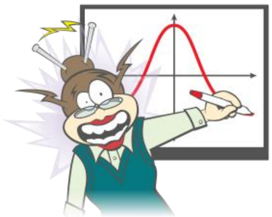
Areas, Volumes and Arc Lengths Revision Sheet



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Each of the questions included here can be solved using the TI-Nspire CX CAS.

Question 1

Consider the function $f:\mathbb{R}^+ \setminus \left\{\frac{1}{4}\right\} \to \mathbb{R}$, $f(x)=\frac{1}{2\sqrt{x}-1}$. Find the area A bounded by the graph of $y=f(x)$, the x-axis and the lines $x=1$ and $x=4$, in the form $\frac{1}{a}\left(a+\log_e(b)\right)$, where $a,b\in\mathbb{Z}^+$.

Response:

Question 2

If $f(x)=\frac{x}{\sqrt{2x+1}}$, $x\in\left(-\frac{1}{2},\infty\right)$ and $g(x)=-\frac{x}{\sqrt{3x+1}}$, $x\in\left(-\frac{1}{3},\infty\right)$, find the area of the region bounded by the graphs of $y=f(x)$, $y=g(x)$ and the line $x=2$, giving your answer in the form $\frac{1}{27}\left(a\sqrt{5}+b\sqrt{7}+c\right)$.

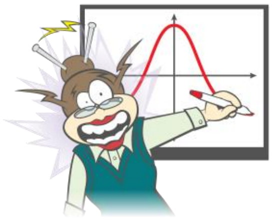
Response:

Question 3

The region bounded by the curve with equation $y=e^x+e^{-x}$, the x-axis and the lines $x=-1$ and $x=1$ is rotated about the x-axis to form a solid of revolution. Find the volume of this solid.

Response:

Question 4



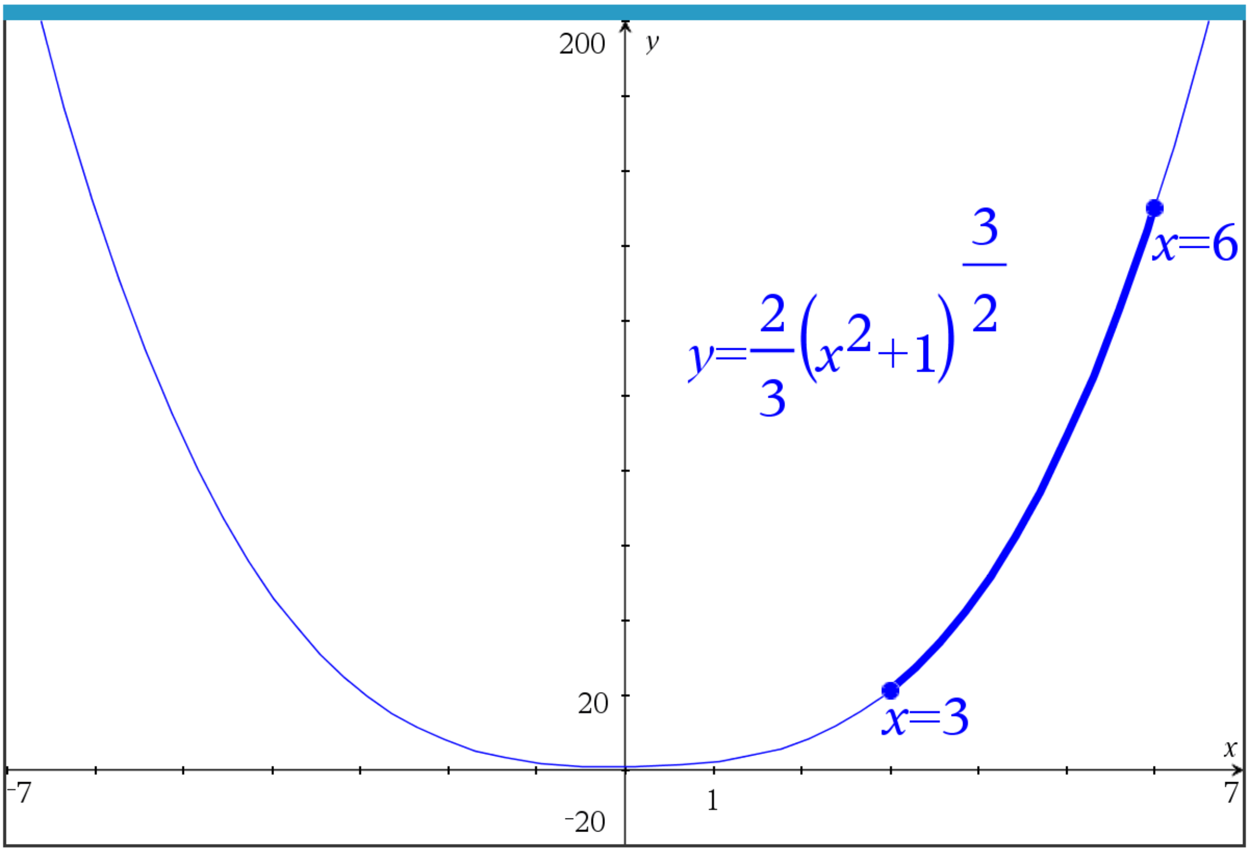
A parabolic bowl whose curved edge is modelled by the curve with equation $y = \frac{1}{5}x^2$ has water poured in to a depth of $12\ \text{cm}$. What is the volume of water in the bowl, in cm^3?

