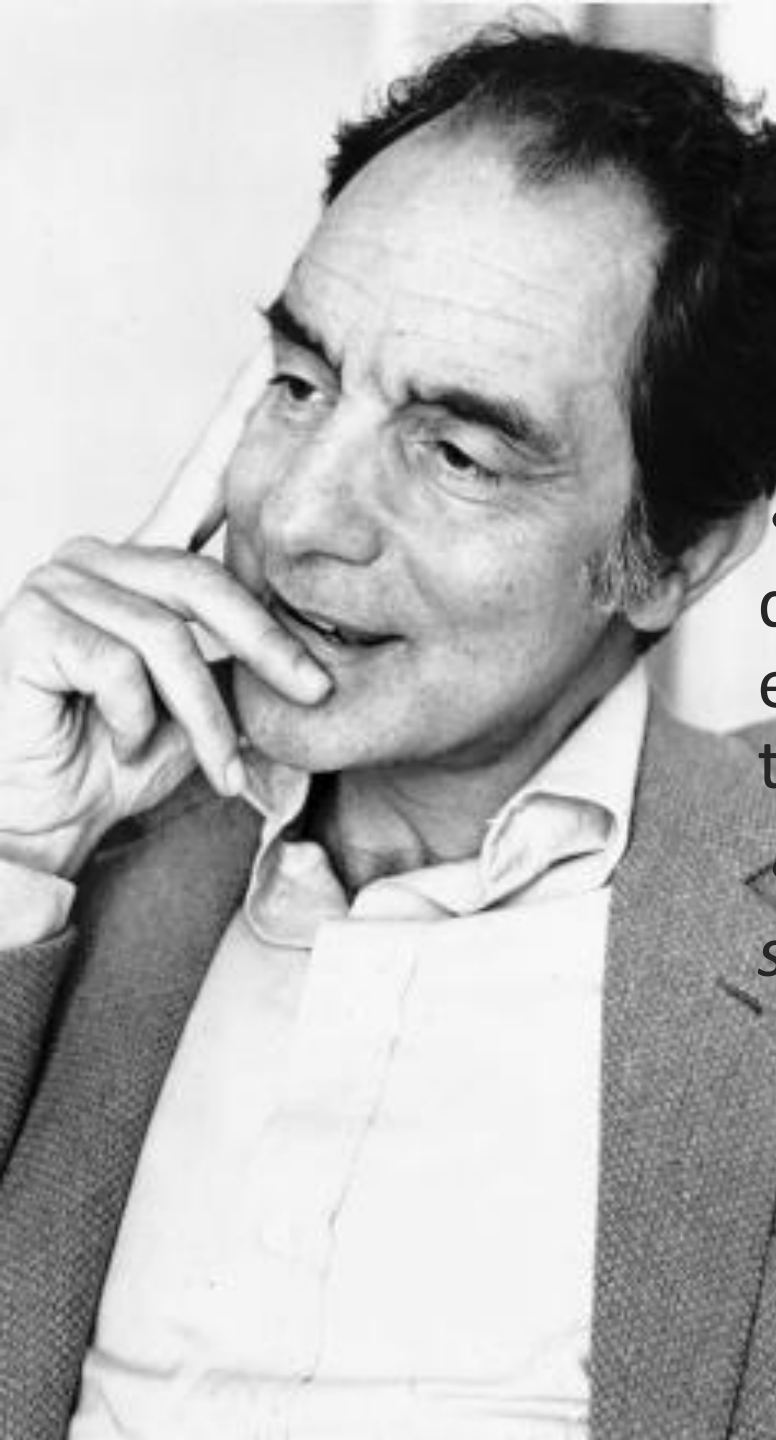


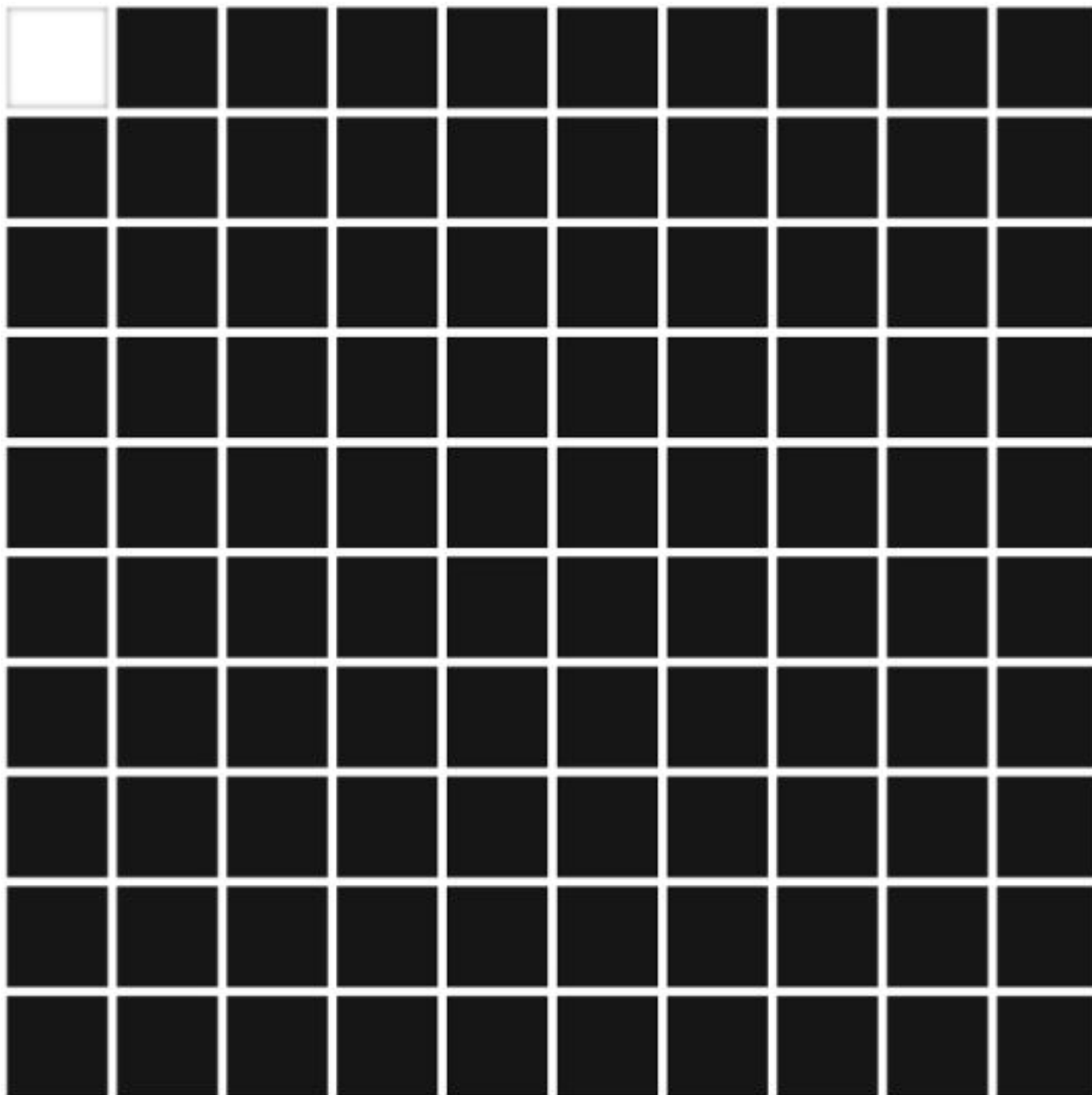
# Creative Solutions for Solving Equations



# Mathematical Style

- “For him, good thinking means quickness, agility in reasoning, economy in argument, but also the use of **imaginative examples**.”
- *Italo Calvino commenting on the style of Galileo*





