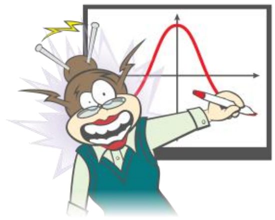
Applications of Differential Equations
Population & Newton's Law of Cooling
Revision Sheet



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Each of the questions included here can be solved using the TI-Nspire CX CAS.

Question 1

The number of moths N in a colony grows at a rate proportional to the current number. Initially there were 90 moths, and after 5 months there were 150 moths present. How many moths are present after 10 months?

Response:

Question 2

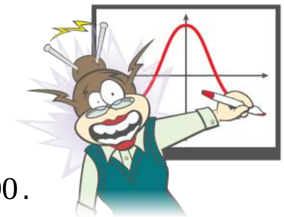
A town's population increases at a rate proportional to the square root of its current population. At the start of a study the population was 1 million, and 8 years later the population was 4 million. What is the population of the town 12 years after the study commenced?

Response:

Question 3

A population P of rabbits t years after observation commenced is modelled by the differential equation $\frac{dP}{dt} = kP(375 - P)$. It is known that the initial population of the rabbits is 25, and after a year the population is 125. Show that $P = \frac{375}{1+14e^{-375kt}}$ where $k = \frac{1}{375}\log_e(7)$. Hence, determine the maximum population of rabbits.

Response:



Question 4

A radioactive substance decays at a rate proportional to the amount Q present at time t. That is, $\frac{dQ}{dt} = -kQ$. Initially, $Q = 500$ and when $t = 30$, $Q = 200$. Find Q when $t = 90$.

Response:

Question 5

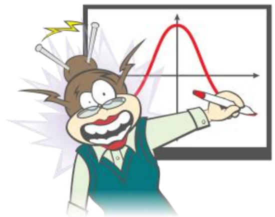
Two substances, A and B, react chemically to form a substance X. Initially there are 4 grams of substance A, 5 grams of substance B and substance X is not present. After 5 minutes, 4 grams of substance X has formed and the reaction rate is $\frac{dx}{dt} = k\left(4 - \frac{x}{2}\right)\left(5 - \frac{x}{2}\right)$ where there is x grams of substance X at time t minutes after the reaction started ($x \in [0,8)$). Find $x(t)$ and hence find the amount of substance X present after 10 minutes.

Response:

Question 6

A container of hot water at 100°C is placed on a kitchen bench with a constant ambient temperature of 28°C. After 3 minutes, the temperature of the water is 76°C. Assuming that Newton's law of Cooling $\frac{d\theta}{dt} = -k\left(\theta - \theta_a\right)$ applies, find the temperature of the water in the container after 6 minutes.

Response:

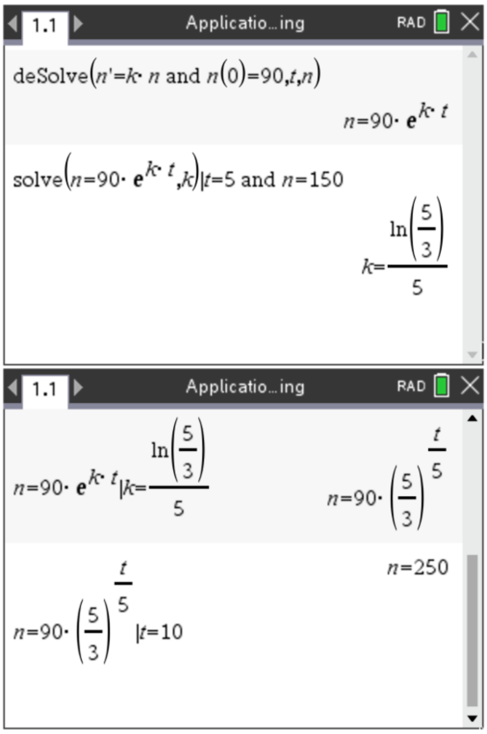


Answers*

Question 1

$N(10) = 250$ moths

In a Calculator application, the deSolve command can be used with the differential equation and initial condition to determine $N(t)$. To find the value of the constant k, the other condition can be used. This value of k can then be substituted back into the solution to fully define $N(t)$, after which $t = 10$ can be substituted in to find $N(10)$.