Response:

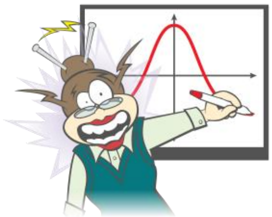
Question 5

The curve shown below has equation $y = \frac{2}{3}\left(x^2+1\right)^{\frac{3}{2}}$. The arc of the curve from $x = 3$ to $x = 6$ is shown in bold.

Find the exact arc length of the curve with equation $y = \frac{2}{3}\left(x^2+1\right)^{\frac{3}{2}}$ over the interval $x \in [3, 6]$.



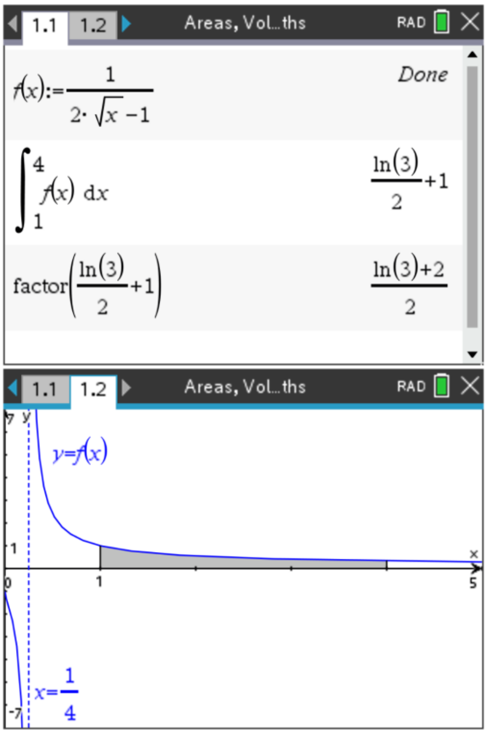
Response:



Answers*

Question 1

$A = \frac{1}{2}\left(2 + \log_e(3)\right)$ square units



Define the function $f(x)$ in a Calculator application, then sketch it in a Graphs application, to visualise the region considered in the question, especially to check if the region is above or below the x-axis.