The Inspiration

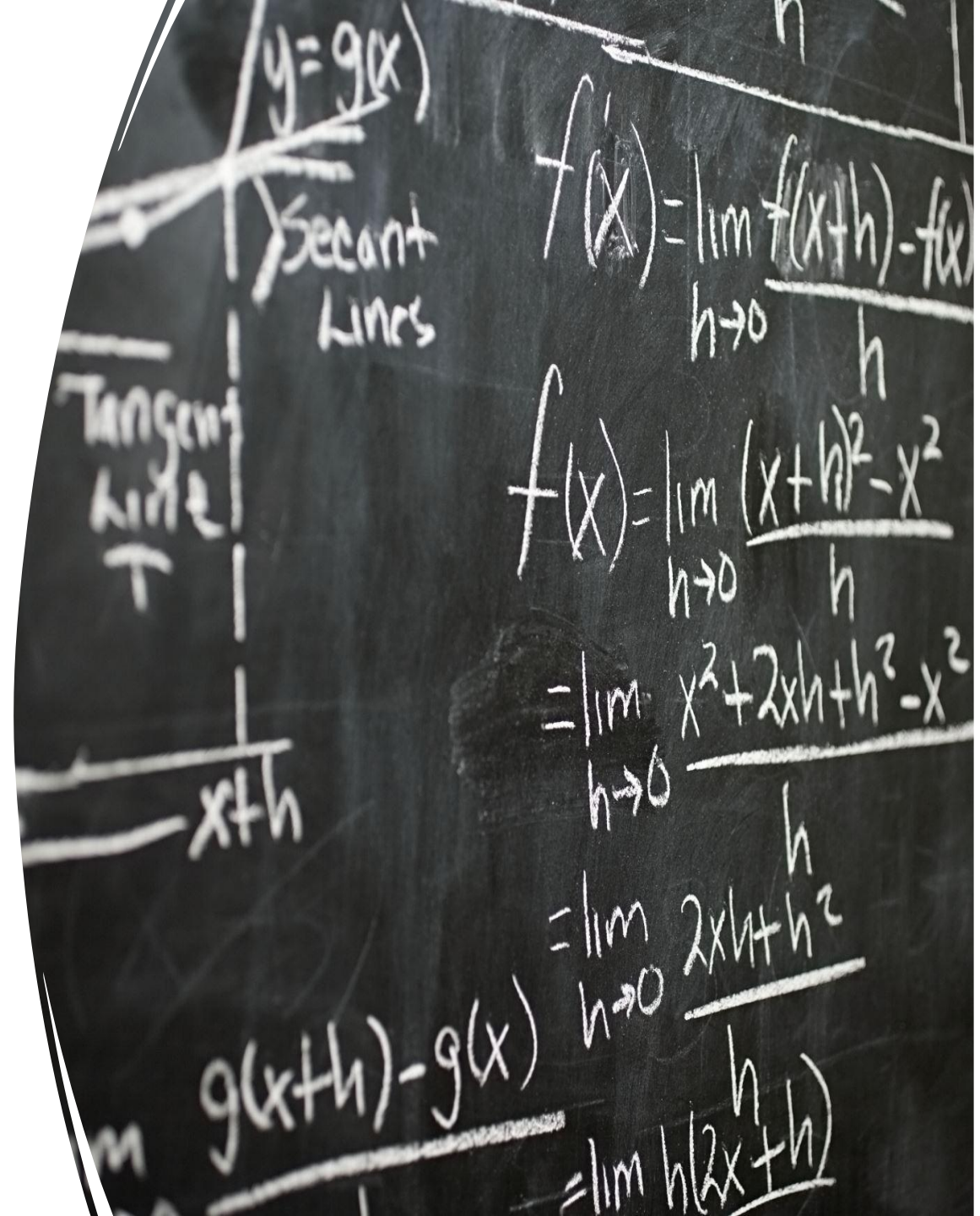
**99 Variations on a Proof** Philip Ording