$$\frac{dN}{dt} = kN \text{ with } N(0) = 90$$

$$N(t) = 90e^{kt}$$

$$k = \frac{1}{5}\log_e\left(\frac{5}{3}\right) \text{ since } N(5) = 150$$

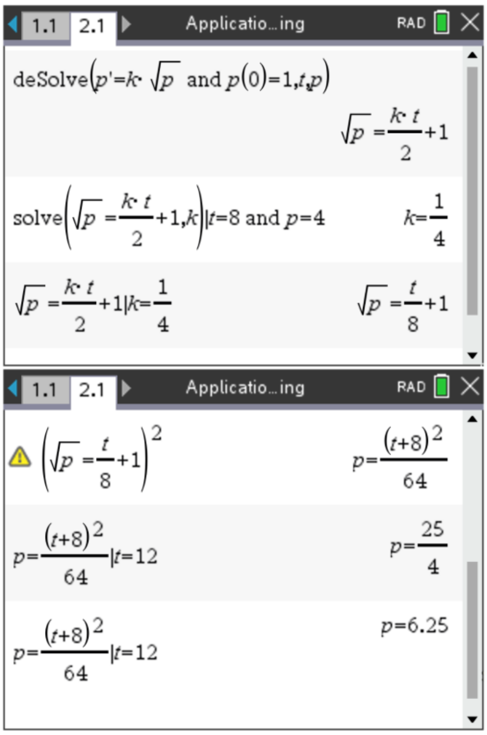
$$N(t) = 90\left(\frac{5}{3}\right)^{\frac{t}{5}}$$

$$N(10) = 250$$

Question 2

6.5 million

In a Calculator application, the deSolve command can be used with the differential equation and initial condition to determine $P(t)$. To find the value of the constant k, the other condition can be used. This value of k can then be substituted back into the solution to fully define $P(t)$, after which $t = 12$ can be substituted to find $P(12)$.



$$\frac{dP}{dt} = k\sqrt{P} \text{ with } P(0) = 1$$

$$\sqrt{P(t)} = \frac{kt}{2} + 1$$

$$k = \frac{1}{4} \text{ since } P(8) = 4$$

$$\sqrt{P(t)} = \frac{t}{8} + 1$$

$$P(t) = \frac{(t+8)^2}{64}$$

$$P(12) = \frac{25}{4} = 6.25$$

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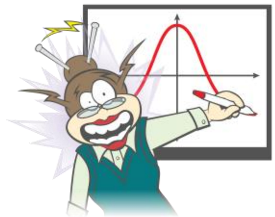
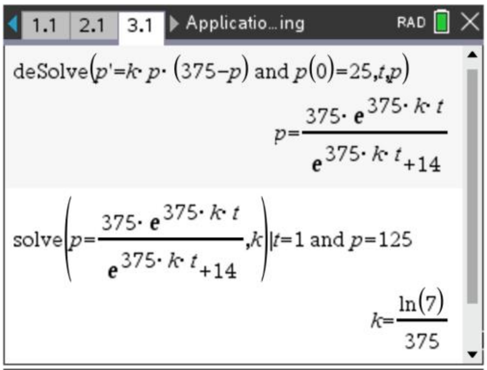
* When using CAS as a calculation and/or algebraic manipulation tool, it is important to set out by-hand working clearly.

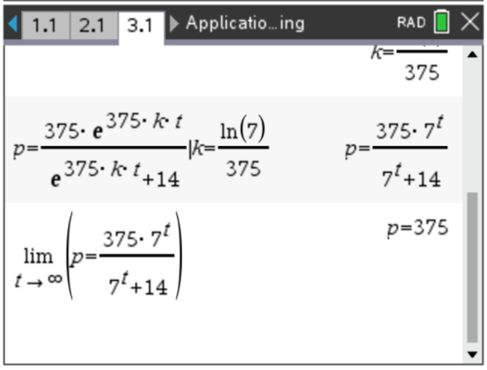


Question 3

$P_{\max} = 375$ rabbits

In a Calculator application, the deSolve command can be used with the differential equation and initial condition to determine $P(t)$. To find the value of the constant k, the other condition can be used. This value of k can then be substituted back into the solution to fully define $P(t)$, after which the limit of $P(t)$ as $t \to \infty$ can be found.