Given that this region is above the x-axis, the area is given by the integral $A = \int_1^4 \frac{1}{2\sqrt{x}-1}\,dx$. Solving this by hand requires a u-substitution, where $u = \sqrt{x}$ and the integral is now

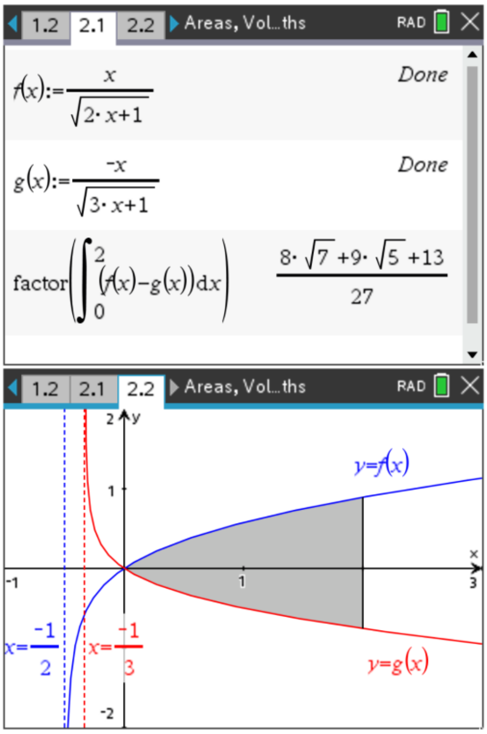
$$A = \int_1^2 \frac{2u}{2u-1}\,du = \int_1^2 \left(\frac{1}{2u-1} + 1\right)du.$$

This integral is able to be calculated by hand.

Using the TI-Nspire CX CAS involves defining $f(x)$ and performing the integration, then using the factor command to convert the answer into the required form.

Question 2

$\frac{1}{27}\left(9\sqrt{5} + 8\sqrt{7} + 13\right)$ square units



Define the functions $f(x)$ and $g(x)$ in a Calculator application, then sketch it in a Graphs application, to visualise the region bounded by the two curves and the line $x = 2$. The function that is "on top" and "below" the region can also be checked.

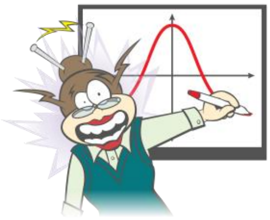
The area of the region is given by the integral

$$\int_0^2 \left(\frac{x}{\sqrt{2x+1}} - \left(-\frac{x}{\sqrt{3x+1}}\right)\right)dx = \int_0^2 \left(\frac{x}{\sqrt{2x+1}} + \frac{x}{\sqrt{3x+1}}\right)dx.$$

This integral is able to be calculated by hand using linear substitution(s).

Using the TI-Nspire CX CAS involves defining $f(x)$ and $g(x)$ and then performing the integration, with the factor command being used to express the answer as a single fraction, as required by the question.

* When using CAS as a calculation and/or algebraic manipulation tool, it is important to set out working clearly.



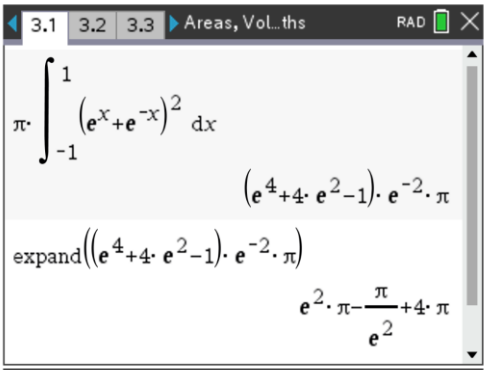
Question 3

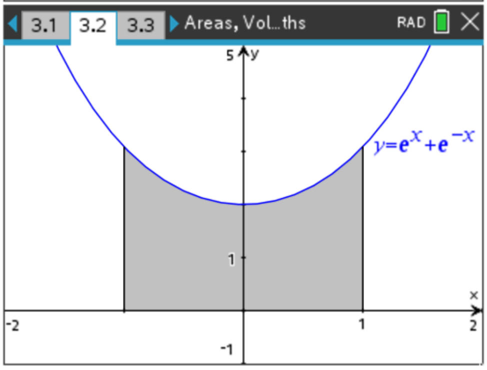
$\frac{\pi}{e^2}\left(e^4+4e^2-1\right)$ cubic units

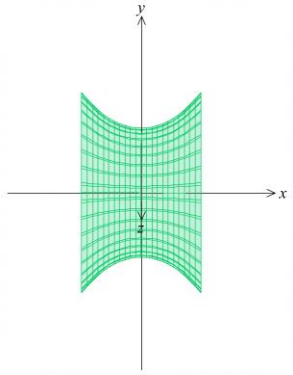
Sketch the curve in a Graphs application, to visualise the region bounded by the curve, the x-axis and the lines $x=-1$ and $x=1$. If desired, the solid of revolution may be sketched in a 3D Graphing page.