Contents	0	1	2	3	4	5	6	7	8	9	
Preface ix	Omitted 1	One-Line 3	Two-Column 5	Illustrated 7	Elementary 9	Puzzle 11	Axiomatic 13	Found 17	Prerequisite 19	Monosyllabic 21	
	10 Wordless 23	11 Exam 25	12 Ruler and Compass 27	13 Reductio ad Absurdum 29	14 Contrapositive 31	15 Matrices 33	16 Ancient 35	17 Interpreted 37	18 Indented 39	19 Jargon 41	
	20 Definitional 43	21 Blackboard 47	22 Substitution 49	23 Symmetry 51	24 Another Symmetry 53	25 Open Collaborative 57	26 Auditory 61	27 Algorithmic 63	28 Flow Chart 65	29 Model 67	
	30 Formulaic 69	31 Counterexample 71	32 Another Counterexample 73	33 Calculus 75	34 Medieval 77	35 Typeset 79	36 Social Media 83	37 Preprint 85	38 Parataxis 87	39 Origami 89	
	40 Induction 91	41 Newsprint 93	42 Analytic 95	43 Screenplay 97	44 Omitted with Condescension 105	45 Verbal 107	46 Cute 109	47 Clever 111	48 Computer Assisted 113	49 Outsider 115	
	50 Chromatic 117	51 Topological 119	52 Antiquity 121	53 Marginalia 125	54 Arborescent 129	55 Prefix 131	56 Postfix 133	57 Calculator 135	58 Inventor's Paradox 137	59 Patented 139	
	60 Geometric 141	61 Modern 143	62 Axonometric 145	63 Back of the Envelope 149	64 Research Seminar 151	65 Tea 153	66 Hand Waving 155	67 Approximate 157	68 Word Problem 159	69 Statistical 161	
	70 Another Medieval 163	71 Blog 167	72 Translated 171	73 Another Translated 173	74 Another Interpreted 175	75 Slide Rule 181	76 Experimental 183	77 Monte Carlo 185	78 Probabilistic 187	79 Intuitionist 189	
	80 Paranoid 191	81 Doggerel 193	82 Inconsistency 195	83 Correspondence 197	84 Tabular 199	85 Exhaustion 201	86 Another Substitution 203	87 Mechanical 207	88 Dialogue 209	89 Interior Monologue 213	
	90 Retrograde 215	91 Mystical 217	92 Refereed 219	93 Neologism 221	94 Authority 223	95 First Person 225	96 Electrostatic 227	97 Psychedelic 229	98 Mondegreen 231	99 Prescribed 233	
											Postscript 235 Acknowledgments 237 Notes 239 Sources 247 Index 257

# The Equation

Solve the cubic equation for all of its REAL solutions:

