Webinar Session
Units 3&4 Specialist Mathematics

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Focus Topic: Solids of Revolution

Units 3&4 Specialist Mathematics

TI-Nspire™ CX CAS technology for understanding

Solids of Revolution

- Part 1 (3 D Graphing of Shapes)
- Part 2 (Determing Volume of Solids of Revolution)

Solids of Revolution

Study Design states :

Volumes of Solids of Revolution of a region about either coordinate axes

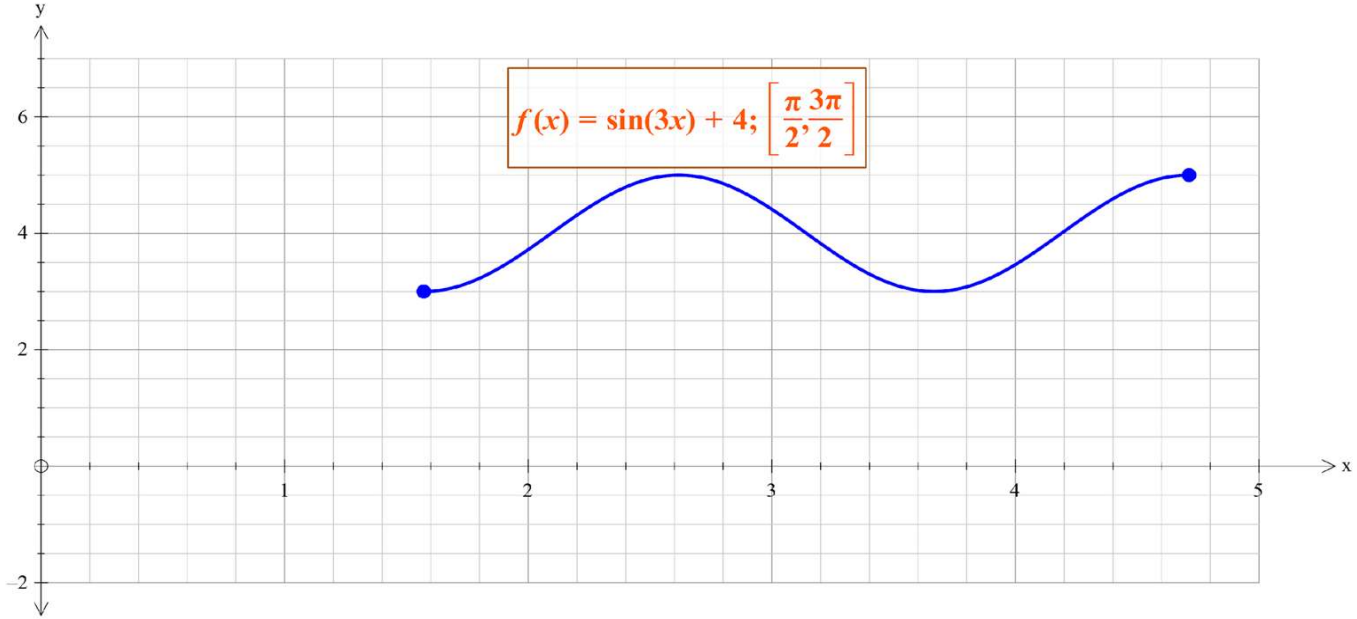
How to we obtain Solids of Revolution ?

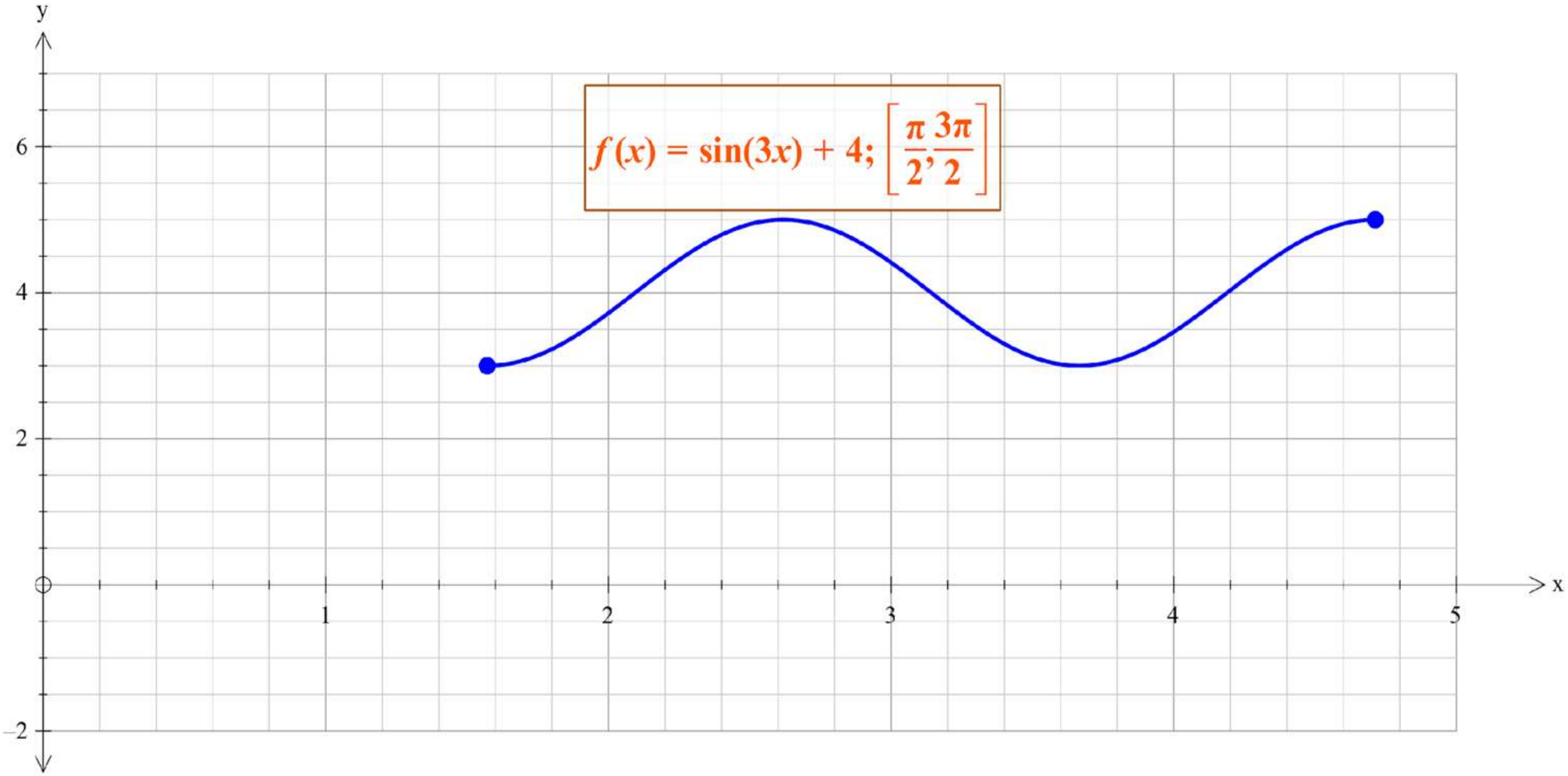
For our Webinar today, we should first define what a solid of Revolution is.

To obtain a Solid of revolution, we start out with a function

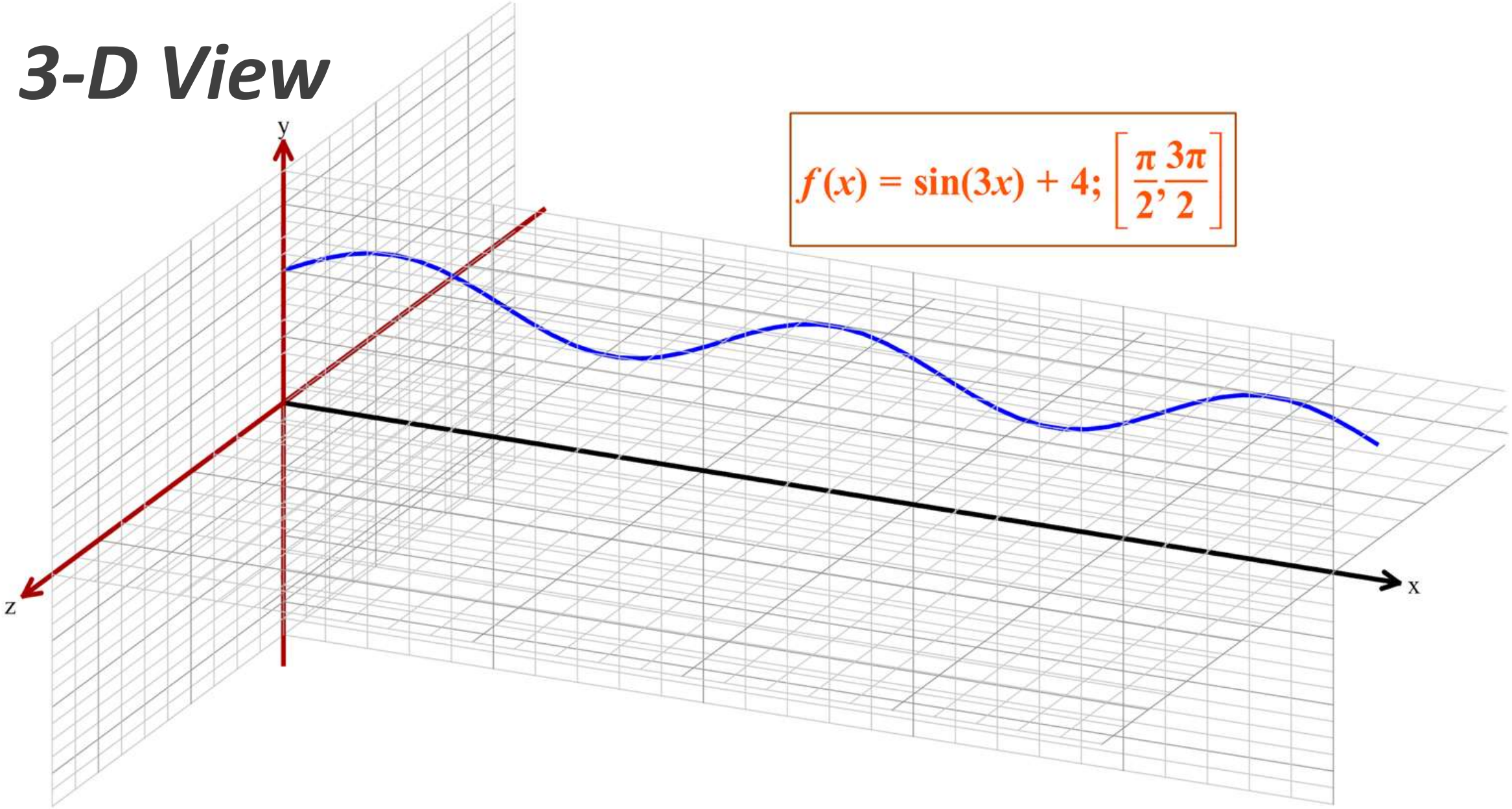
We then rotate this curve (function) about a given axis or line.

Doing so gives us a 3-Dimensional shape




$f(x) = \sin(3x) + 4; \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$
y
x
6
4
2
−2
1
2
3
4
5

3-D View



$$f(x) = \sin(3x) + 4; \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$$

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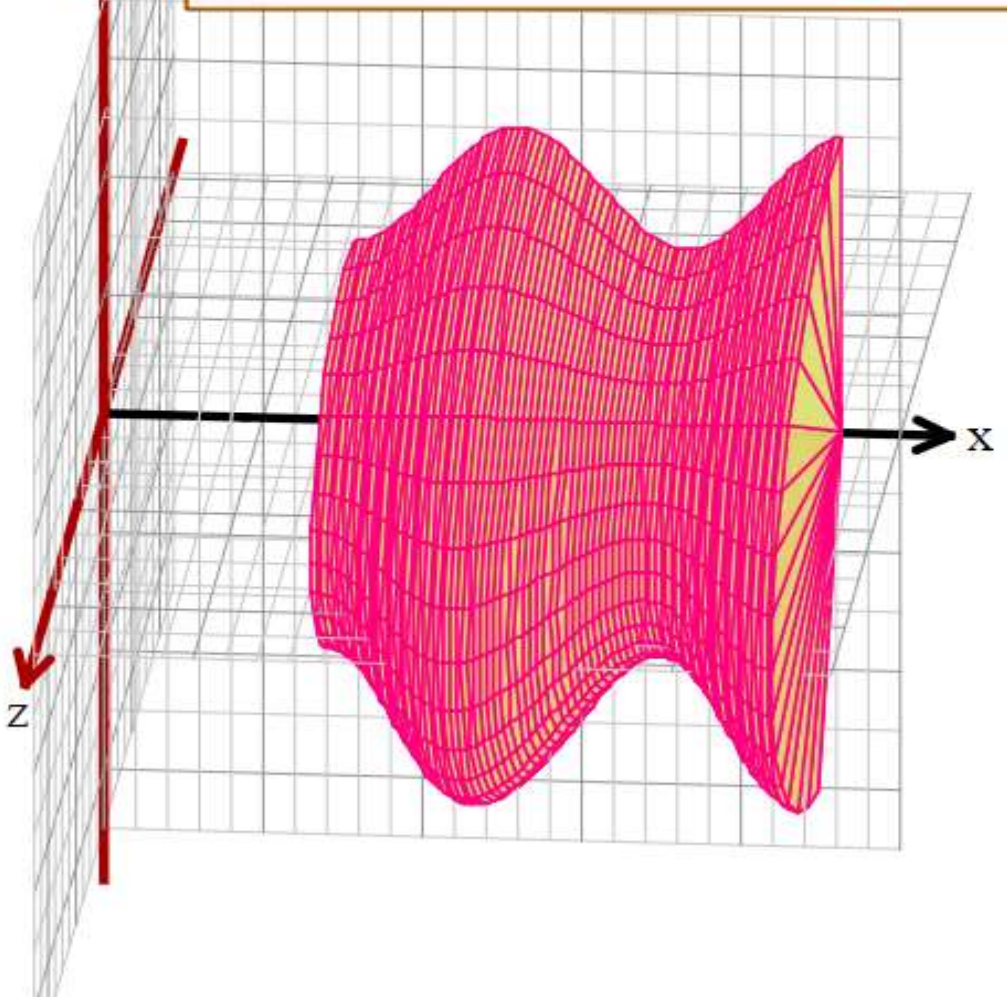
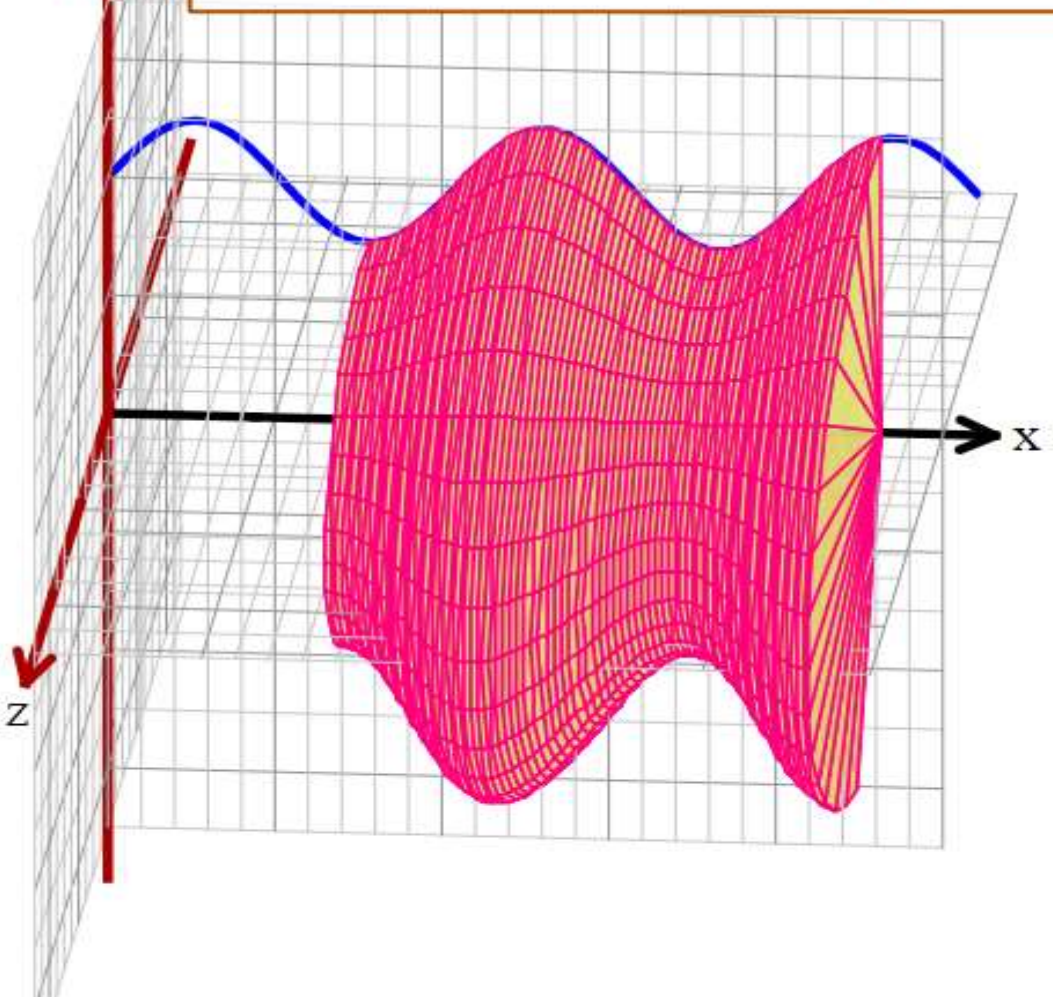


