

Applications of Differential Equations Population & Newton's Law of Cooling Revision Sheet

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Each of the questions included here can be solved using the TI-Nspire CX CAS.

Question 1

The number of moths N in a colony grows at a rate proportional to the current number. Initially there were 90 moths, and after 5 months there were 150 moths present. How many moths are present after 10 months?

Response:

Question 2

A town's population increases at a rate proportional to the square root of its current population. At the start of a study the population was 1 million, and 8 years later the population was 4 million. What is the population of the town 12 years after the study commenced?

Response:

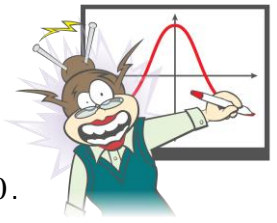
Question 3

A population P of rabbits t years after observation commenced is modelled by the differential equation

$\frac{dP}{dt} = kP(375 - P)$. It is known that the initial population of the rabbits is 25, and after a year the population is

125. Show that $P = \frac{375}{1 + 14e^{-375kt}}$ where $k = \frac{1}{375} \log_e(7)$. Hence, determine the maximum population of rabbits.

Response:



Question 4

A radioactive substance decays at a rate proportional to the amount Q present at time t .

That is, $\frac{dQ}{dt} = -kQ$. Initially, $Q = 500$ and when $t = 30$, $Q = 200$. Find Q when $t = 90$.

Response:

Question 5

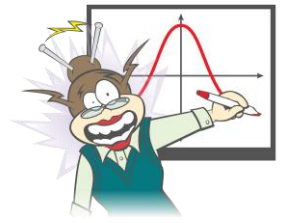
Two substances, A and B , react chemically to form a substance X . Initially there are 4 grams of substance A , 5 grams of substance B and substance X is not present. After 5 minutes, 4 grams of substance X has formed and the reaction rate is $\frac{dx}{dt} = k\left(4 - \frac{x}{2}\right)\left(5 - \frac{x}{2}\right)$ where there is x grams of substance X at time t minutes after the reaction started ($x \in [0, 8)$). Find $x(t)$ and hence find the amount of substance X present after 10 minutes.

Response:

Question 6

A container of hot water at 100°C is placed on a kitchen bench with a constant ambient temperature of 28°C . After 3 minutes, the temperature of the water is 76°C . Assuming that Newton's law of Cooling $\frac{d\theta}{dt} = -k(\theta - \theta_a)$ applies, find the temperature of the water in the container after 6 minutes.

Response:



Answers*

Question 1

$$N(10) = 250 \text{ moths}$$

In a Calculator application, the deSolve command can be used with the differential equation and initial condition to determine $N(t)$. To find the value of the constant k , the other condition can be used. This value of k can then be substituted back into the solution to fully define $N(t)$, after which $t = 10$ can be substituted in to find $N(10)$.

$$\frac{dN}{dt} = kN \text{ with } N(0) = 90$$

$$N(t) = 90e^{kt}$$

$$k = \frac{1}{5} \log_e \left(\frac{5}{3} \right) \text{ since } N(5) = 150$$

$$N(t) = 90 \left(\frac{5}{3} \right)^{\frac{t}{5}}$$

$$N(10) = 250$$

```

1.1 Application RAD
deSolve(n'=k*n and n(0)=90,t,n)
n=90*e^{k*t}
solve(n=90*e^{k*t},k)|t=5 and n=150
ln(5/3)
k=-----
5
  
```

```

1.1 Application RAD
n=90*e^{k*t}|k=ln(5/3)/5
n=90*(5/3)^{t/5}
n=250
n=90*(5/3)^{t/5}|t=10
  
```

Question 2

$$6.5 \text{ million}$$

In a Calculator application, the deSolve command can be used with the differential equation and initial condition to determine $P(t)$. To find the value of the constant k , the other condition can be used. This value of k can then be substituted back into the solution to fully define $P(t)$, after which $t = 12$ can be substituted to find $P(12)$.

$$\frac{dP}{dt} = k\sqrt{P} \text{ with } P(0) = 1$$

$$\sqrt{P(t)} = \frac{kt}{2} + 1$$

$$k = \frac{1}{4} \text{ since } P(8) = 4$$

$$\sqrt{P(t)} = \frac{t}{8} + 1$$

$$P(t) = \frac{(t+8)^2}{64}$$

$$P(12) = \frac{25}{4} = 6.25$$

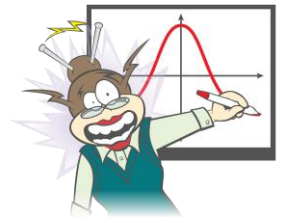
```

1.1 2.1 Application RAD
deSolve(p'=k*sqrt(p) and p(0)=1,t,p)
sqrt(p)=k*t/2+1
solve(sqrt(p)=k*t/2+1,k)|t=8 and p=4
k=1/4
sqrt(p)=k*t/2+1|k=1/4
sqrt(p)=t/8+1
  
```

```

1.1 2.1 Application RAD
((sqrt(p)=t/8+1)^2)
p=(t+8)^2/64
p=(t+8)^2/64|t=12
p=25/4
p=(t+8)^2/64|t=12
p=6.25
  
```

* When using CAS as a calculation and/or algebraic manipulation tool, it is important to set out by-hand working clearly.