$$x^3 - 6x^2 + 11x - 6 = 2x - 2$$



0

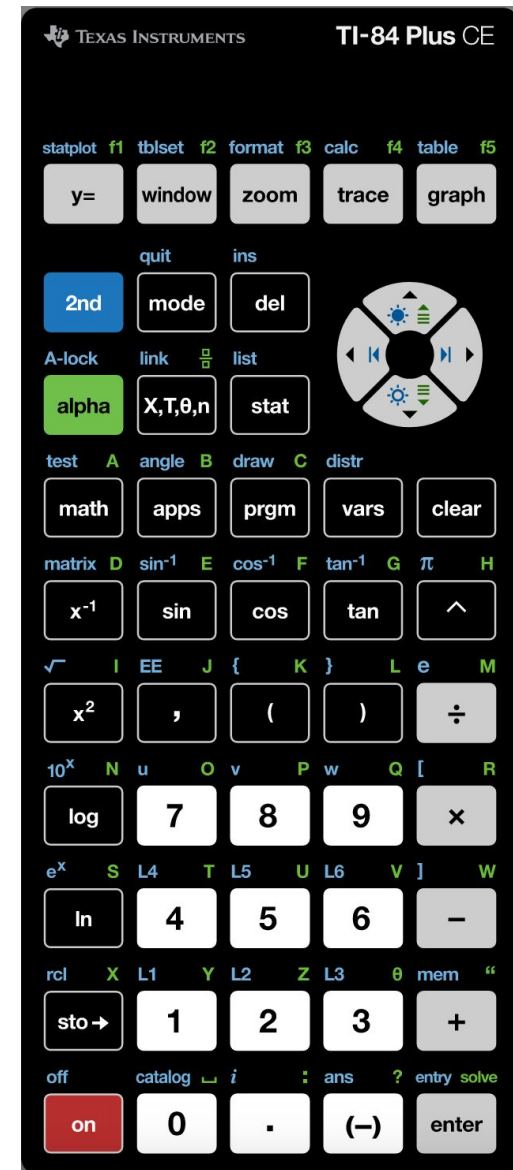
Omitted

Theorem. If  $x^3 - 6x^2 + 11x - 6 = 2x - 2$ , then  
 $x = 1$  *or*  $x = 4$

Since the solution is trivial . . . The proof is left to the reader.