$$f(x) = \sin(3x) + 4; \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$$



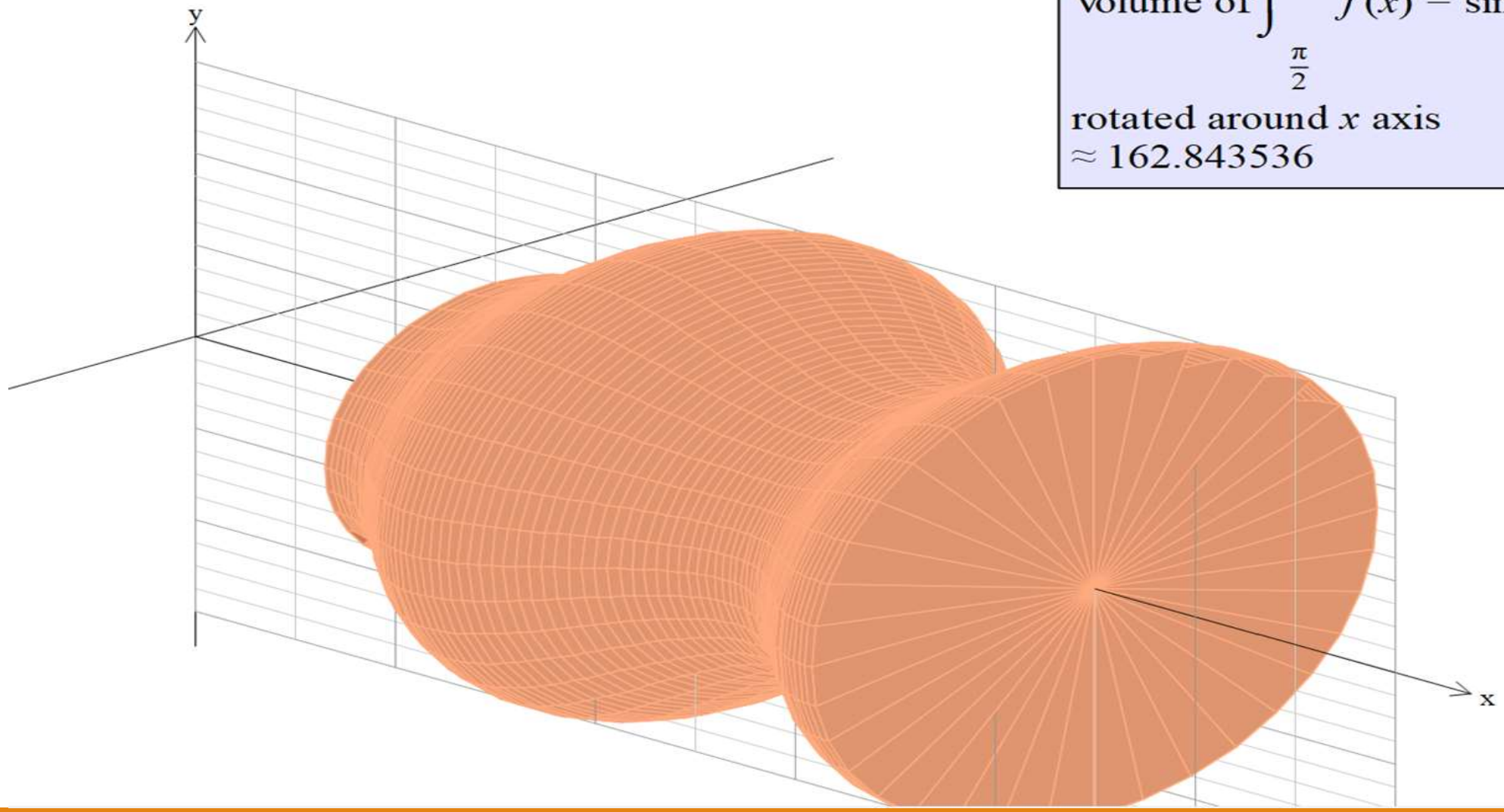
- ***Volumes of Solids of Revolution***
  1. Method of Rings / Discs
  2. Method of Cylinders (will not cover in this webinar)

- ***Also volume enclosed by 2 curves***
  or Intersecting curves

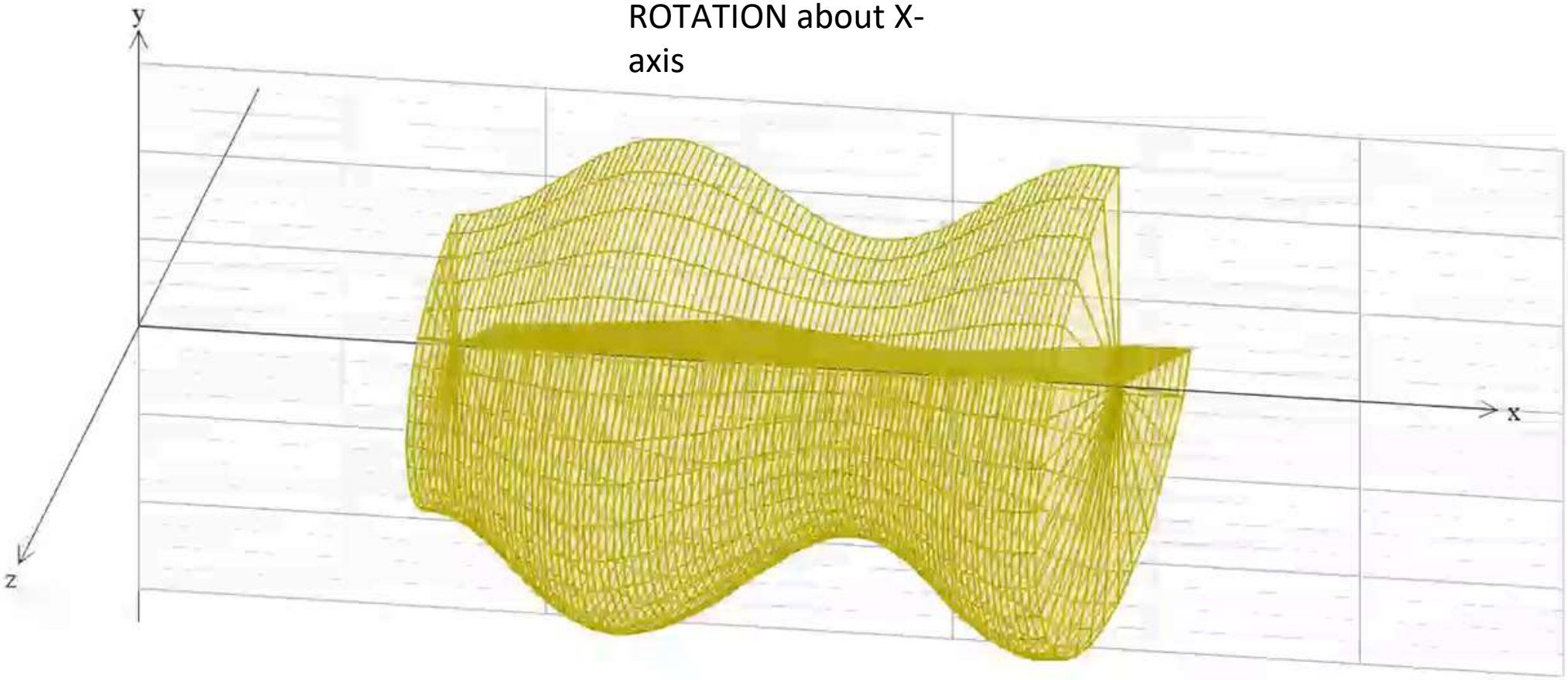
$$\text{Volume of} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x) = \sin(3x) + 4; \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \quad dx$$

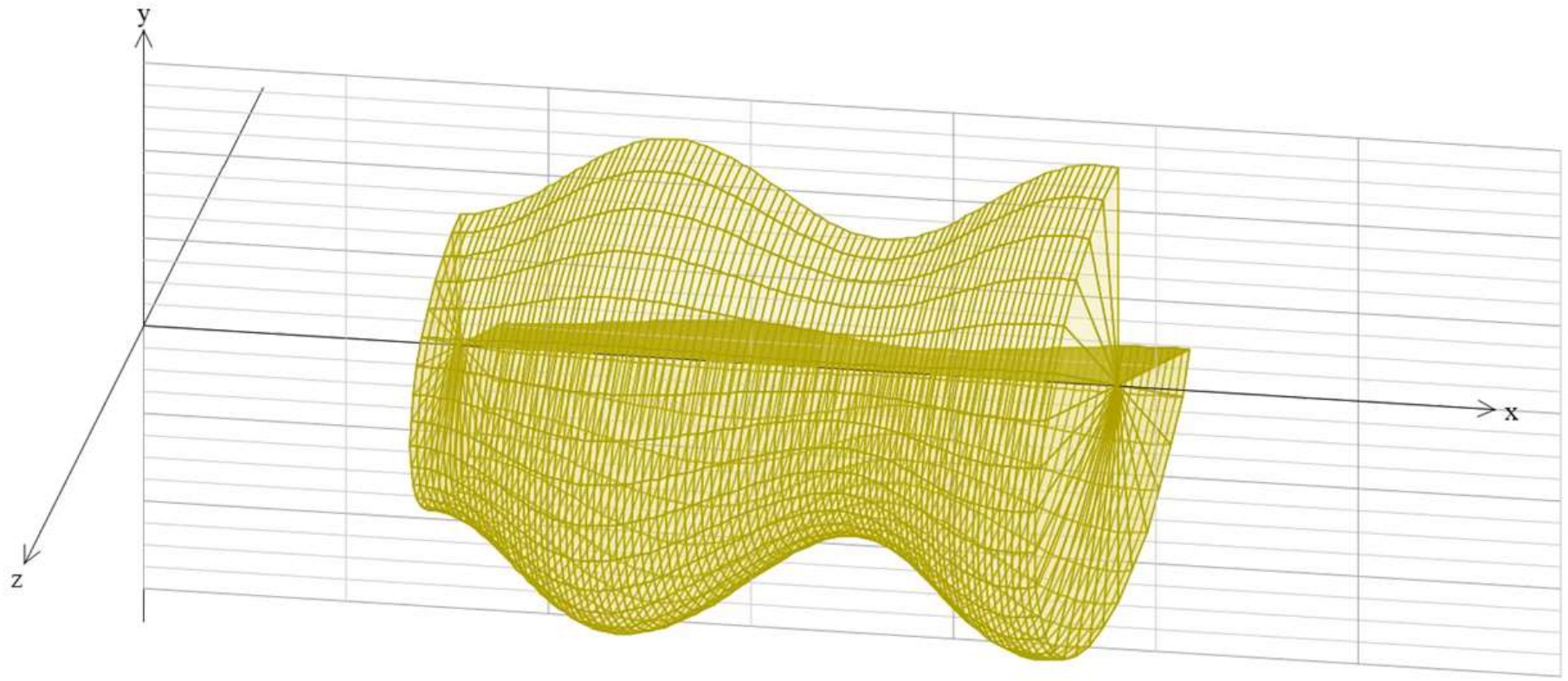
rotated around x axis

≈ 162.843536



ROTATION about X-axis




y
x
z

[redacted]

How to do a 3-D graph of our function $f(x) = \sin(3x) + 4; \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

We will Parametrise the Function

$y = f(x)$

To generate a surface of revolution out of any 2-dimensional scalar function $y = f(x)$

make t the function's parameter

set the axes of rotation's function to simply t

then use u to rotate the function around the axis by setting the other two functions equal to

$\boxed{f(t)\sin(u) \text{ and } f(t)\cos(u)}$

For example

to rotate a function $y = f(x)$ around thex-axis starting from the top of the xz-*plane*,

parameterise it as $\boxed{\vec{r}(t,u) = \left(t,\ f(t)\sin(u)\ ,\ f(t)\cos(u)\right)}$

for t $= x$ and u $\in [0, 2\pi]$

$f(x) = \sin(3x) + 4$

let $x = t$

set the axes of rotation's function to simply t

then use u to rotate the function around the axis by setting the other two functions equal to

$$\boxed{\begin{aligned} f(t)\sin(u) &\rightarrow \left(\sin(3t) + 4\right) \times \sin(u) \\ f(t)\cos(u) &\rightarrow \left(\sin(3t) + 4\right) \times \sin(u) \end{aligned}}$$

for t $= x$ and u $\in [0, 2\pi]$

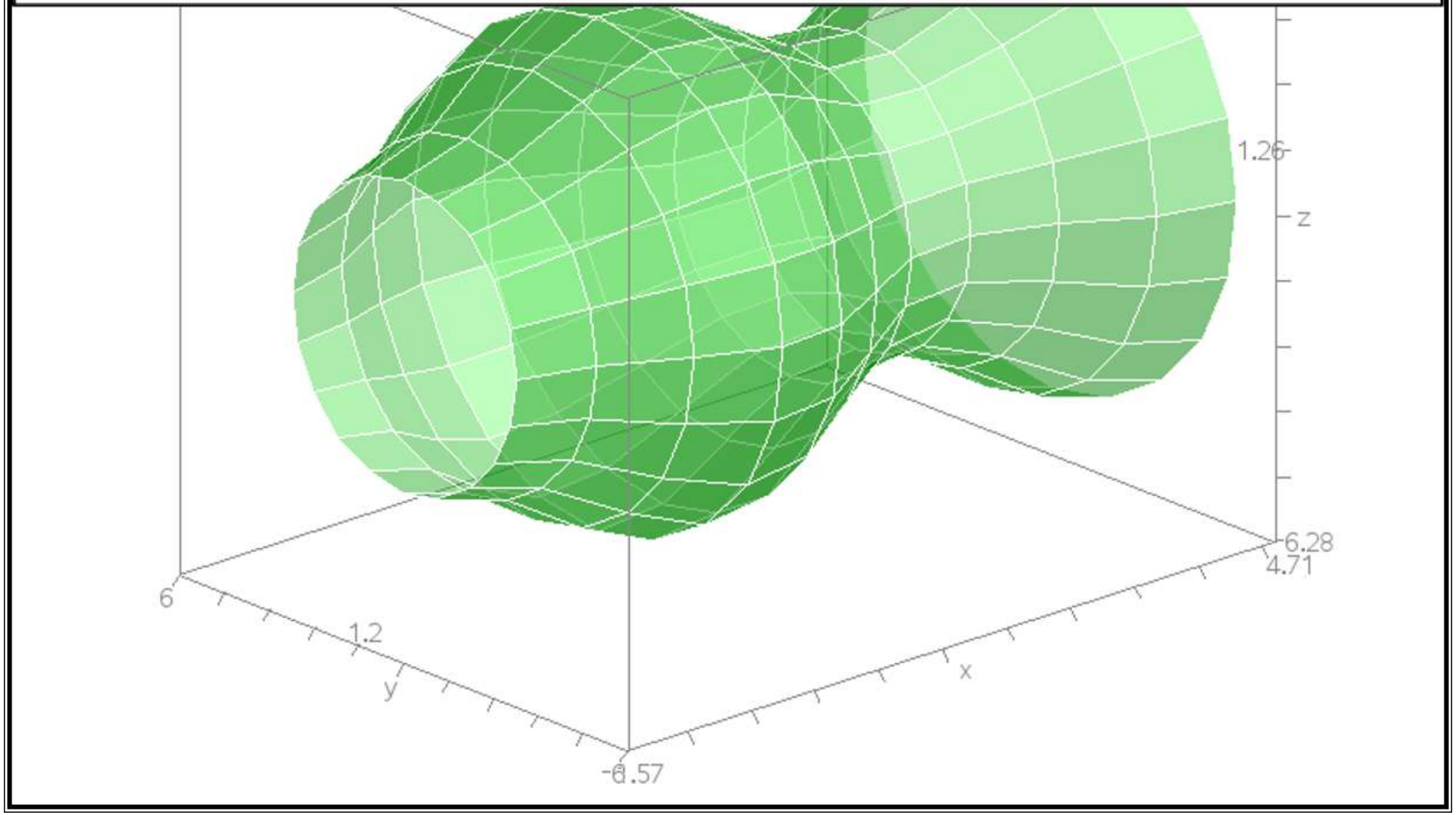
on CAS

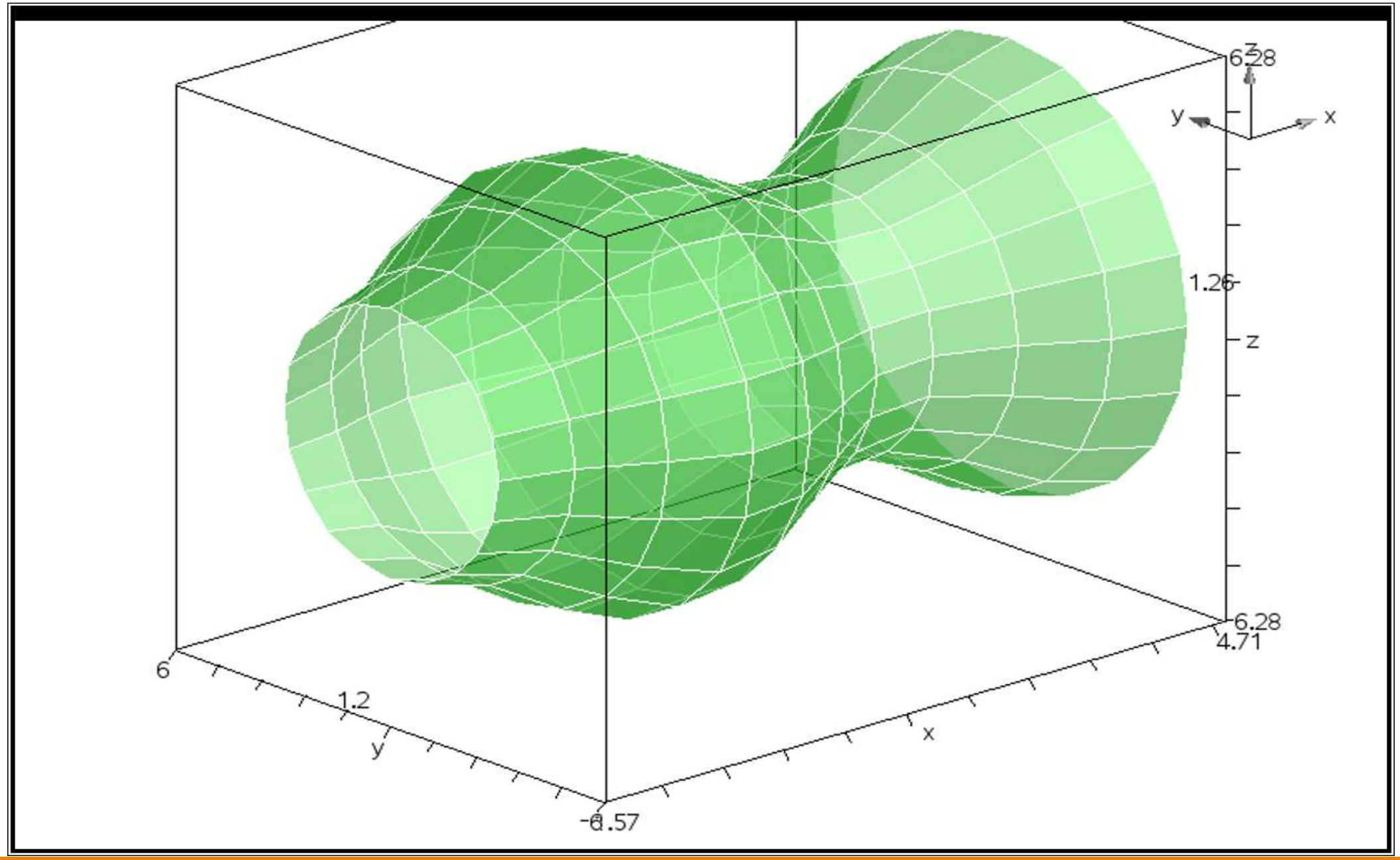
$$\boxed{\begin{aligned} xpl(t,u) &= t \\ ypl(t,u) &= \left(\sin(3t) + 4\right) \times \sin(u) \\ zpl(t,u) &= \left(\sin(3t) + 4\right) \times \cos(u) \end{aligned}}$$

xp1 (t,u) = t

yp1 (t,u) = $(\sin(3t)+4)\cdot\sin(u)$

zp1 (t,u) = $(\sin(3t)+4)\cdot\cos(u)$




z
6.28
y
x
1.26
z
6.28
4.71
6
1.2
y
x
-6.57

xp1	(t,u) =	t
yp1	(t,u) =	$(\sin(3t)+4)\cdot\sin(u)$
zp1	(t,u) =	$(\sin(3t)+4)\cdot\cos(u)$