$$\frac{dP}{dt} = kP(375 - P) \text{ with } P(0) = 25$$

$$P(t) = \frac{375e^{375kt}}{e^{375kt} + 14}, \text{ now multiply by } \frac{e^{-375kt}}{e^{-375kt}}:$$

$$= \frac{375}{1 + 14e^{-375kt}}$$

$$k = \frac{1}{375}\log_e(7) \text{ since } P(1) = 125$$

$$P(t) = \frac{375}{1 + 14e^{-\log_e(7)t}}$$

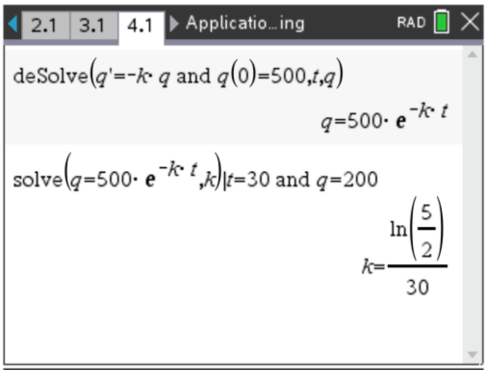
$$= \frac{375}{1 + 14(7^{-t})}$$

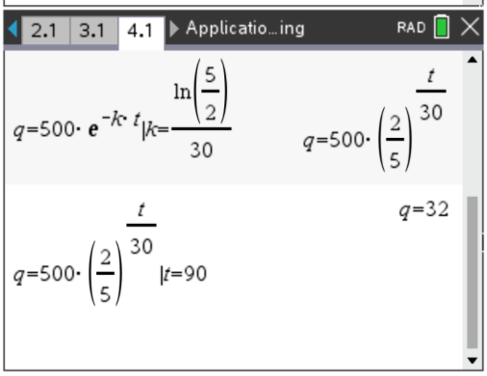
$$P_{\max} = \lim_{t\to\infty} P(t) = 375$$

Question 4

$Q(90) = 32$

In a Calculator application, the deSolve command can be used with the differential equation and initial condition to determine $Q(t)$. To find the value of the constant k, the other condition can be used. This value of k can then be substituted back into the solution to fully define $Q(t)$, after which $t = 90$ can be substituted to find $Q(90)$.





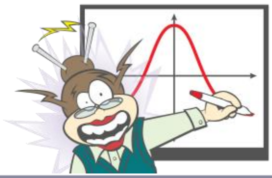
$$\frac{dQ}{dt} = -kQ \text{ with } Q(0) = 500$$