Question 3

$$P_{\max} = 375 \text{ rabbits}$$

In a Calculator application, the deSolve command can be used with the differential equation and initial condition to determine $P(t)$. To find the value of the constant k , the other condition can be used. This value of k can then be substituted back into the solution to fully define $P(t)$, after which the limit of $P(t)$ as $t \rightarrow \infty$ can be found.

$$\frac{dP}{dt} = kP(375 - P) \text{ with } P(0) = 25$$

$$P(t) = \frac{375e^{375kt}}{e^{375kt} + 14}, \text{ now multiply by } \frac{e^{-375kt}}{e^{-375kt}}:$$

$$= \frac{375}{1 + 14e^{-375kt}}$$

$$k = \frac{1}{375} \log_e(7) \text{ since } P(1) = 125$$

$$P(t) = \frac{375}{1 + 14e^{-\log_e(7)t}}$$

$$= \frac{375}{1 + 14(7^{-t})}$$

$$P_{\max} = \lim_{t \rightarrow \infty} P(t) = 375$$

The first screenshot shows the command: `deSolve(p'=k*p*(375-p) and p(0)=25,t,p)`. The result is $p = \frac{375 \cdot e^{375 \cdot k \cdot t}}{e^{375 \cdot k \cdot t} + 14}$. Below it, the command `solve(p = \frac{375 \cdot e^{375 \cdot k \cdot t}}{e^{375 \cdot k \cdot t} + 14}, k | t=1 and p=125)` is shown, resulting in $k = \frac{\ln(7)}{375}$.

The second screenshot shows the command `k = \frac{\ln(7)}{375}` and the resulting expression $p = \frac{375 \cdot e^{375 \cdot k \cdot t}}{e^{375 \cdot k \cdot t} + 14}$ with $k = \frac{\ln(7)}{375}$. This simplifies to $p = \frac{375 \cdot 7^t}{7^t + 14}$. The final command `lim(p = \frac{375 \cdot 7^t}{7^t + 14}, t \to \infty)` results in $p = 375$.

Question 4

$$Q(90) = 32$$

In a Calculator application, the deSolve command can be used with the differential equation and initial condition to determine $Q(t)$. To find the value of the constant k , the other condition can be used. This value of k can then be substituted back into the solution to fully define $Q(t)$, after which $t = 90$ can be substituted to find $Q(90)$.

$$\frac{dQ}{dt} = -kQ \text{ with } Q(0) = 500$$

$$Q(t) = 500e^{-kt}$$

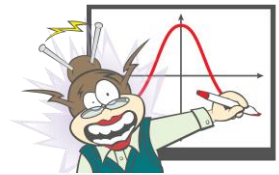
$$k = \frac{1}{30} \log_e\left(\frac{5}{2}\right) \text{ since } Q(30) = 200$$

$$Q(t) = 500 \left(\frac{2}{5}\right)^{\frac{t}{30}}$$

$$Q(90) = 32$$

The first screenshot shows the command: `deSolve(q'=-k*q and q(0)=500,t,q)`. The result is $q = 500 \cdot e^{-k \cdot t}$. Below it, the command `solve(q = 500 \cdot e^{-k \cdot t}, k | t=30 and q=200)` is shown, resulting in $k = \frac{\ln(\frac{5}{2})}{30}$.

The second screenshot shows the command `k = \frac{\ln(\frac{5}{2})}{30}` and the resulting expression $q = 500 \cdot e^{-k \cdot t}$ with $k = \frac{\ln(\frac{5}{2})}{30}$. This simplifies to $q = 500 \cdot \left(\frac{2}{5}\right)^{\frac{t}{30}}$. The final command `q = 500 \cdot \left(\frac{2}{5}\right)^{\frac{t}{30}} | t=90` results in $q = 32$.



Question 5

$$x(10) = 5.5 \text{ grams}$$

In a Calculator application, the deSolve command can be used with the differential equation and initial condition to determine $x(t)$. To find the value of the constant k , the other condition can be used. This value of k can then be substituted back into the solution to fully define $x(t)$, after which $t = 10$ can be substituted to find $x(10)$.

$$\frac{dx}{dt} = k \left(4 - \frac{x}{2} \right) \left(5 - \frac{x}{2} \right) \text{ with } x(0) = 0$$

$$x(t) = \frac{40 \left(e^{\frac{kt}{2}} - 1 \right)}{5e^{\frac{kt}{2}} - 4}$$

$$k = -\frac{2}{5} \log_e \left(\frac{5}{6} \right) \text{ since } x(5) = 4$$

$$x(t) = \frac{40 \left(\sqrt[5]{5^t} - \sqrt[5]{6^t} \right)}{4\sqrt[5]{5^t} - 5\sqrt[5]{6^t}}$$

$$x(10) = 5.5$$

Question 6

$$\theta(6) = 60^\circ \text{ C}$$

In a Calculator application, the deSolve command can be used with the differential equation and initial condition to determine $\theta(t)$. To find the value of the constant k , the other condition can be used. This value of k can then be substituted back into the solution to fully define $\theta(t)$, after which $t = 6$ can be substituted to find $\theta(6)$.

$$\frac{d\theta}{dt} = -k(\theta - 28) \text{ with } \theta(0) = 100$$

$$\theta(t) = 72e^{-kt} + 28$$

$$k = \frac{1}{3} \log_e \left(\frac{3}{2} \right) \text{ since } \theta(3) = 76$$

$$\theta(t) = 72 \left(\frac{2}{3} \right)^{\frac{t}{3}} + 28$$

$$\theta(6) = 60$$