Now try graphing

	xp2	(t,u) =	$30\cos(t)\sin(u)$
☑	yp2	(t,u) =	$30\cos(t)\cos(u)$
	zp2	(t,u) =	$10\sin(t)$

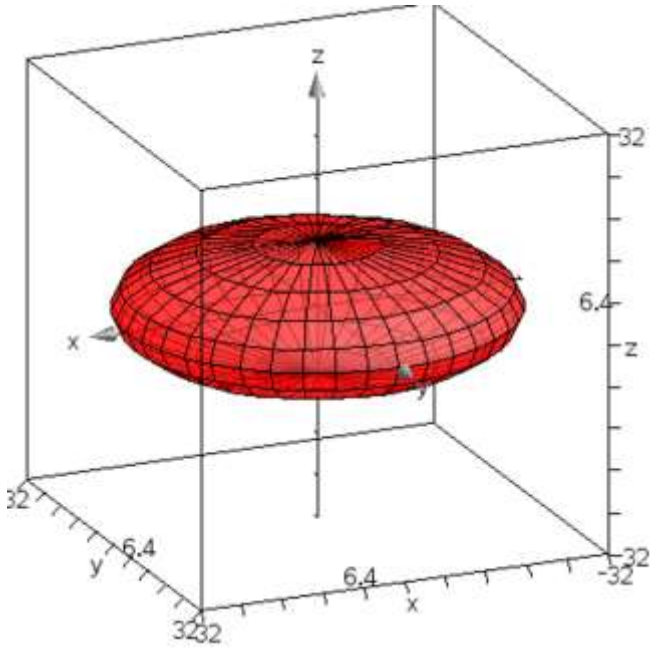
Predict the shape?

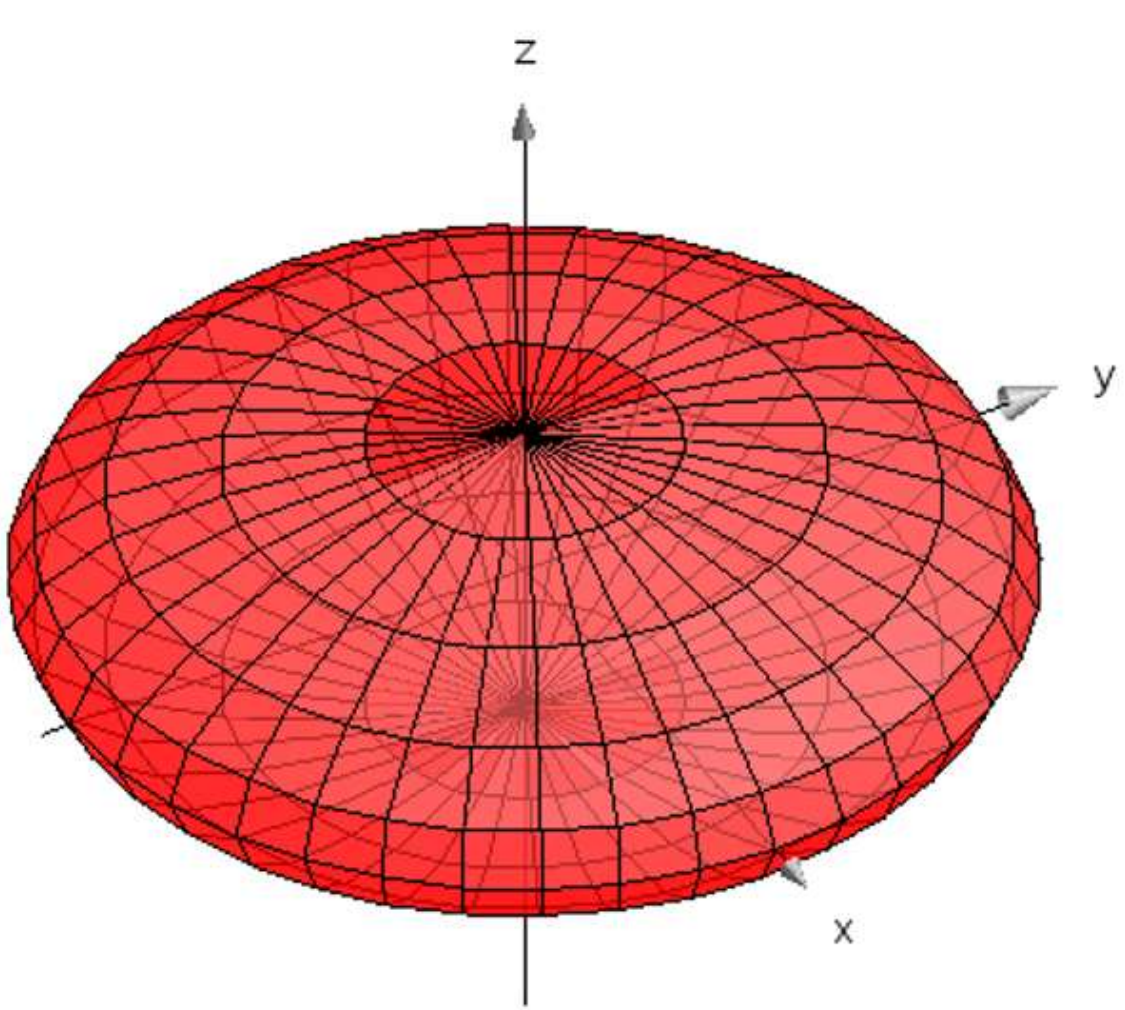
Change 10sin(t) to 50sin(t) – predict ?

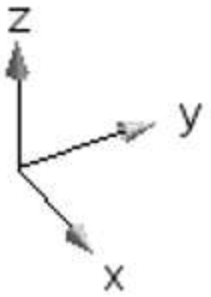
$xp2(t,u)=30\cdot\cos(t)\cdot\sin(u)$

$yp2(t,u)=30\cdot\cos(t)\cdot\cos(u)$

$zp2(t,u)=10\cdot\sin(t)$



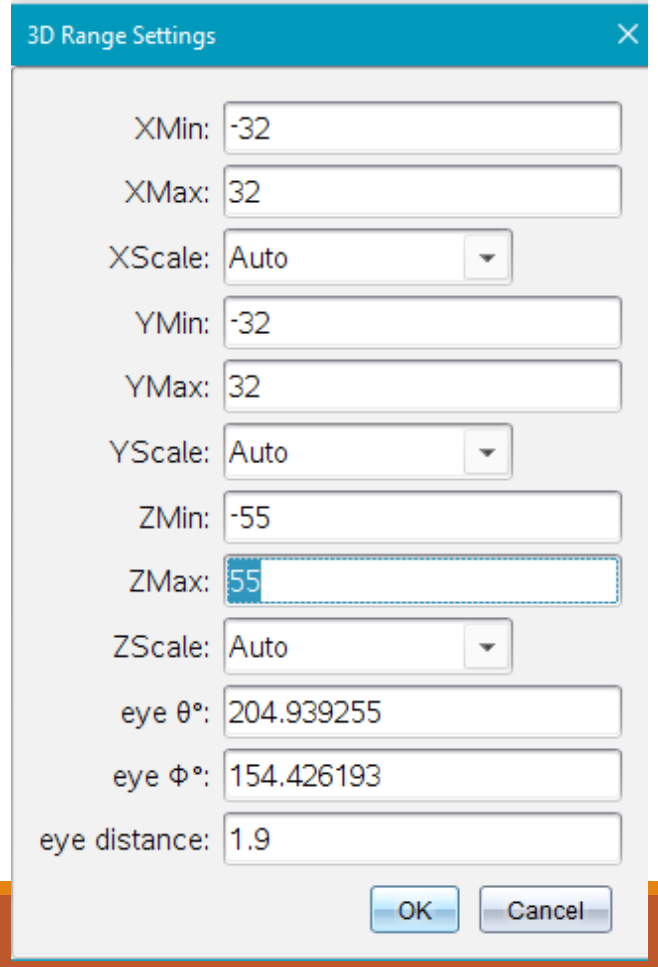


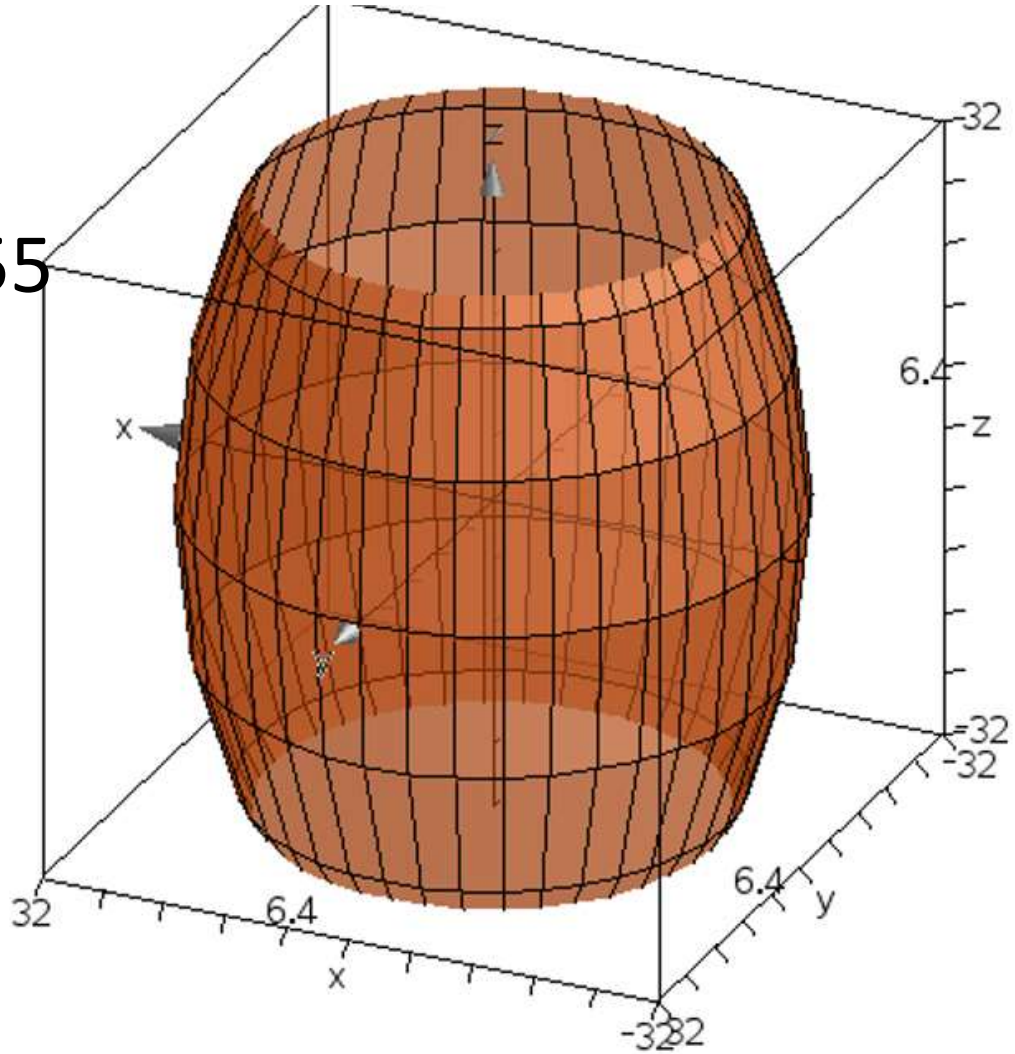


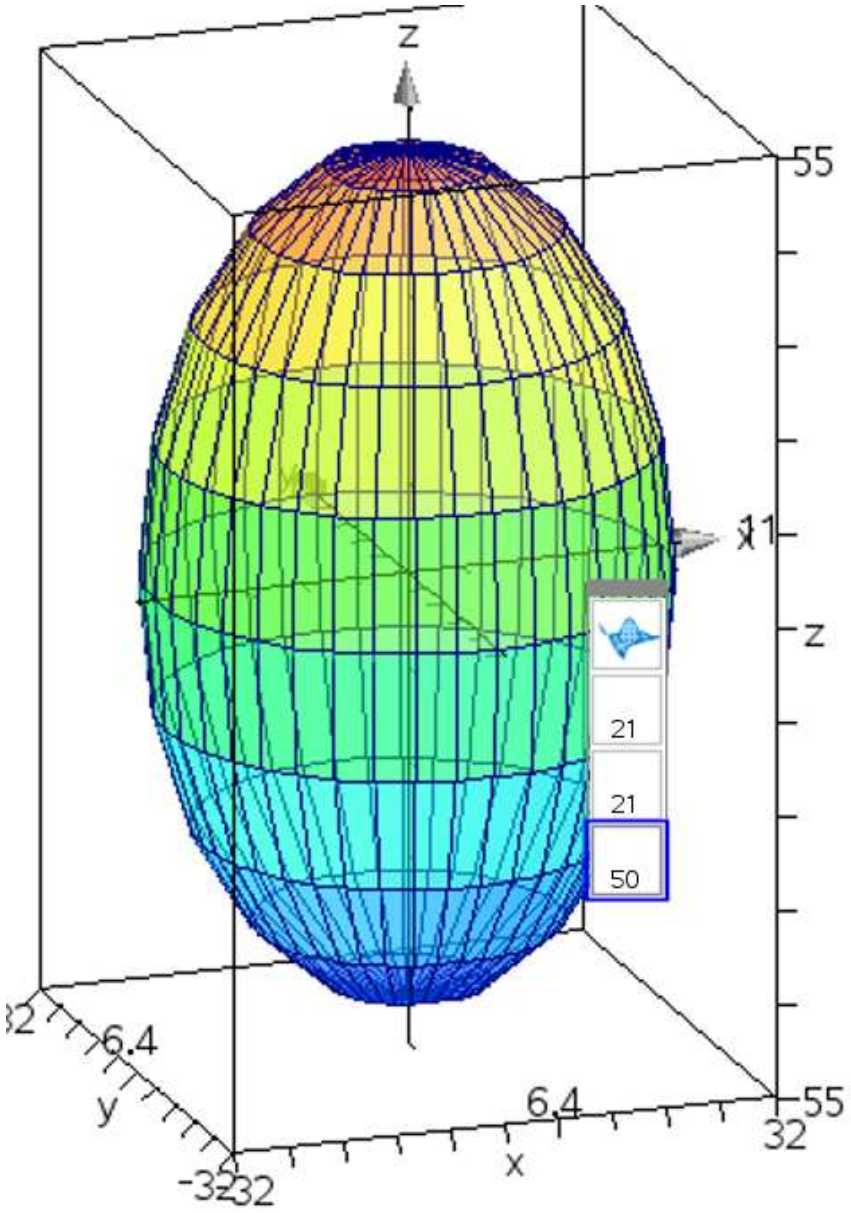
$$xp2(t,u)=30\cdot\cos(t)\cdot\sin(u)$$
$$yp2(t,u)=30\cdot\cos(t)\cdot\cos(u)$$
$$zp2(t,u)=50\cdot\sin(t)$$

Next

Change the z-axis scale from -55 to 55





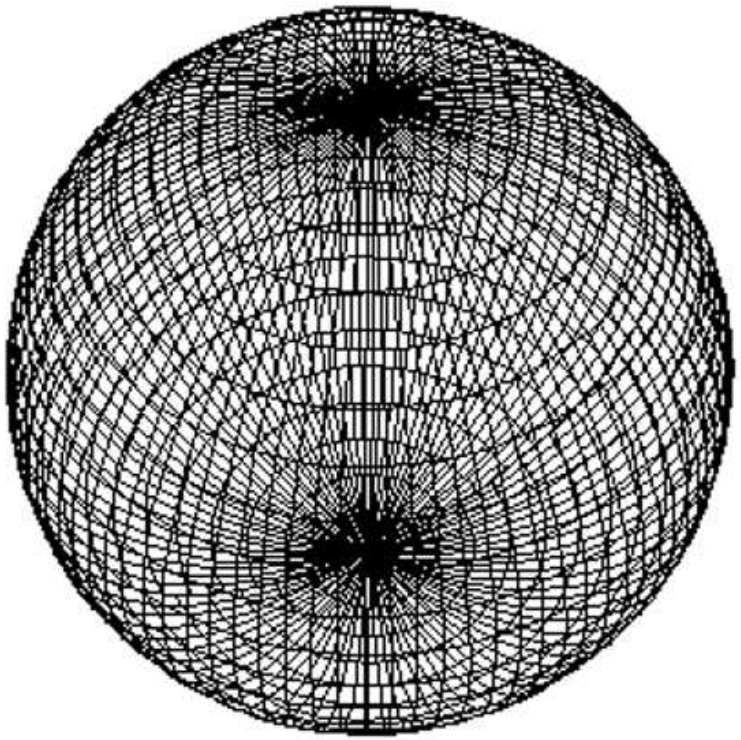


xp2	(t,u) =	$\mathbf{a}\cdot\cos(t)\cdot\sin(u)$
yp2	(t,u) =	$\mathbf{b}\cdot\cos(t)\cdot\cos(u)$
zp2	(t,u) =	$\mathbf{c}\cdot\sin(t)$

a =80.

b =80.

c =80.