$$Q(t) = 500e^{-kt}$$

$$k = \frac{1}{30}\log_e\left(\frac{5}{2}\right) \text{ since } Q(30) = 200$$

$$Q(t) = 500\left(\frac{2}{5}\right)^{\frac{t}{30}}$$

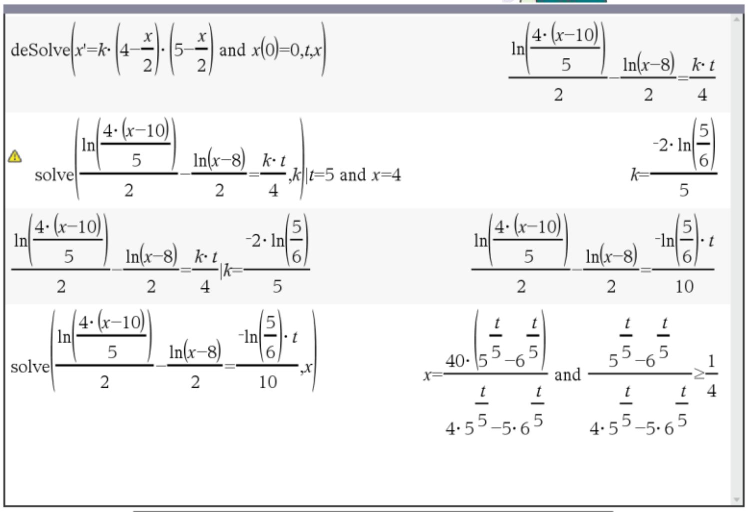
$$Q(90) = 32$$

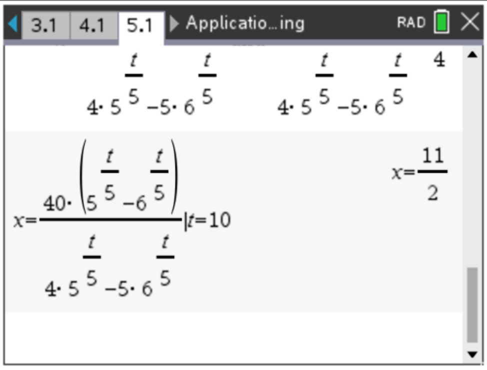


Question 5

$x(10) = 5.5$ grams

In a Calculator application, the deSolve command can be used with the differential equation and initial condition to determine $x(t)$. To find the value of the constant k, the other condition can be used. This value of k can then be substituted back into the solution to fully define $x(t)$, after which $t = 10$ can be substituted to find $x(10)$.





$$\frac{dx}{dt} = k\left(4-\frac{x}{2}\right)\left(5-\frac{x}{2}\right) \text{ with } x(0)=0$$

$$x(t) = \frac{40\left(e^{\frac{kt}{2}}-1\right)}{5e^{\frac{kt}{2}}-4}$$

$$k = -\frac{2}{5}\log_e\left(\frac{5}{6}\right) \text{ since } x(5)=4$$

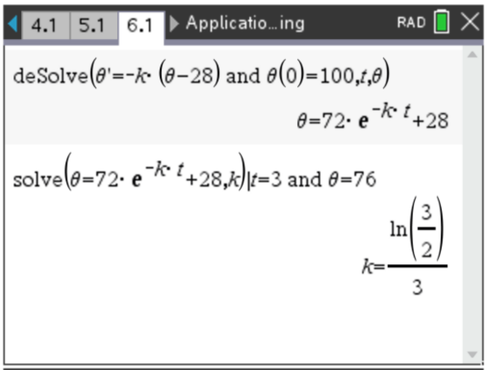
$$x(t) = \frac{40\left(\sqrt[5]{5^t}-\sqrt[5]{6^t}\right)}{4\sqrt[5]{5^t}-5\sqrt[5]{6^t}}$$

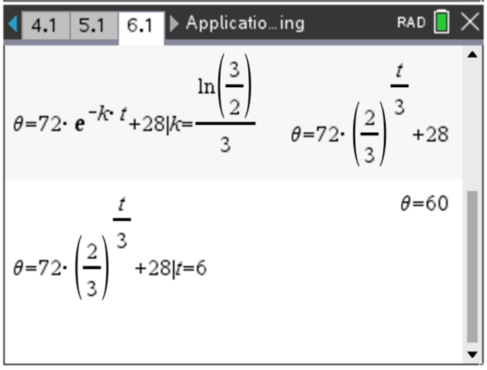
$$x(10) = 5.5$$

Question 6

$\theta(6) = 60^\circ\mathrm{C}$

In a Calculator application, the deSolve command can be used with the differential equation and initial condition to determine $\theta(t)$. To find the value of the constant k, the other condition can be used. This value of k can then be substituted back into the solution to fully define $\theta(t)$, after which $t = 6$ can be substituted to find $\theta(6)$.





$$\frac{d\theta}{dt} = -k(\theta-28) \text{ with } \theta(0)=100$$

$$\theta(t) = 72e^{-kt}+28$$

$$k = \frac{1}{3}\log_e\left(\frac{3}{2}\right) \text{ since } \theta(3)=76$$

$$\theta(t) = 72\left(\frac{2}{3}\right)^{\frac{t}{3}}+28$$

$$\theta(6) = 60$$