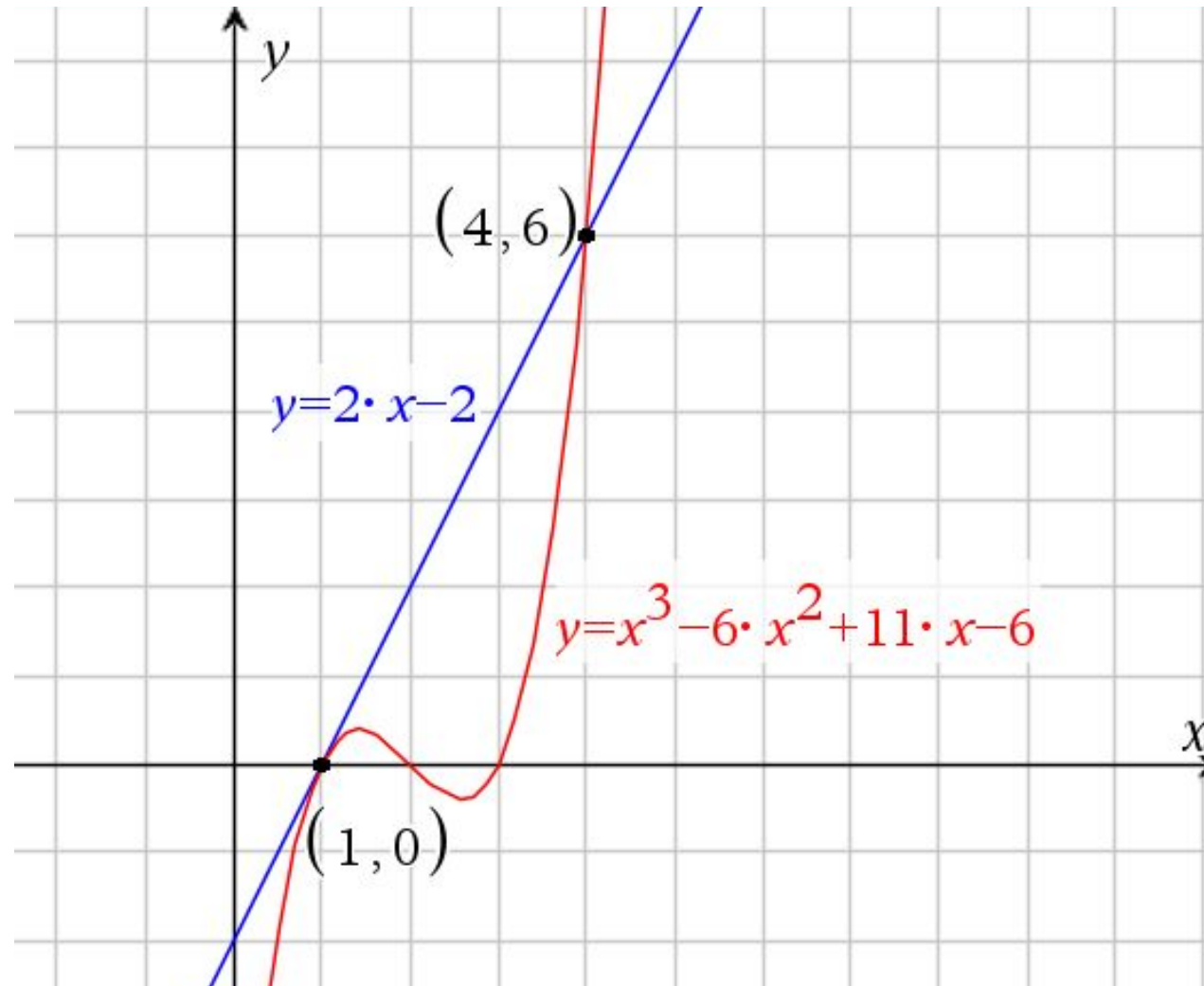
57

Calculator



3

Illustrated



The two points of intersection of the cubic  $y = x^3 - 6x^2 + 11x - 6$  and the line  $y = 2x - 2$  occur at  $(1, 0)$  and  $(4, 6)$ .