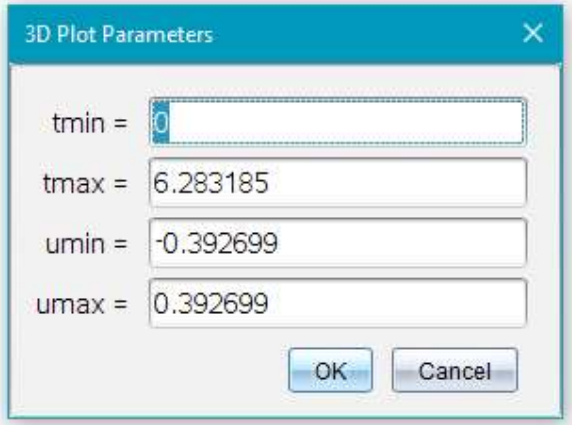


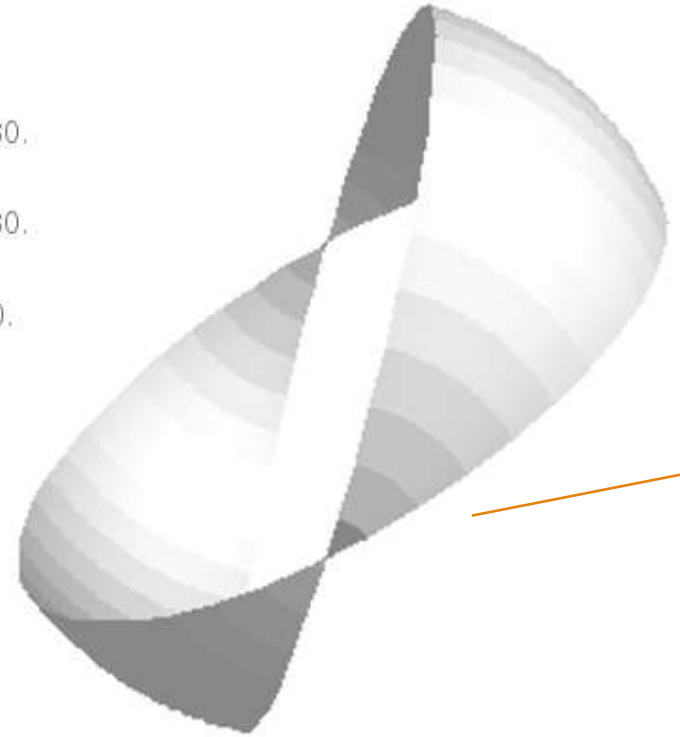
$$xp2(t,u) = a \cdot \cos(t) \cdot \sin(u)$$
$$yp2(t,u) = b \cdot \cos(t) \cdot \cos(u)$$
$$zp2(t,u) = c \cdot \sin(t)$$

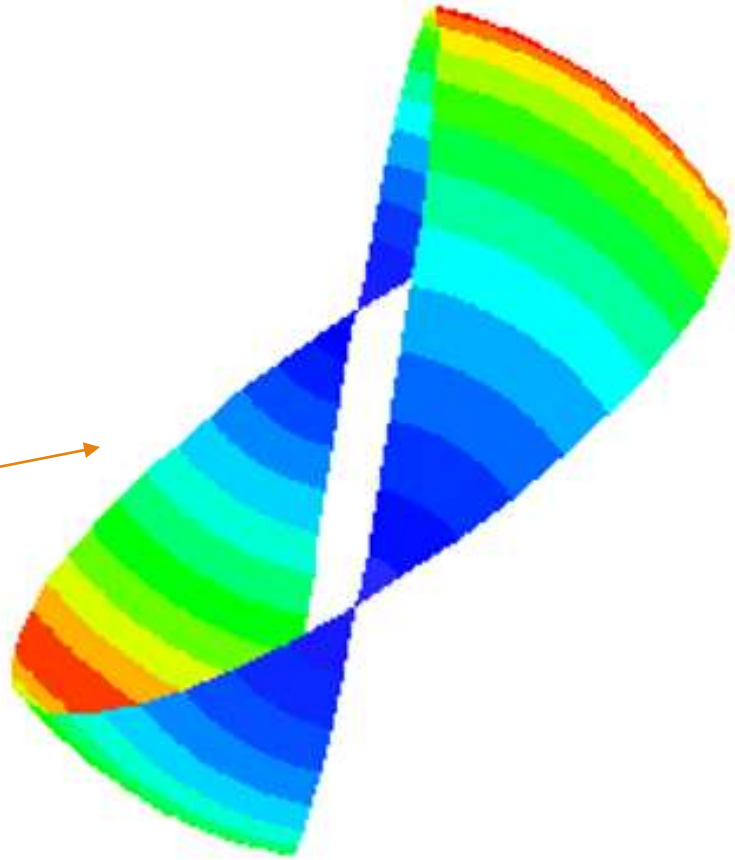
a =80.

b =80.

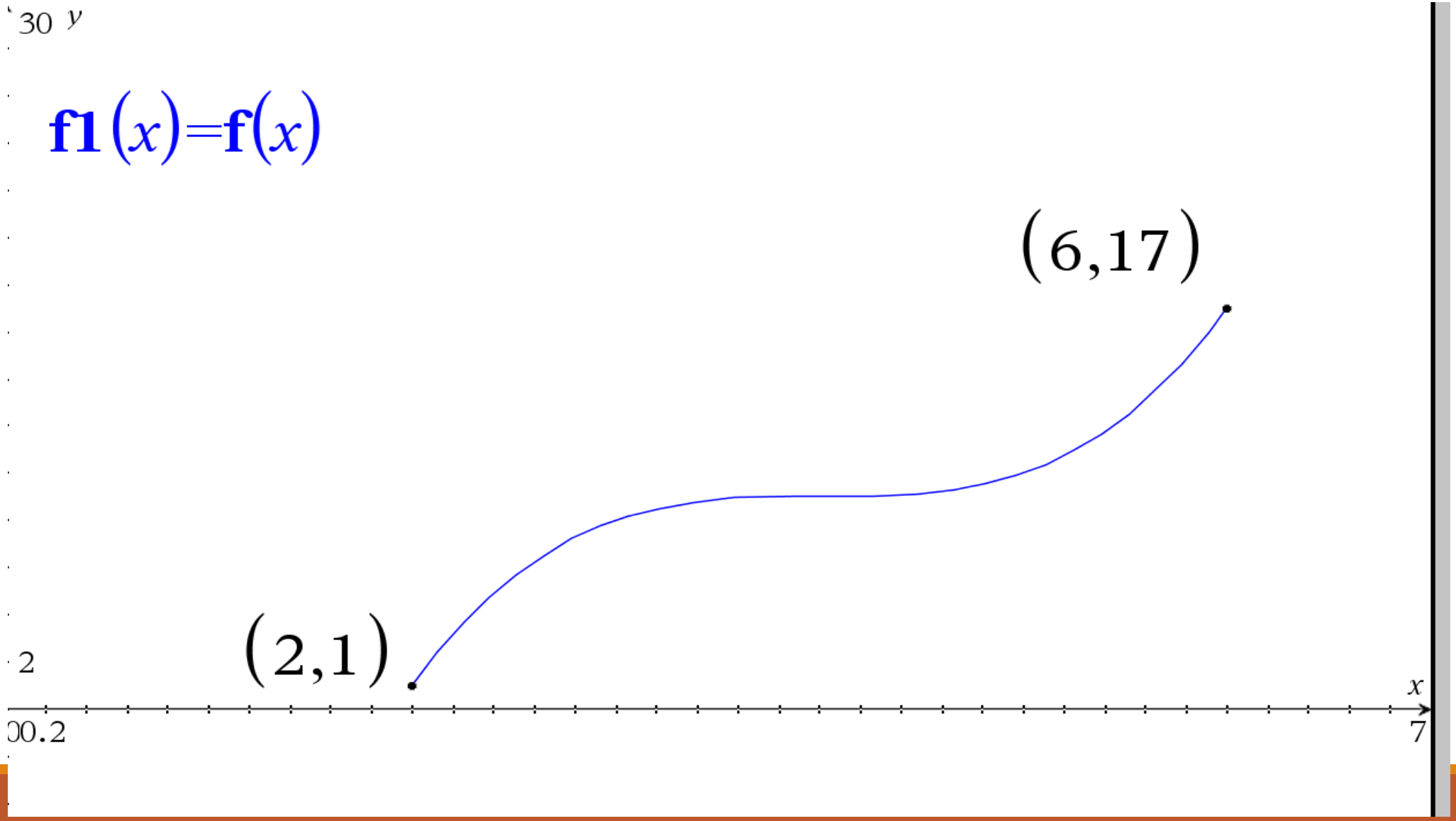
c =80.







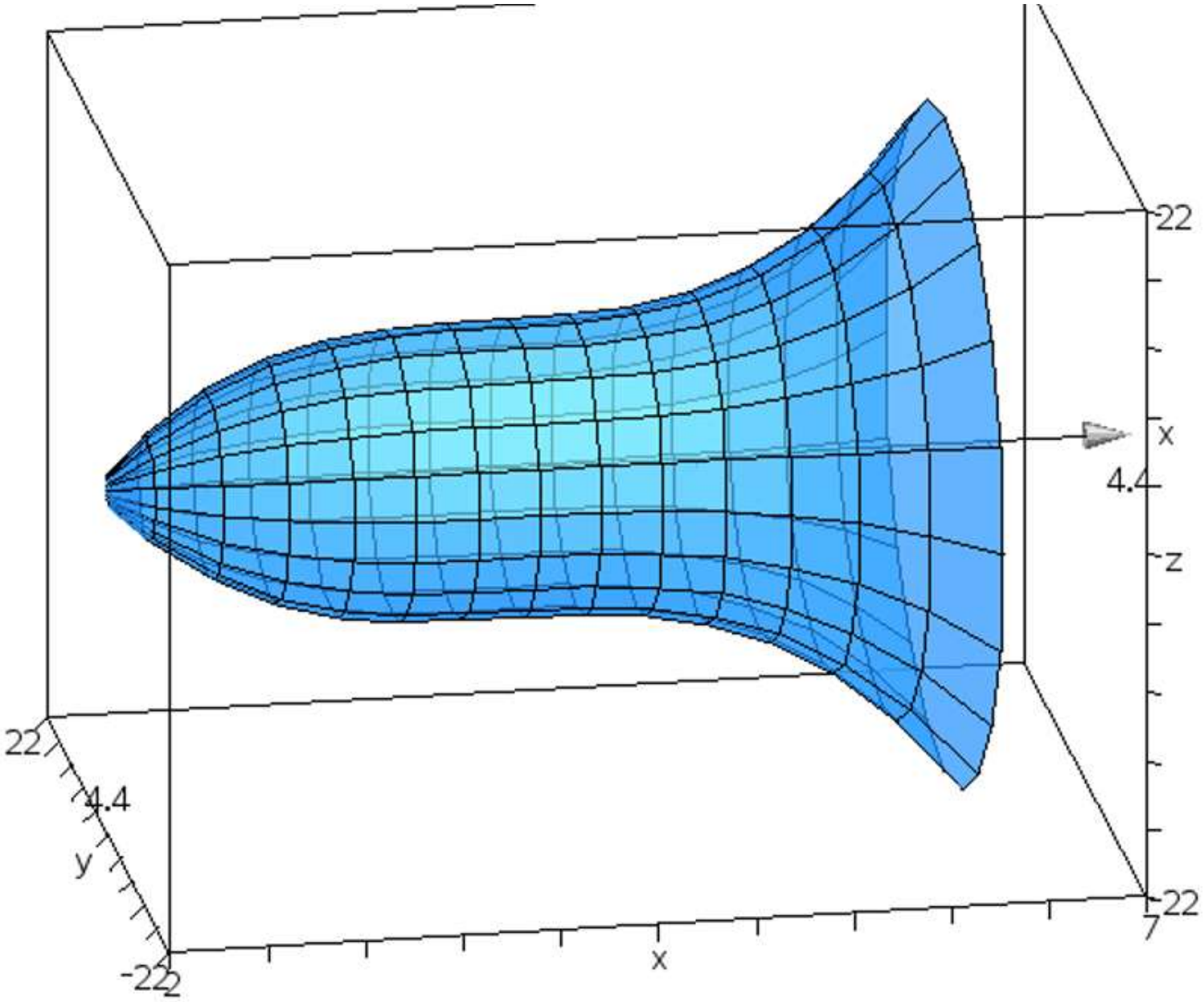
$\text{Define } \mathbf{f}(t)=(t-4)^3+9 \mid 2\le t\le 6$ ▸ *Done*



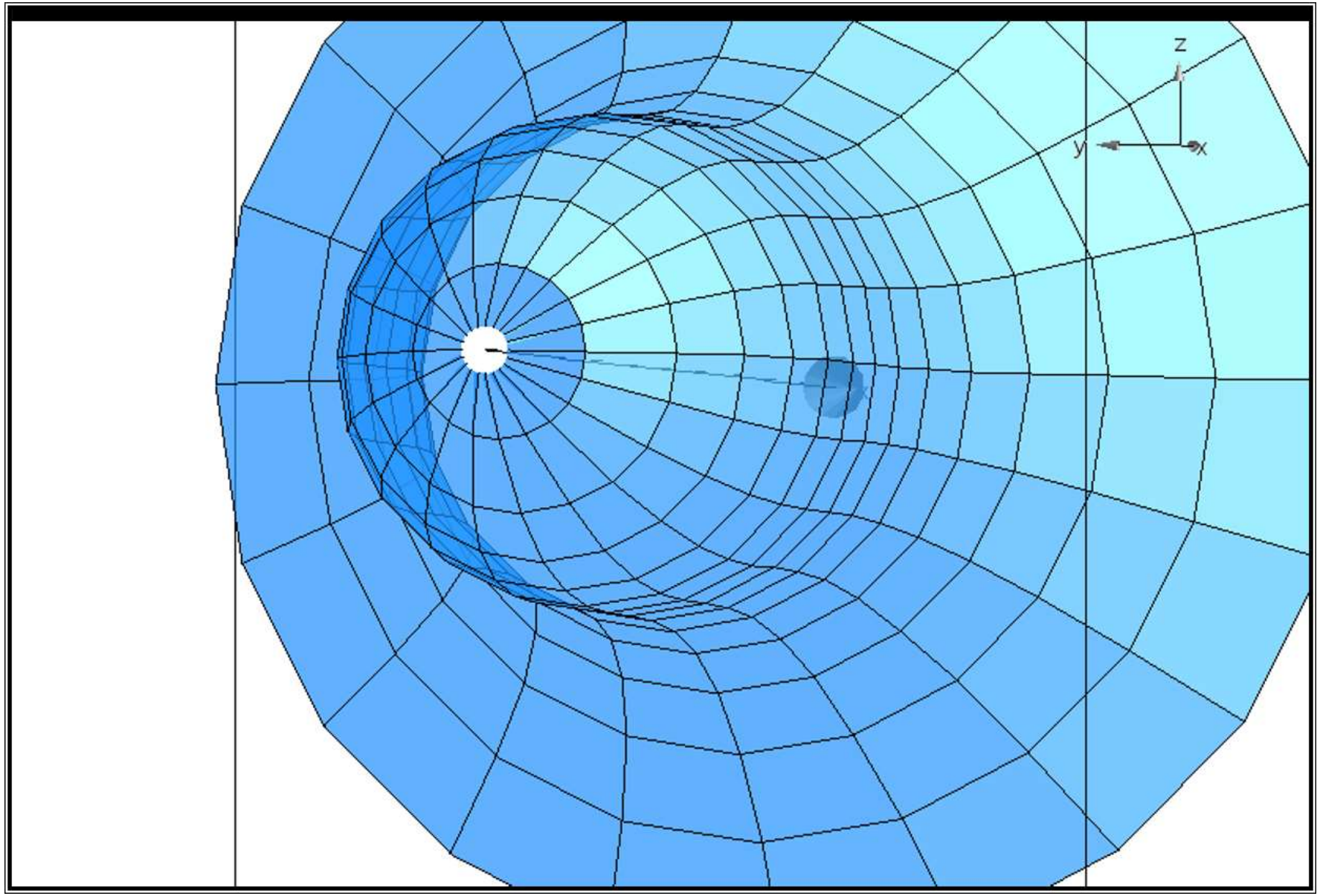
xp2 (t,u) = u

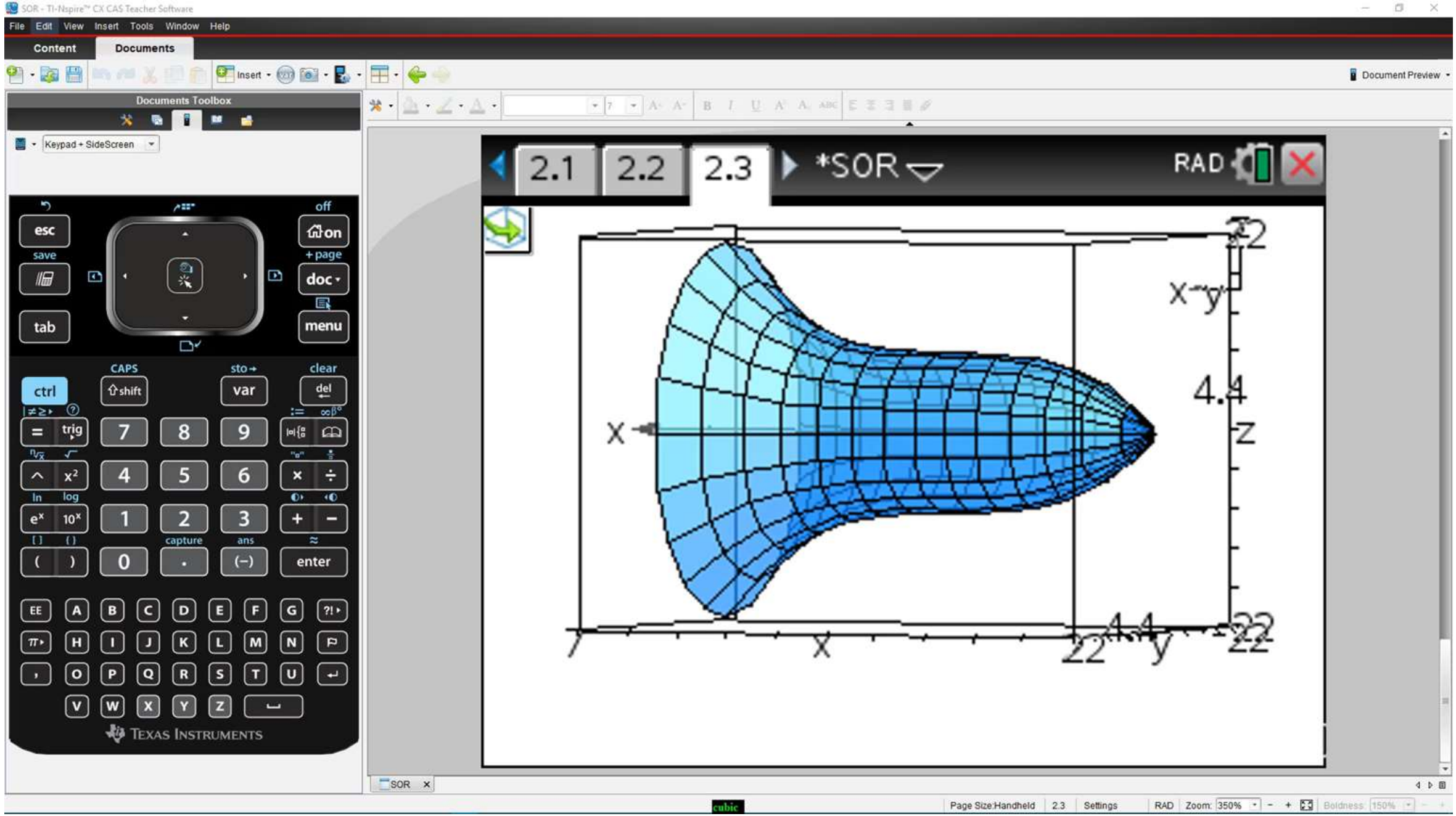
yp2 (t,u) = $\left((u-4)^3+9\right)\cdot\sin(t)$

