interval for which $\mathbf{f}'(c)$ is equal to the average rate of change over this interval.

   c. Is this a violation of the Mean Value Theorem? Explain why or why not.

   Answer: This is not a contradiction of the theorem because while the function is continuous on [–5, –3], the function is not differentiable on the interval (–5, –3) and thus does not meet the initial conditions of the MVT. Specifically, the function is not differentiable at $x = –4$.

   Teacher Tip: While the graph provides visual evidence that the function is not differentiable at –4 and 1, students should be cautioned against determining the differentiability of the functions provided in this activity based solely on inspection. Students should be encouraged to zoom in on the graph or investigate the function rule, $\mathbf{f}(x) = \left|\frac{x^2}{4} + \frac{3x}{4} - 1\right|$ in this case, to determine differentiability. You might also want to have a whole class review the concept of differentiability before moving to the next question.

   d. Drag points a and b to display each of the intervals and complete the table below:

Interval	Does a parallel tangent line exist?	Is the function differentiable on the open interval?
[–3, 0]	**yes**	**yes**
[–3, 2]	**yes**	**no**
[–1, 2]	**no**	**no**

e. Explain what these results tell you about applying the Mean Value Theorem.

Sample answer: A function must be differentiable on the interval in order to accurately apply the Mean Value Theorem. Although it is often possible to find points where the instantaneous rate of change, **f'**(x), is equal to the average rate of change in intervals for which the function is not everywhere differentiable, this is not always the case. Further, the converse of the Mean Value Theorem is not necessarily true.

TI-Nspire Navigator Opportunity: *Quick Poll* (*Multiple Choice* or *Open Response*)
See Note 2 at the end of this lesson.

Wrap Up

Upon completion of the discussion, the teacher should ensure that students understand:

- How the Mean Value Theorem can be expressed symbolically, graphically (as slopes of secants and tangent lines), and verbally (comparing instantaneous and average rates of change).
- The conditions that must be met in order to apply the Mean Value Theorem.

Assessment

Brian got a speeding ticket driving to the Kansas State football game on Saturday. He left at 1:00 p.m. and was issued a speeding ticket at 1:40 p.m. after driving for 45 miles. Brian claims he never drove over the 65 mile per hour speed limit. Use the Mean Value Theorem to plead Brian's case or to argue against him.

TI-Nspire Navigator

Note 1

Question 2, *Class Capture*: Use Class Capture to display several student screens to illustrate that a tangent line is displayed for every interval. To check for understanding of the interval notation, you might assign specific intervals for students to display as in question 1. Alternatively, you could ask students to display an interval in which the rate of change is negative, positive, and/or zero.

Note 2

Question 5, *Quick Poll*: The question stated below can be written as an open response or multiple choice item to assess student understanding of the Mean Value Theorem.
(It could also be added as Page 4.1 to the TI-Nspire document.)

Use the terms provided to complete the following statement:

"If the initial conditions are met, that is, if **f**(x) is ____(1)____ on the interval [a,b] and ____(2)____ on (a,b), the Mean Value Theorem guarantees that there is a point in the interval where the instantaneous rate of change is equal to the ____(3)____ over the interval. Graphically this means that there is a point in the interval where the ____(4)____ line is ____(5)____ to the secant line."

Terms:

- **a.** average rate of change
- **b.** parallel
- **c.** differentiable
- **d.** continuous
- **e.** tangent