The volume of the solid is given by the integral $\pi\int_0^2\left(e^x+e^{-x}\right)^2 dx$.
This integral is able to be calculated by hand by expanding the integrand and integrating term-by-term.

Using the TI-Nspire CX CAS involves typing out the integral including the terminals, ensuring the π is preceding the integral, and finally using the expand command to express the result in a different form.







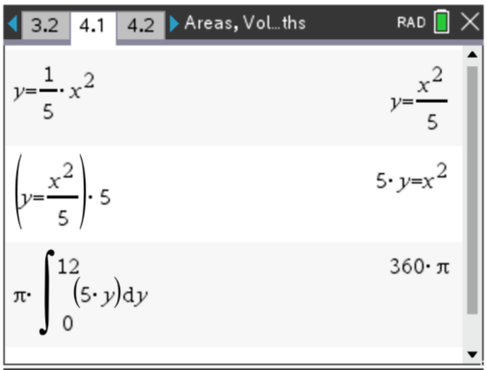
Question 4

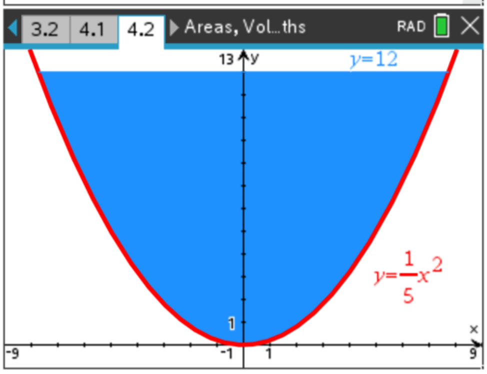
360π cm^3

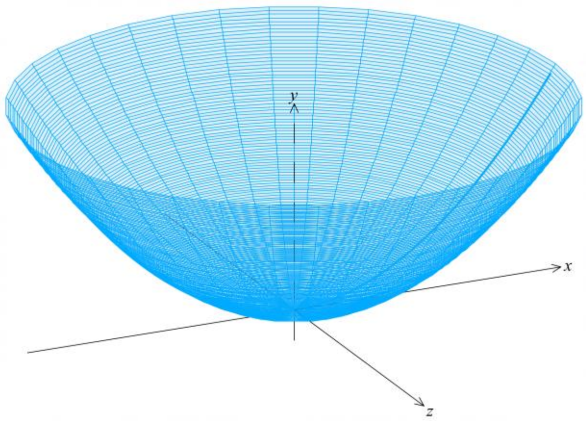
Rearrange the equation of the bowl's curve to make x^2 the subject in terms of y. Then, in a Graphs application, sketch the curve to visualise the bowl and the water within, up to a height of 12 cm.

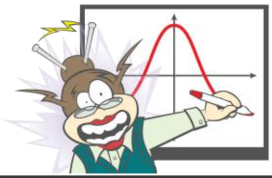
The volume of water is given by the integral $\pi\int_0^{12} 5y\,dy$. This integral is able to be calculated by hand using standard techniques.

Using the TI-Nspire CX CAS involves typing out the integral including the terminals, ensuring the π is preceding the integral.









Question 5

129 units

Define the equation of the curve as a function $f(x)$ in a Calculator application, then define the derivative as $df(x)$.

The required arc length is given by the integral $\int_3^6 \sqrt{1+\left(f'(x)\right)^2}\,dx$.

This integral is able to be calculated by hand using various algebraic and calculus techniques.

Using the TI-Nspire CX CAS involves performing the integration with the appropriate defined derivative function, ensuring the terminals and variable of integration are correct.