zp2 (t,u) = $\left((u-4)^3+9\right)\cdot\cos(t)$



ROTATION AROUND X-AXIS


z
y
x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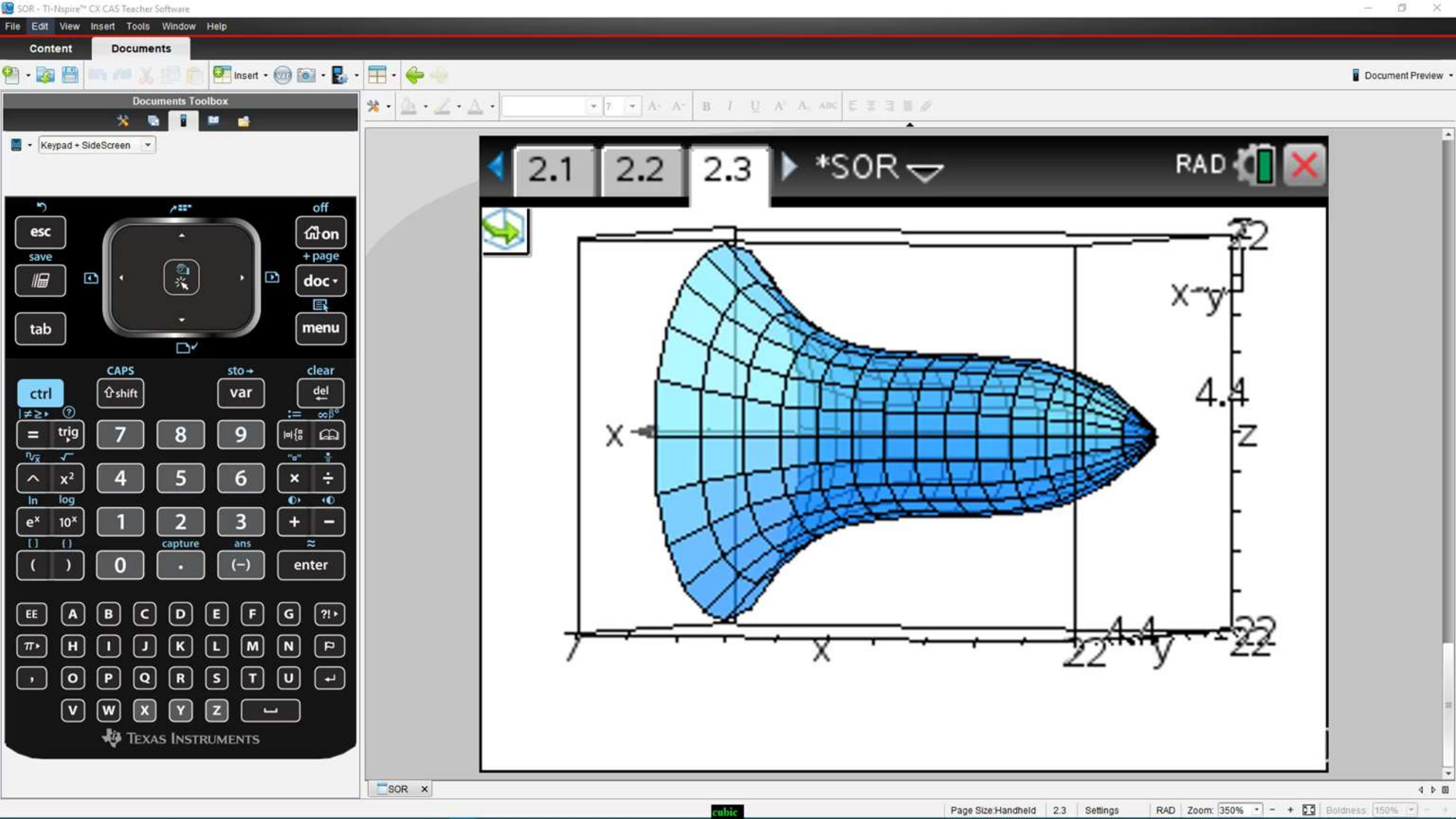
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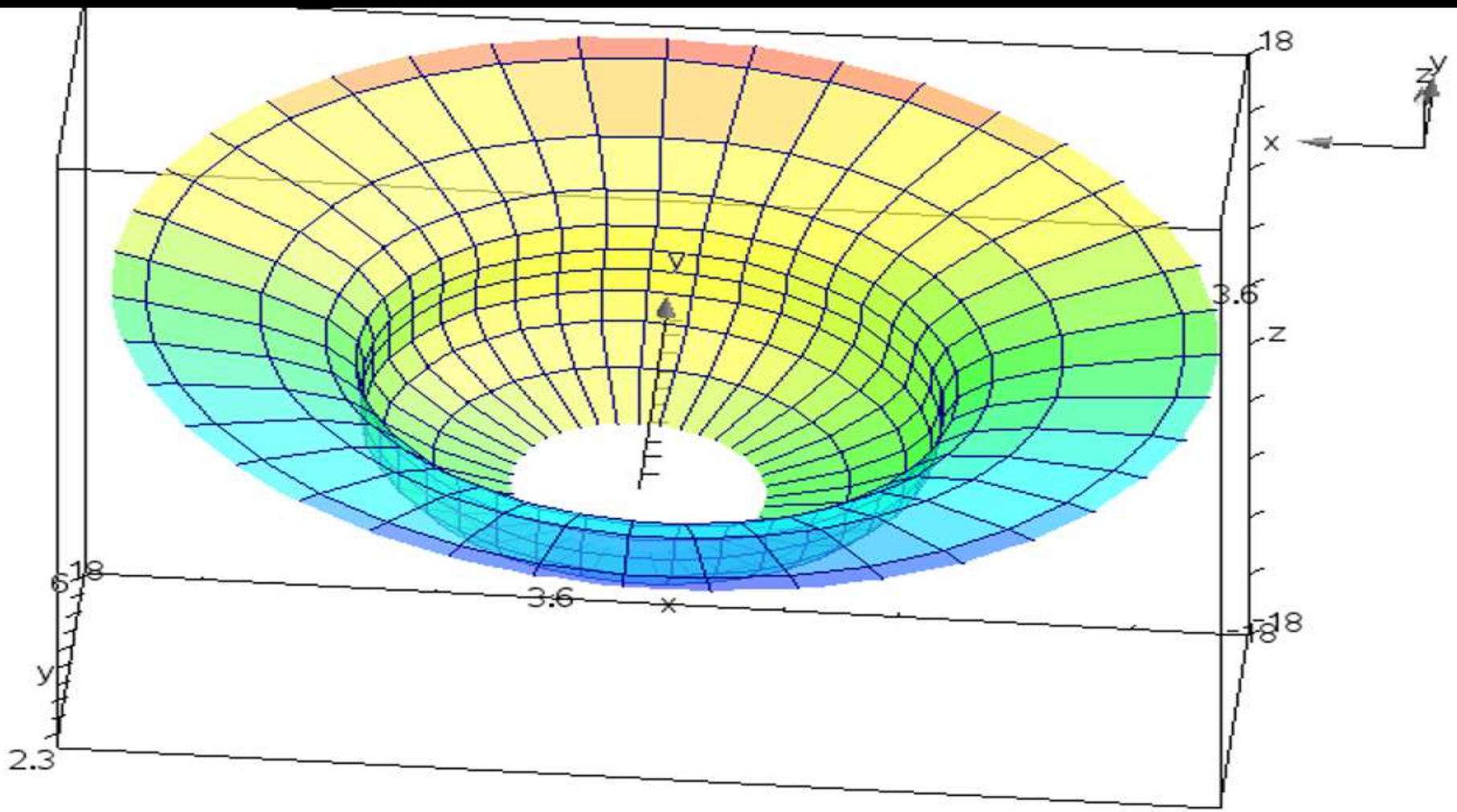
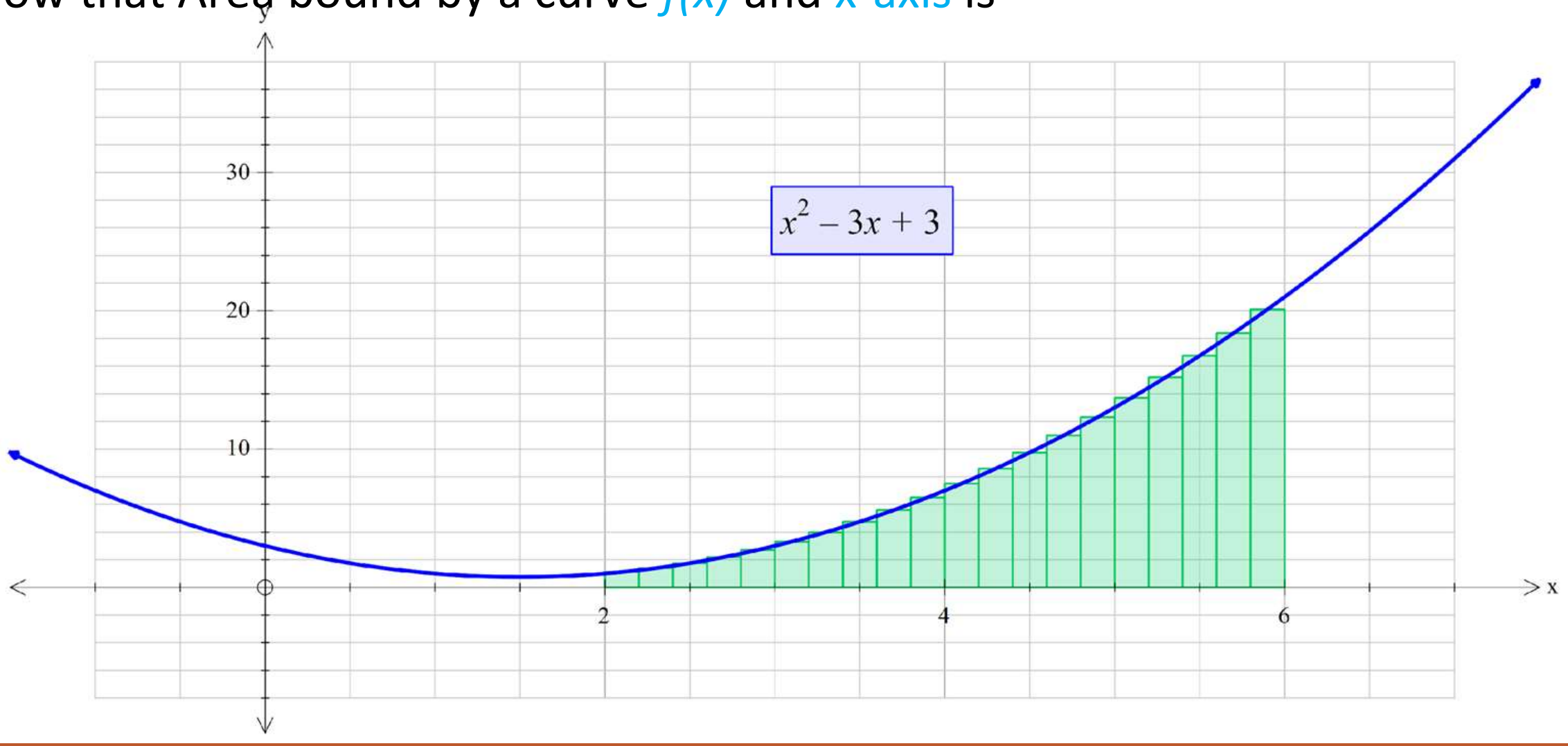


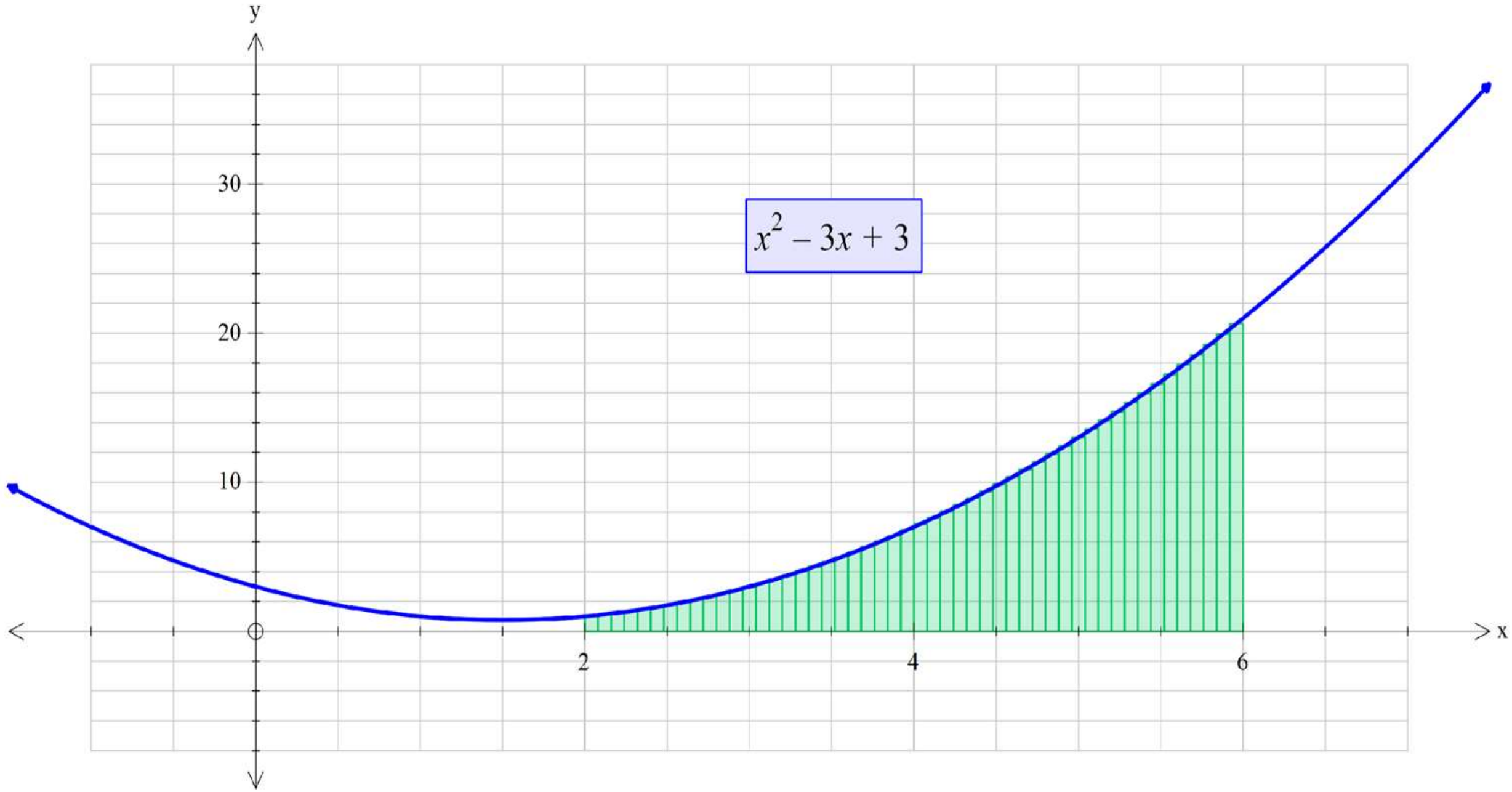
ROTATION AROUND Y-AXIS

- ***Volumes of Solids of Revolution***
  1. Method of Rings / Discs
  2. Method of Cylinders

From the study of Integral Calculus (Rectangle mid-point method or Trapezium method)

we know that Area bound by a curve *f(x)* and x-axis is




y
30
20
10
$x^2 - 3x + 3$
2
4
6
x

$$A = l \times w$$

Sum of Area of rectangles formed

$$A \approx sum\left(f(x_i) \times \frac{b-a}{n} \right)$$

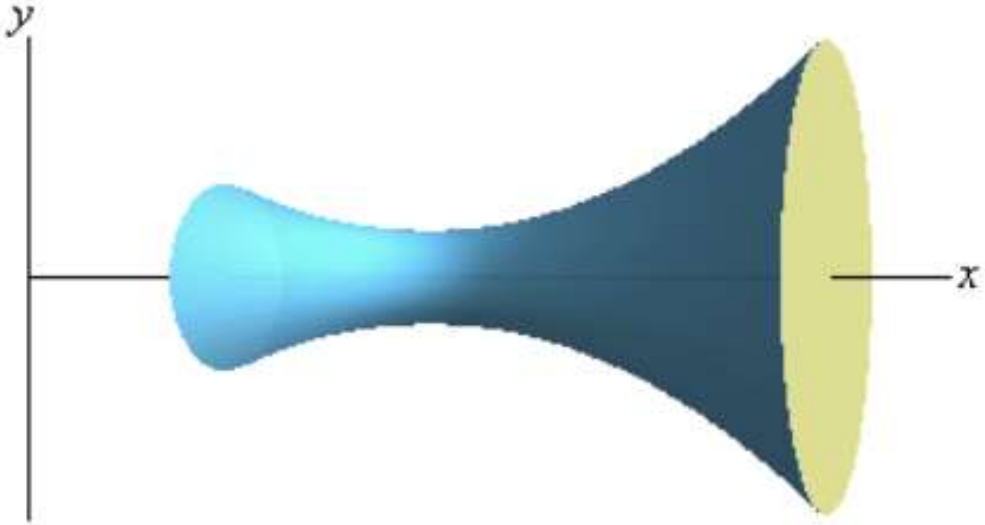
$$A \approx sum\left(f(x_i) \right) \times \triangle x$$

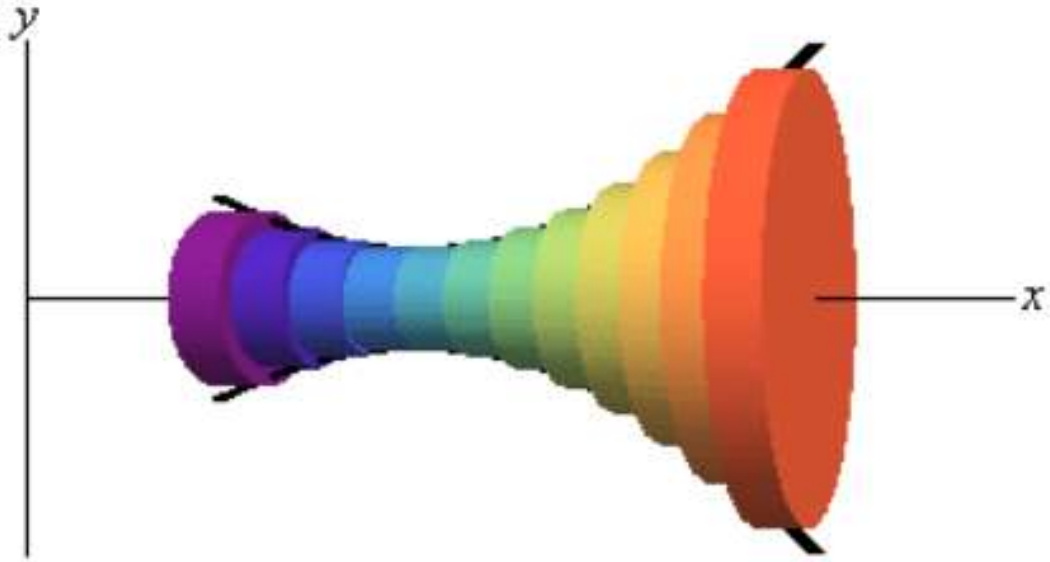
$$A \approx \sum_{i=1}^{n} f(x_i) \times \triangle x$$

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \times \triangle x \qquad \boxed{when\ \mathrm{n} \to \infty,\ \triangle x \to 0}$$

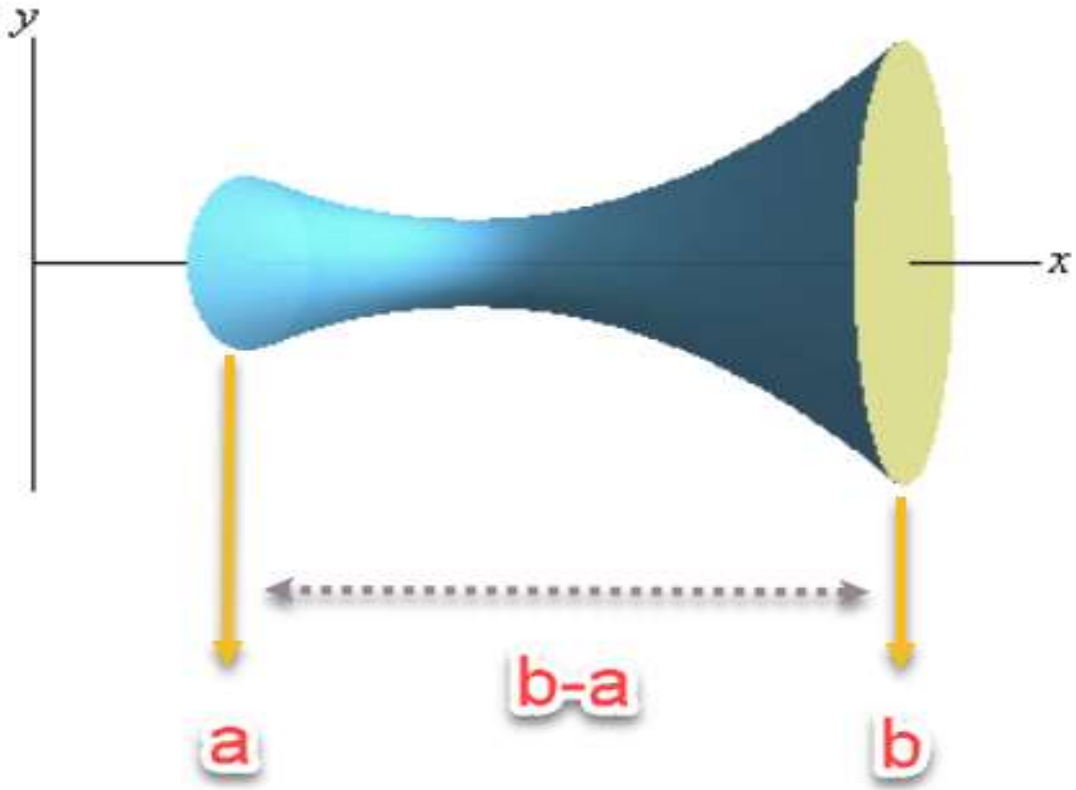
$$A = \int_a^b f(x) \times dx$$

The same logic can be extended to finding Volumes for Solids of Revolutions





The 3 – D shape so formed can be Approximated to being made of several discs of width approaching zero



Each disc so formed , the Area of Cross-section is a Circle

Area of a Circle

$A = \pi r^2$ or $A = \pi\left(f(x)^2\right)$

$\because$ since the radius of each disc is $f(x)$

$Also\ \Delta x = \dfrac{b-a}{n}$ (the width of each disc)

Volume of each disc $V = A \times$ width

$V_i = A(x_i) \times \Delta x$

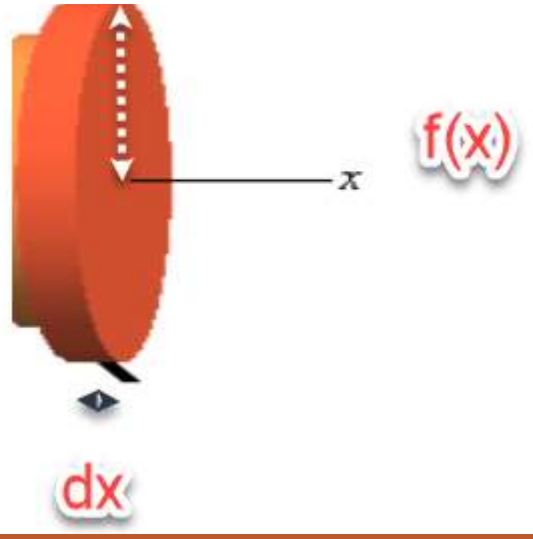
$\Delta x \to 0$

then Volume of all discs

$V \approx \sum_{i=1}^{n} A(x_i) \times \Delta x$ or $V \approx \sum_{i=1}^{n} \pi \times (f(x_i))^2 \times \Delta x$

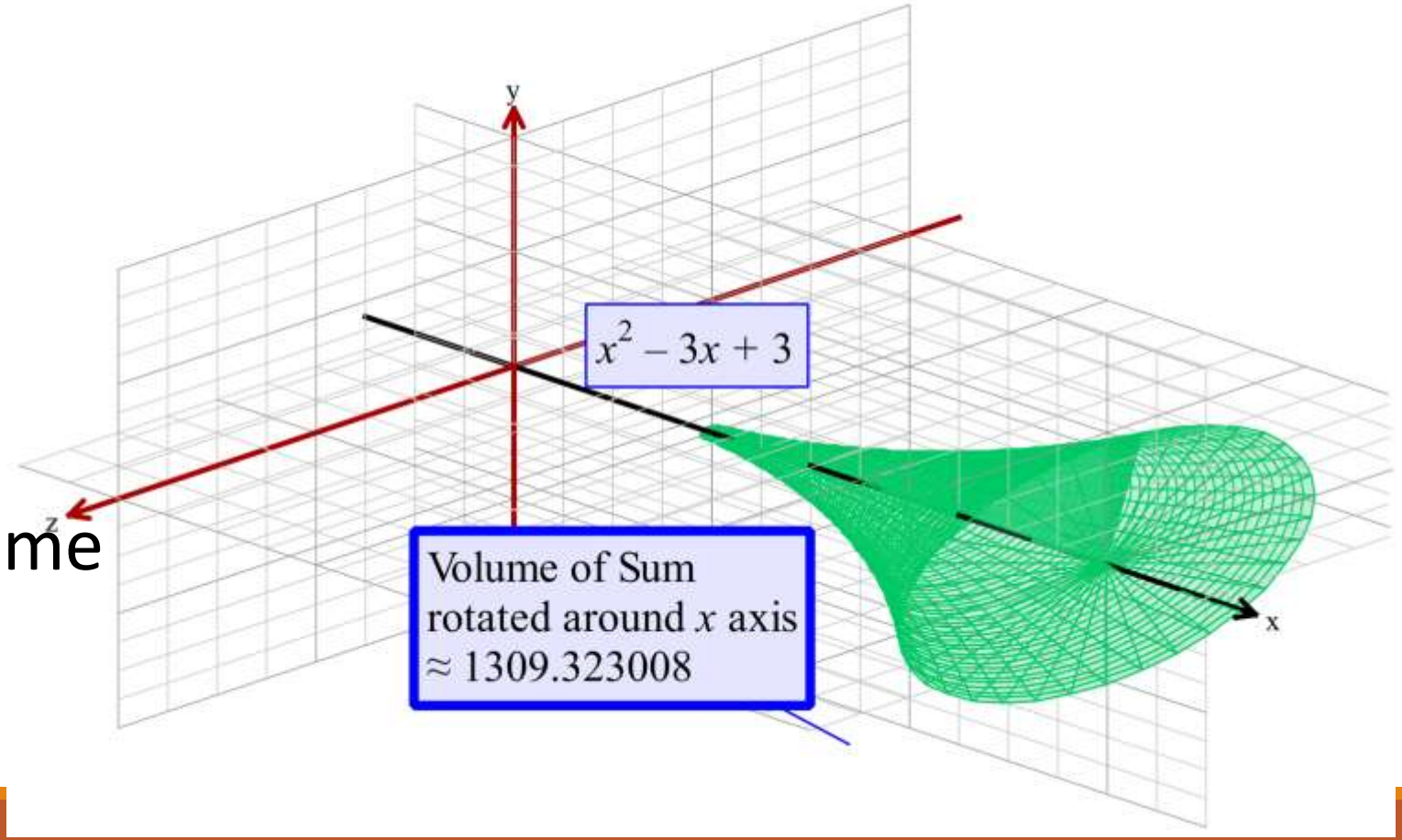
$V = \lim_{n \to \infty} \sum_{i=1}^{n} \pi \times (f(x_i))^2 \times \Delta x$

$V = \pi \int_a^b \left(f(x)^2\right) \times dx$



$$f(x) = x^2 - 3x + 3$$

Find the Volume of the Solid shape so formed when the curve is rotated around the x - $axis$ in the the interval [2,6]



We can see that the Volume is approximately 1309.3 cubic units

Define $f(x)=x^2-3\cdot x+3$ ▸ *Done*

List and Spread Sheet Method

$0.02\rightarrow dx$ ▸ 0.02

$sum(vol)$ ▸ 1323.34103211

	A xi	B rad	C vol
=	=seq(x,x,2,6,dx)	=f(xi)	=π*rad^2*dx
166	5.3	15.19	14.4975767336
167	5.32	15.3424	14.7899420016
168	5.34	15.4956	15.0867836522
169	5.36	15.6496	15.3881478892
170	5.38	15.8044	15.6940811581

Enter the following in the formula for each of the columns

$xi:=seq(x,x,2,6,dx)$

$rad:=f(xi)$

$vol:=\pi\cdot rad^2\cdot dx$

$$\text{Define } \mathbf{vf}(a,b,n)=\pi\cdot\sum_{i=1}^{n}\left(\frac{b-a}{n}\cdot\left(\mathbf{f}\left(a+i\cdot\frac{b-a}{n}\right)\right)^2\right)$$

▸ *Done*

$$\mathbf{vf}(2,6,1000) \blacktriangleright \frac{4078911145833\cdot\pi}{9765625000}$$

$$\text{round}(\mathbf{vf}(2,6,1000),2) \blacktriangleright 1312.18$$

Using formula

$$\int_{2}^{6} \left(\pi \cdot \left(\mathrm{f}(x)\right)^{2}\right) \mathrm{d}x \blacktriangleright 1309.41581802$$

Some more examples

Determine the Volume of the Solid obtained by rotating the region bounded by

$x^2 - 4x + 5$ at

$x = \frac{3}{2}$ *and* $x = 5$

and the x - *axis*

Rotated about the x-axis

$$a(x) := \pi \cdot \left(x^2 - 4 \cdot x + 5\right)^2 \blacktriangleright \textit{Done}$$

$$\text{expand}(a(x)) \blacktriangleright \pi \cdot x^4 - 8 \cdot \pi \cdot x^3 + 26 \cdot \pi \cdot x^2 - 40 \cdot \pi \cdot x + 25 \cdot \pi$$

$$v(x) = \int_{\frac{3}{2}}^{5} a(x)\, dx \blacktriangleright v(x) = 220.507079359$$

$$v(x) = \int_{\frac{3}{2}}^{5} a(x)\, dx \blacktriangleright v(x) = \frac{33691 \cdot \pi}{480}$$

$$\pi \cdot \int_{\frac{3}{2}}^{5} \left(x^2 - 4 \cdot x + 5\right)^2 \, dx \blacktriangleright \frac{33691 \cdot \pi}{480}$$

Determine the Volume of the solid shaped obtained by rotating the region bound by

$$f(x) = x^{\frac{2}{3}} \quad and \quad g(x) = \frac{x}{3}$$

that lie in Quadrant 1 and rotated about the y-axis

The rings method will be used here

$$\pi \cdot \int_0^9 \left((3 \cdot y)^2 - \left(y^{\frac{3}{2}} \right)^2 \right) dy \blacktriangleright \frac{2187 \cdot \pi}{4}$$

$$f(x) = x^{\frac{2}{3}} \quad \Rightarrow \quad y = x^{\frac{2}{3}}$$

or $\boxed{x = y^{\frac{3}{2}}}$

also $y = \frac{x}{3} \Rightarrow \boxed{x = 3y}$

$$A(x) = \pi \left((3y)^2 - \left(y^{\frac{3}{2}} \right)^2 \right)$$

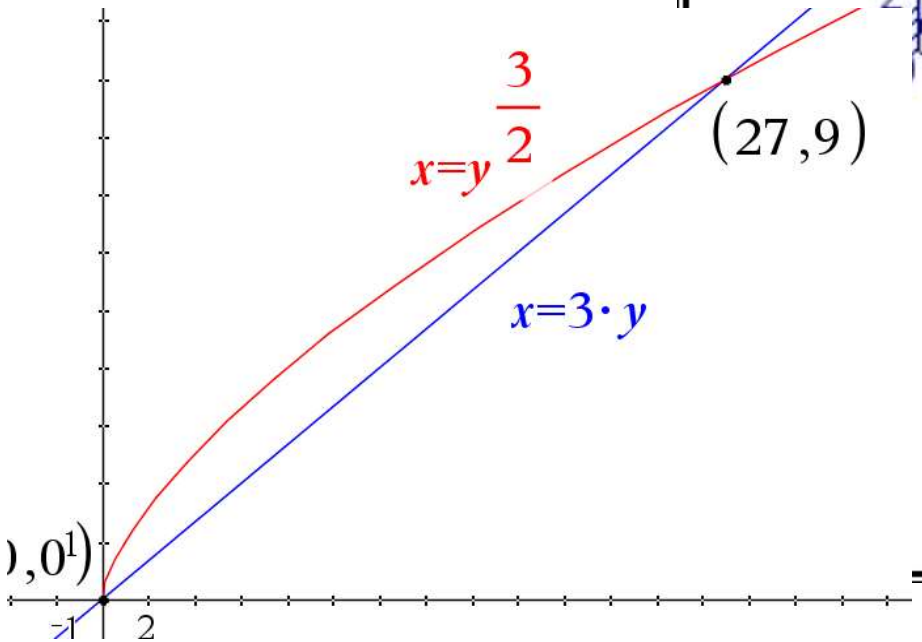
$$A(x) = \pi \left((9y^2 - y^3 \right)$$

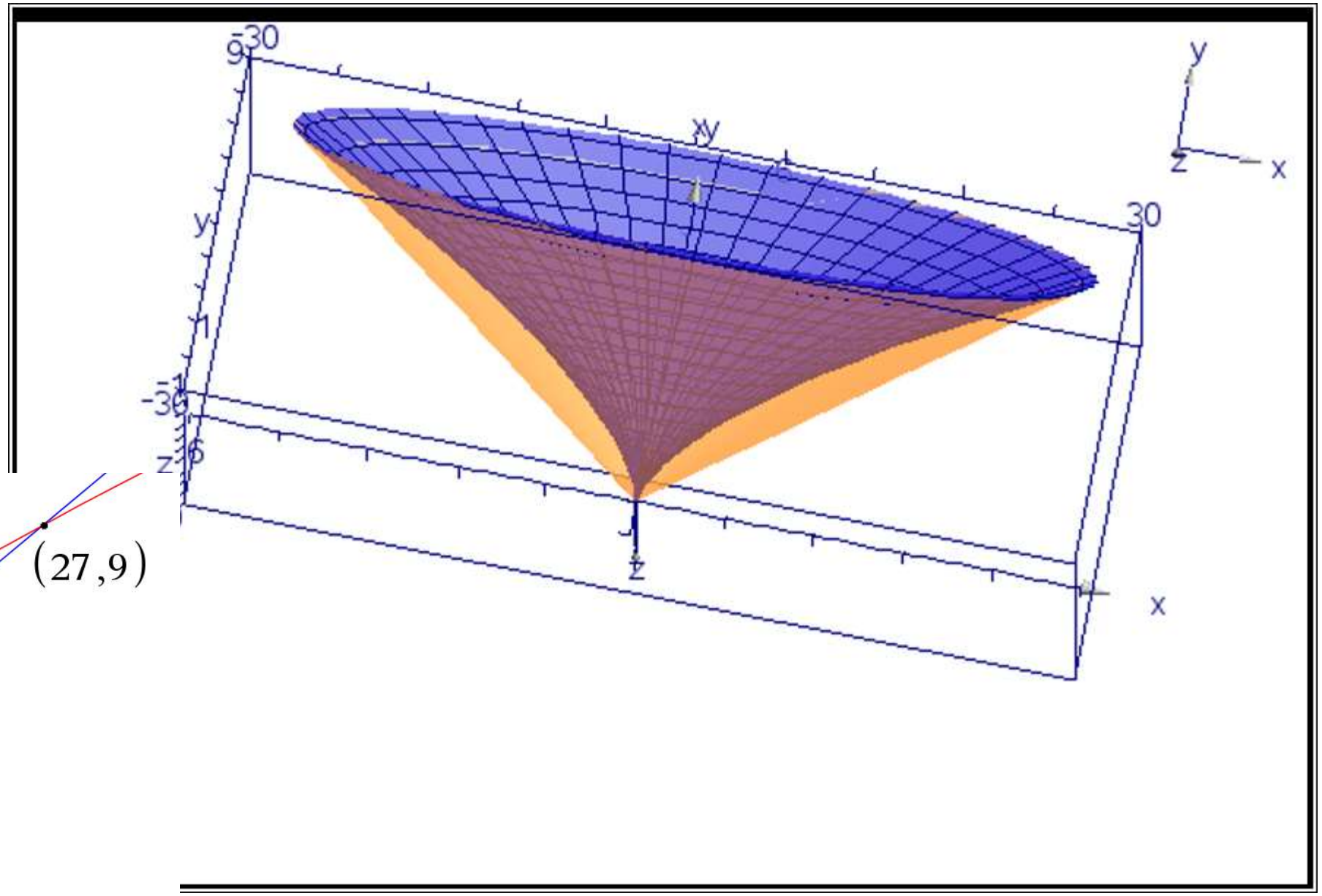
$$V = \pi \int_0^3 \left((9y^2 - y^3 \right)$$

$$V = \pi \left[\frac{9y^3}{3} - \frac{y^4}{4} \right]_0^9$$

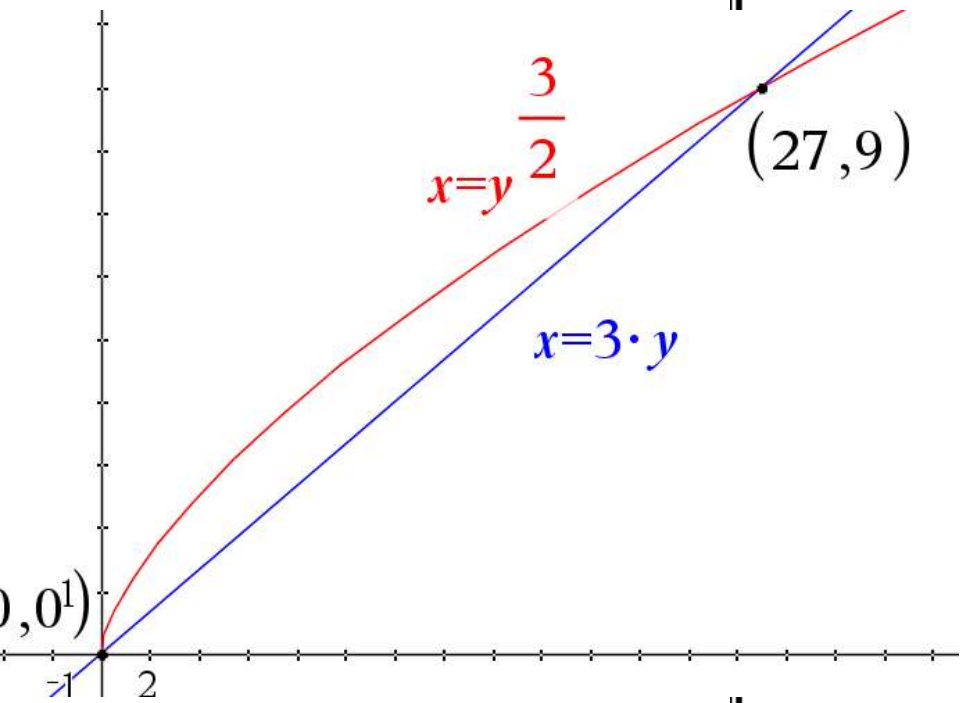
$$V = \pi \left[2187 - \frac{6561}{4} \right] \rightarrow \pi \left[\frac{2187}{4} \right]$$

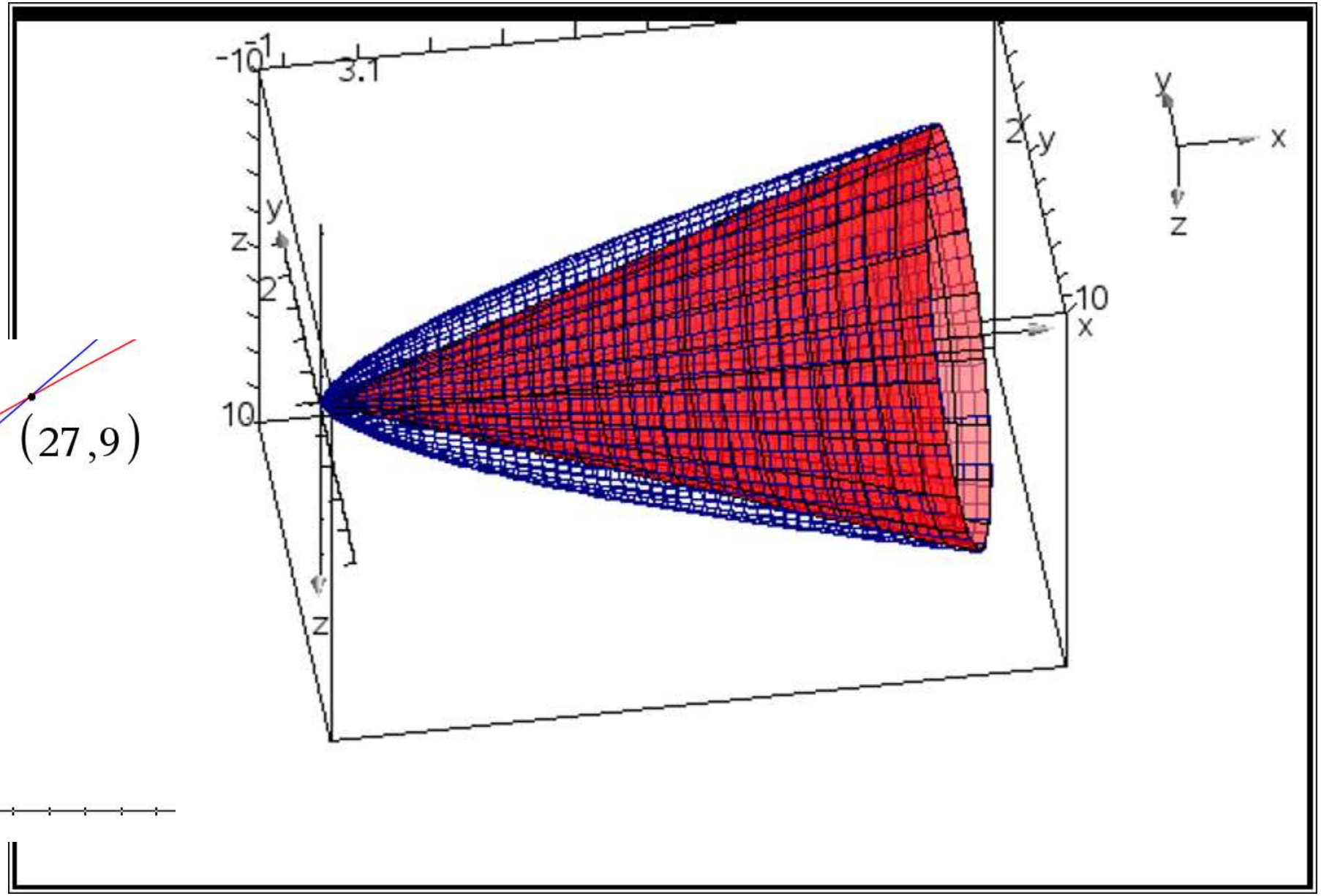
When rotated about Y-axis





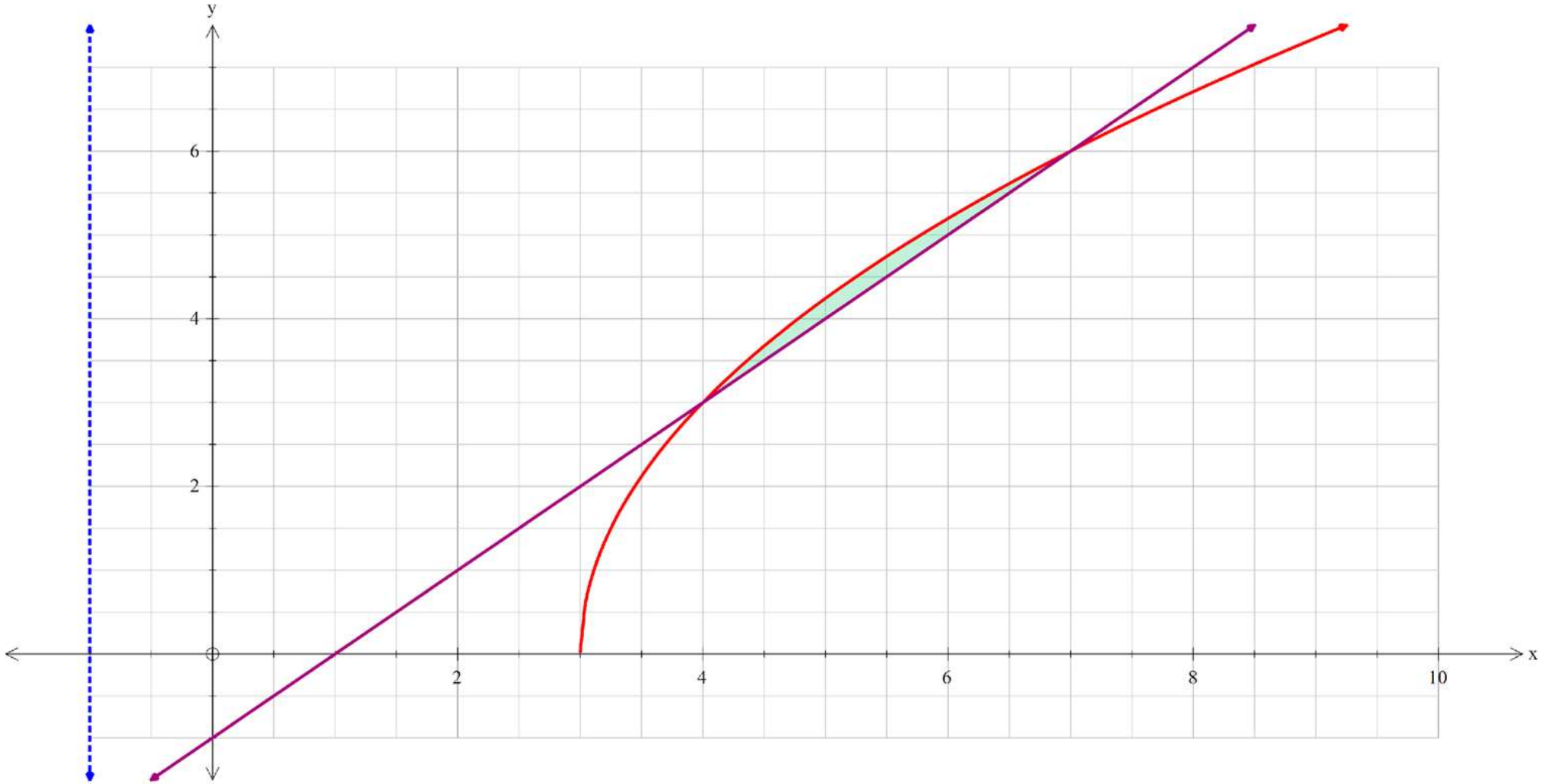
When rotated about X-axis





Determine the Volume of solid obtained by rotating the region bound by

$f(x) = 3\sqrt{x-3}$ and $g(x) = x - 1$
when rotated about the line $x = -1$



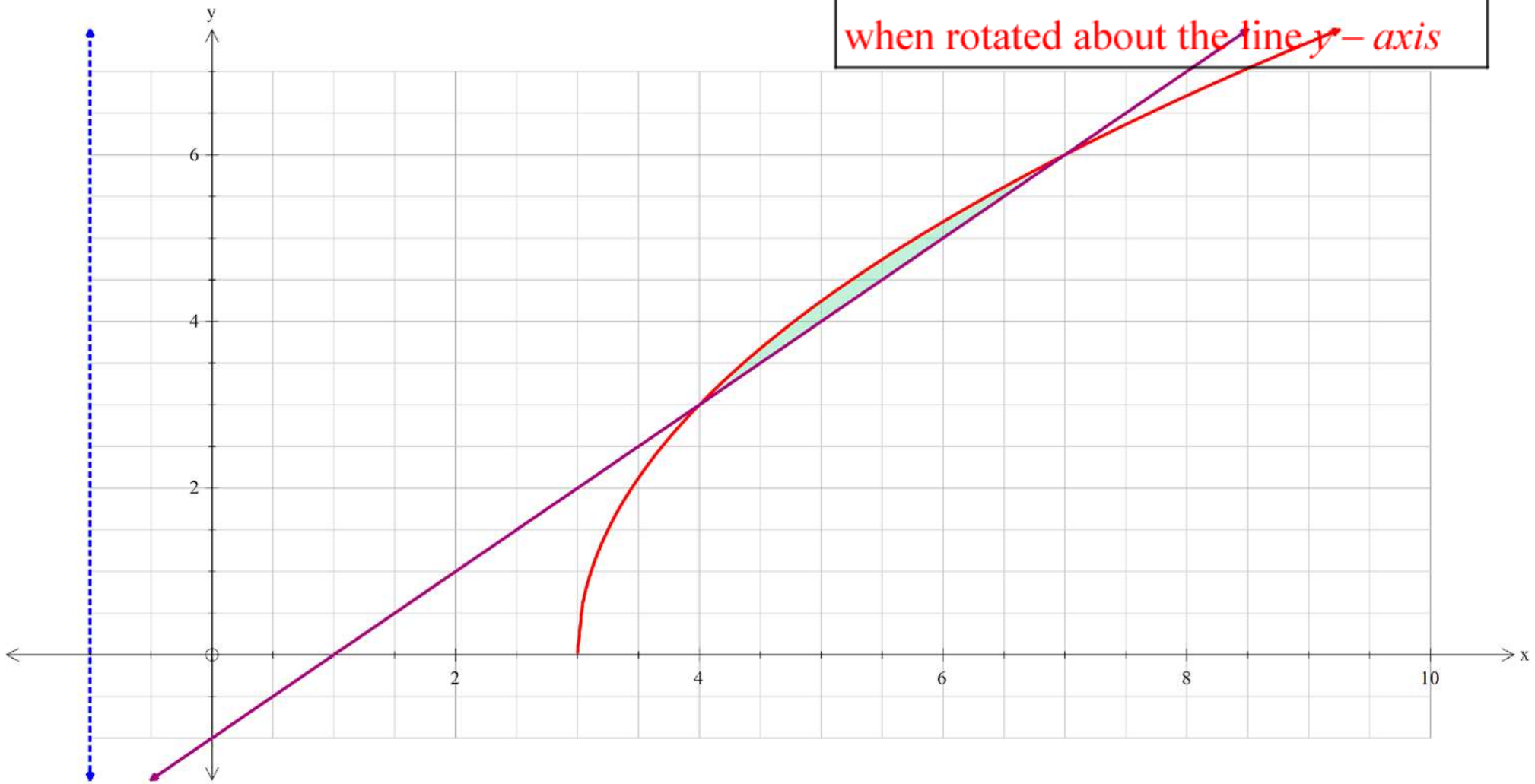
Both equations can undergo a horizontal translation and then rotated along the y-axis

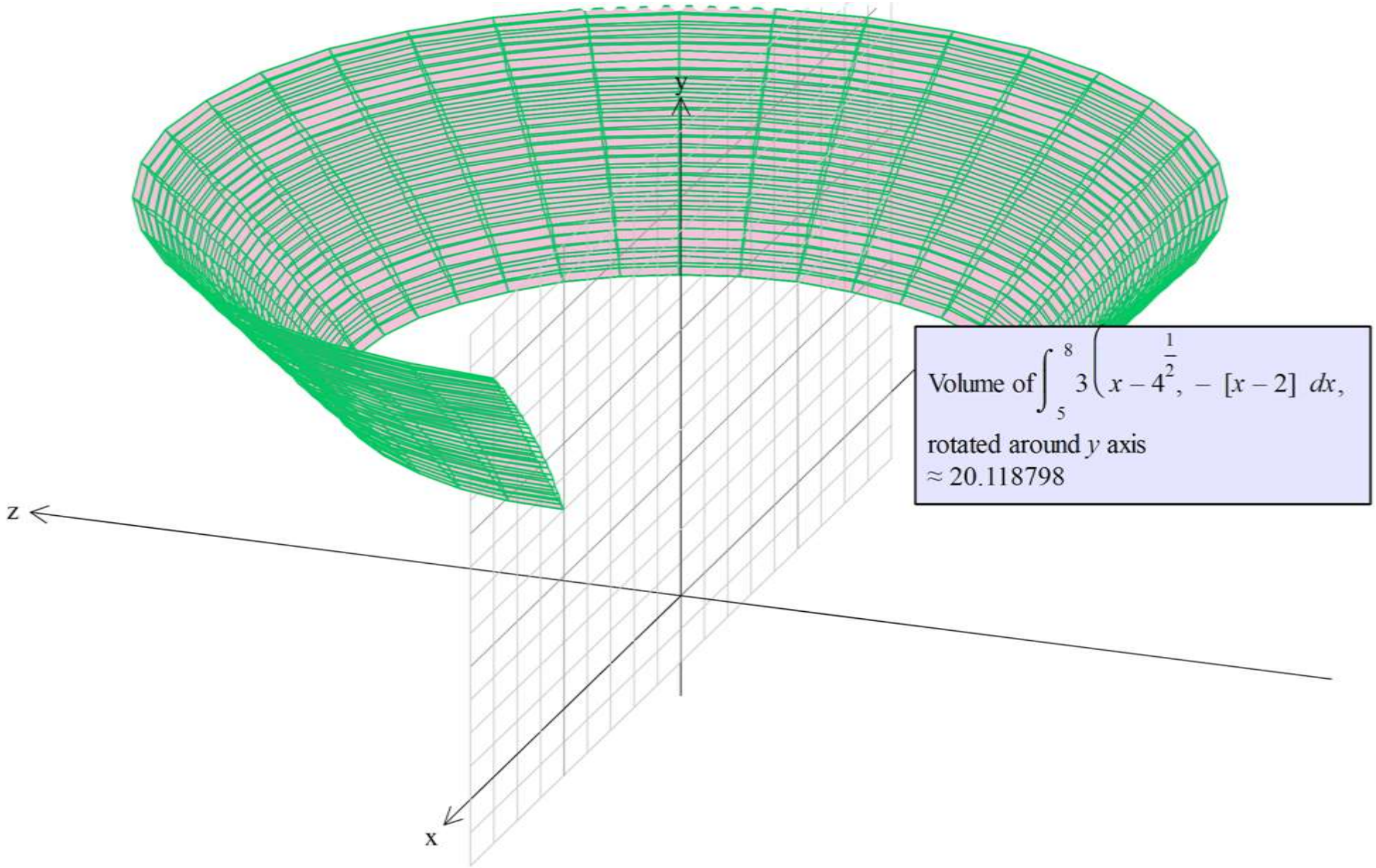
$f(x) = 3\sqrt{x-3-1}$ and $g(x) = x - 1 - 1$

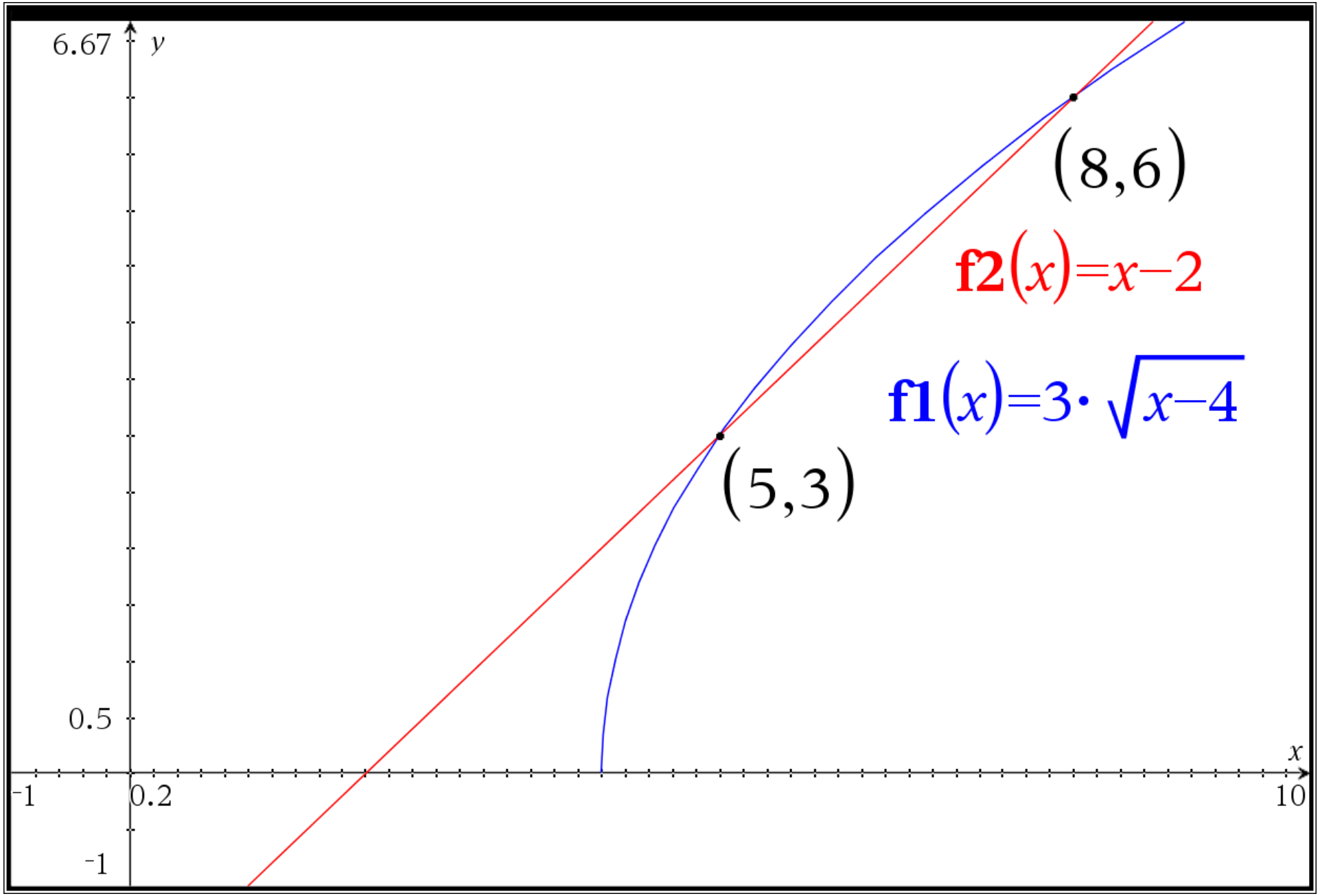
when rotated about the line $x = 0$

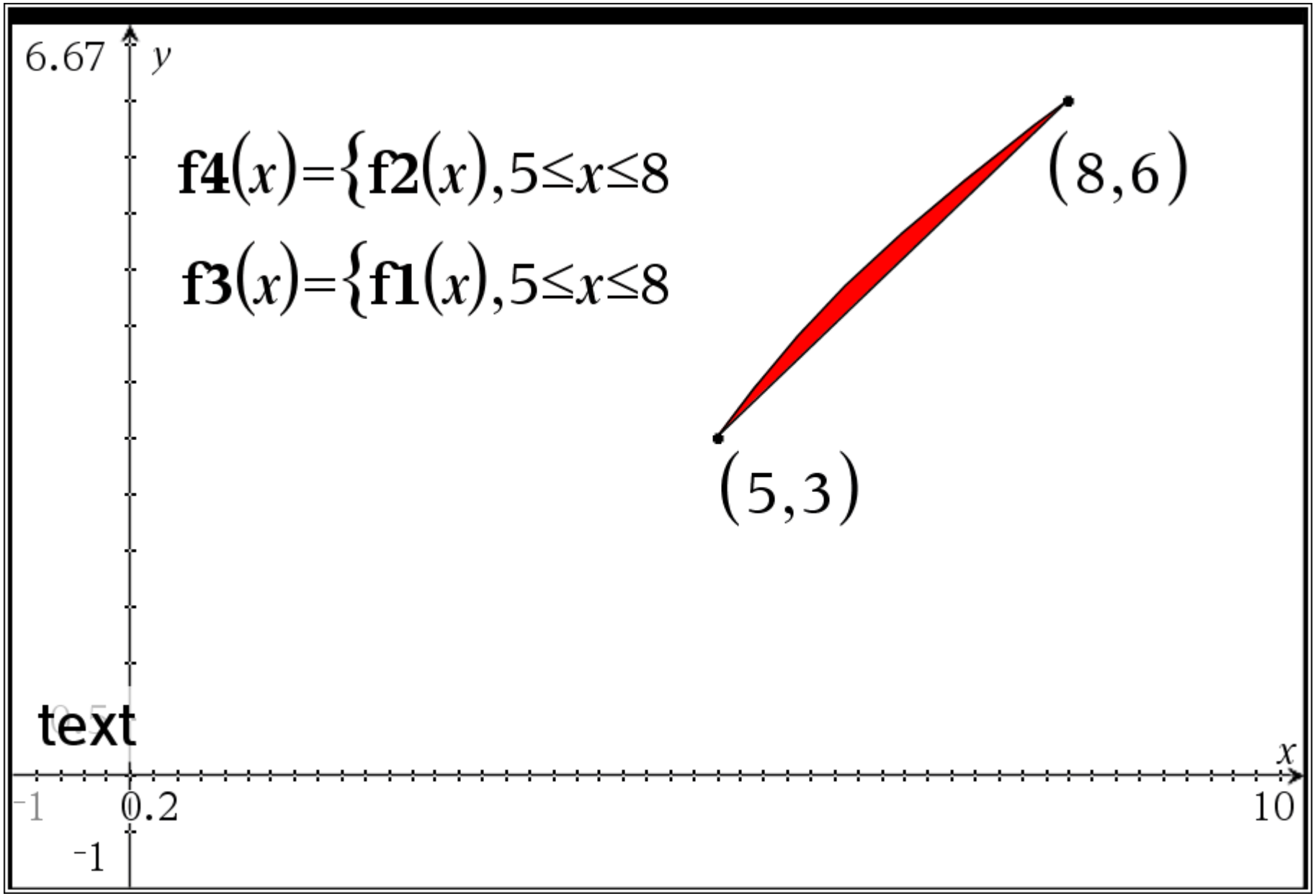
$f(x) = 3\sqrt{x-4}$ and $g(x) = x - 2$

when rotated about the line $y - axis$




y
z
x
Volume of $\int_5^8 3\left(x-4^{\frac{1}{2}},\ -\ [x-2]\right)\ dx$,
rotated around y axis
≈ 20.118798


6.67
y
(8,6)
f2(x)=x−2
f1(x)=3· √(x−4)
(5,3)
0.5
x
-1
0.2
10
-1


6.67
y
f4(x)={f2(x),5≤x≤8
f3(x)={f1(x),5≤x≤8
(8,6)
(5,3)
text
x
-1
0.2
10
-1

$xp1(t,u)=(t-2)\cdot\sin(u)$

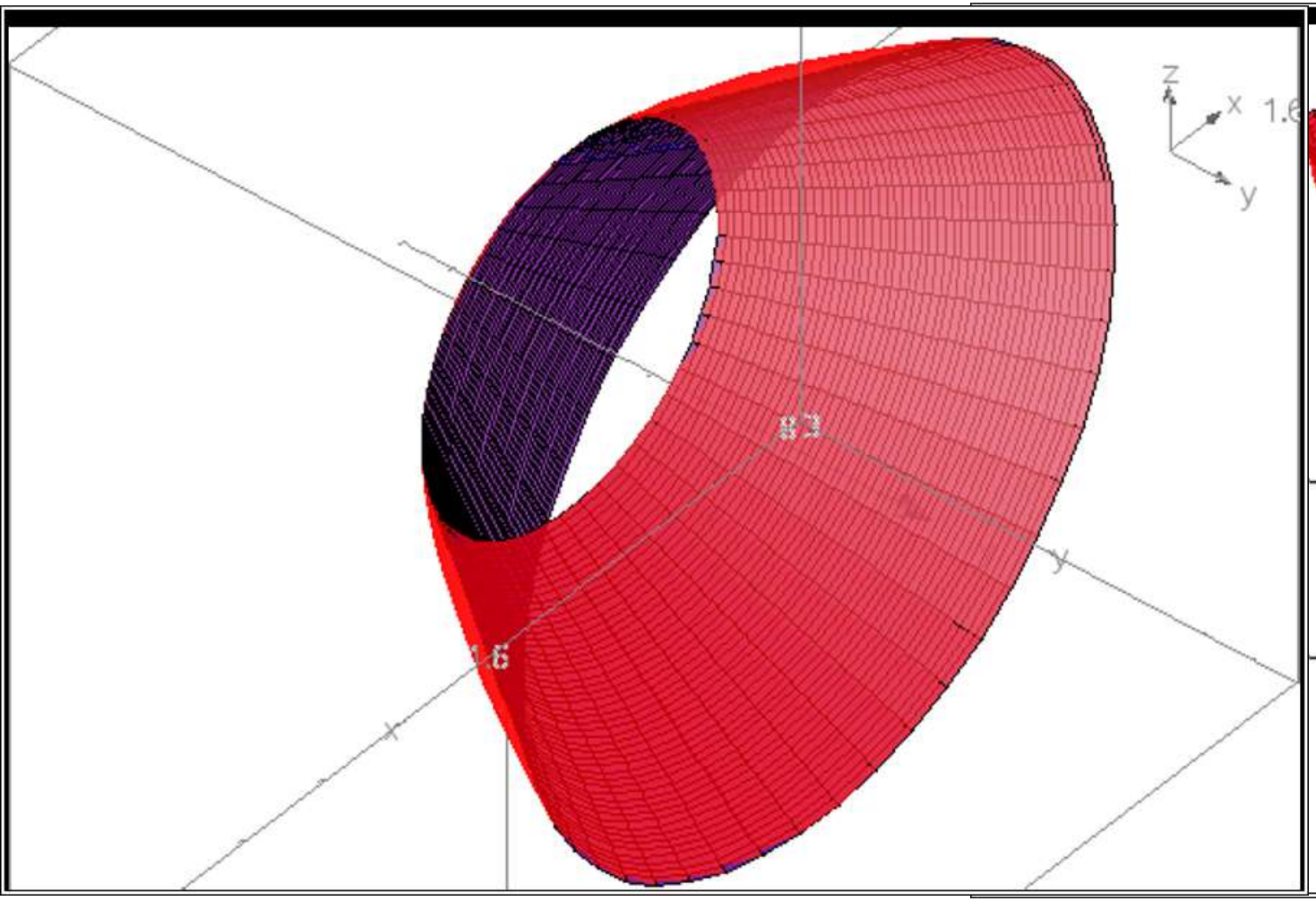
$yp1(t,u)=t$

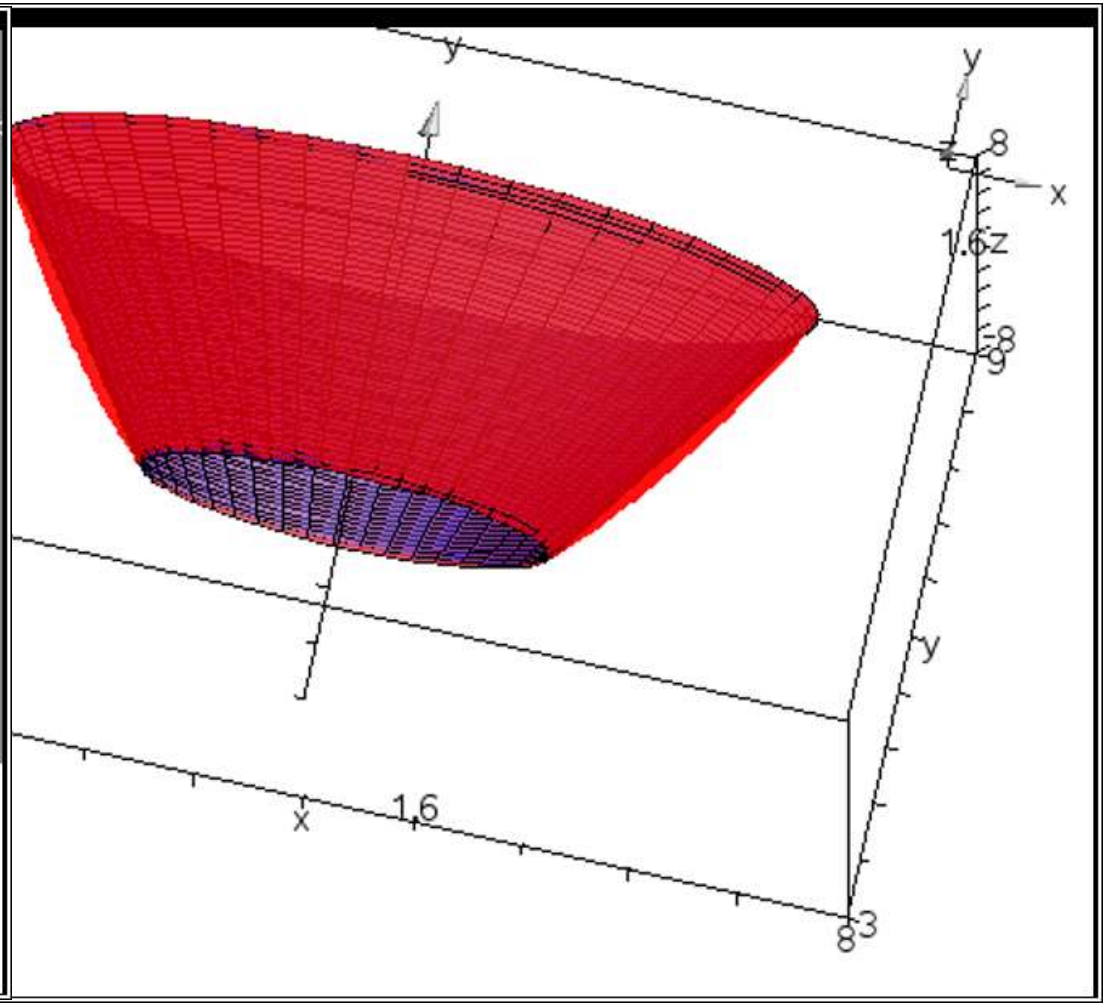
$zp1(t,u)=(t-2)\cdot\cos(u)$

$xp2(t,u)=3\cdot\sqrt{t-4}\cdot\sin(u)$

$yp2(t,u)=t$

$zp2(t,u)=3\cdot\sqrt{t-4}\cdot\cos(u)$





$$y = 3\sqrt{x-3} \quad \rightarrow x = \frac{y^2}{9} + 3$$

$$y = x - 1 \quad \rightarrow x = y + 1$$

Outer radius y+1+1 → y+2

Inner radius $\frac{y^2}{9} + 3 + 1 \rightarrow \frac{y^2}{9} + 4$

Cross-sectional area is

$$A(y) = \pi\left((y+2)^2 - \left(\frac{y^2}{9} + 4\right)^2\right)$$

$$A(Y) = \pi\left(-\frac{y^4}{81} + \frac{y^2}{9} + 4y - 12\right)$$

$$V = \pi\int_3^6 (-\frac{y^4}{81} + \frac{y^2}{9} + 4y - 12)dy$$

$$V = \frac{32\pi}{5}$$

Volume of $\int_5^8 3\left(x - 4^{\frac{1}{2}}, - [x-2]\right) dx,$

rotated around y axis

≈ 20.118798

$$\pi \cdot \int_{3}^{6} \left(\frac{-y^4}{81} + \frac{y^2}{9} + 4 \cdot y - 12 \right) dy \blacktriangleright \frac{32 \cdot \pi}{5}$$

$$\pi \cdot \int_{3}^{6} \left(\frac{-y^4}{81} + \frac{y^2}{9} + 4 \cdot y - 12 \right) dy \blacktriangleright 20.106192983$$

Volume of $\int_{5}^{8} 3 \left(x - 4^{\frac{1}{2}}, - [x-2] \right) dx$,

rotated around y axis

≈ 20.118798