22

Substitution

**Theorem.** *If  $x^3 - 6x^2 + 11x - 6 = 2x - 2$ , then  $x = 1$  or  $x = 4$ .*

*Proof:*

$$x^3 - 6x^2 + 11x - 6 = 2x - 2$$

$$x^3 - 6x^2 + 9x - 4 = 0$$

Let  $x = m + 1$        $(m + 1)^3 - 6(m + 1)^2 + 9(m + 1) - 4 = 0$

$$(m^3 + 3m^2 + 3m + 1) - 6(m^2 + 2m + 1) + 9(m + 1) - 4 = 0$$

$$m^3 + 3m^2 + 3m + 1 - 6m^2 - 12m - 6 + 9m + 9 - 4 = 0$$

$$m^3 - 3m^2 = 0$$

$$m^2(m - 3) = 0$$

$$m = 0 \text{ or } m = 3$$

$$x = 0 + 1 \text{ or } x = 3 + 1$$

$$x = 1 \text{ or } x = 4$$

50

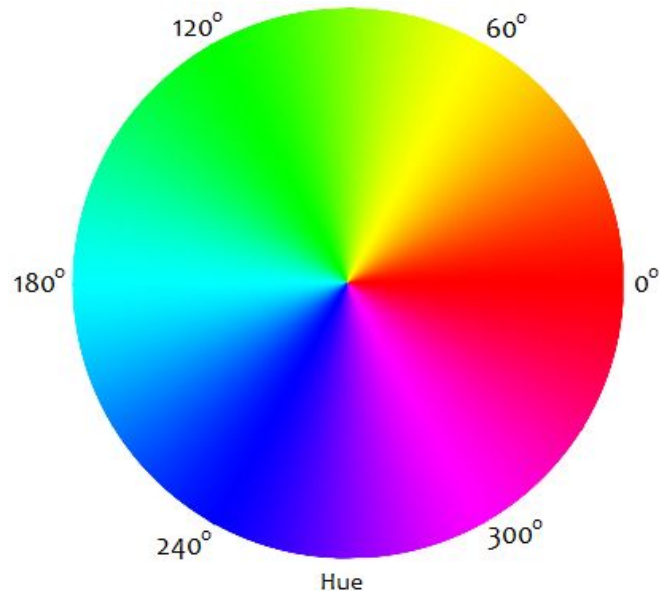
Chromatic

# Chromatic

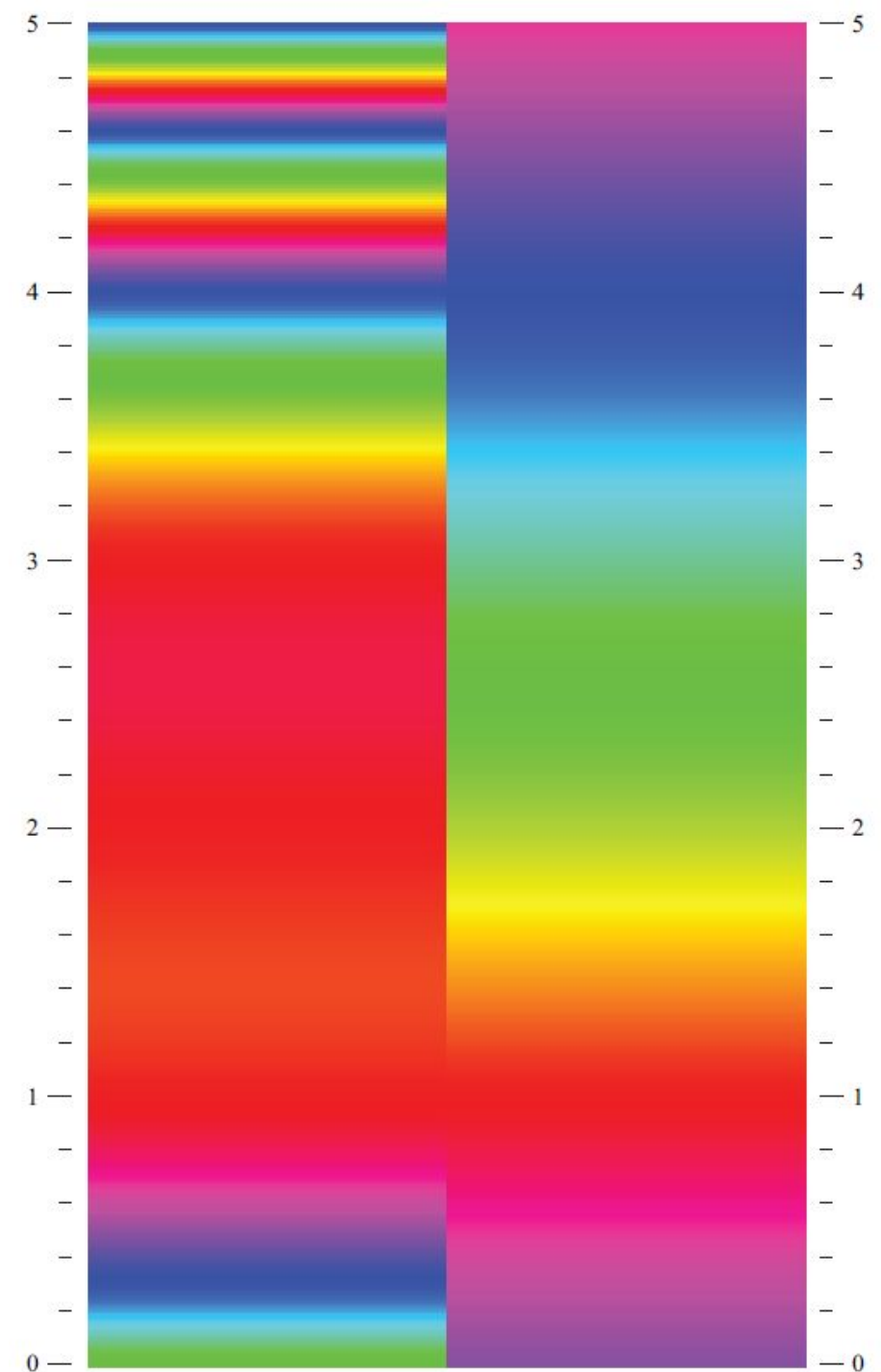
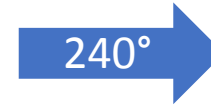
- The two spectra represent both sides of the equation

$$x^3 - 6x^2 + 11x - 6 = 2x - 2$$

- At height  $x$  the hue on the left is proportional to the left side of the equation and the hue on the right to the right side of the equation.
- The **red** band ( $0^\circ$ ) at  $x = 1$  and the **blue** band ( $240^\circ$ ) at  $x = 4$  are the two solutions.



1 unit =  $80^\circ$  in hue



26

Auditory

♩ = 52



## Cubic function

**♩ = 52**

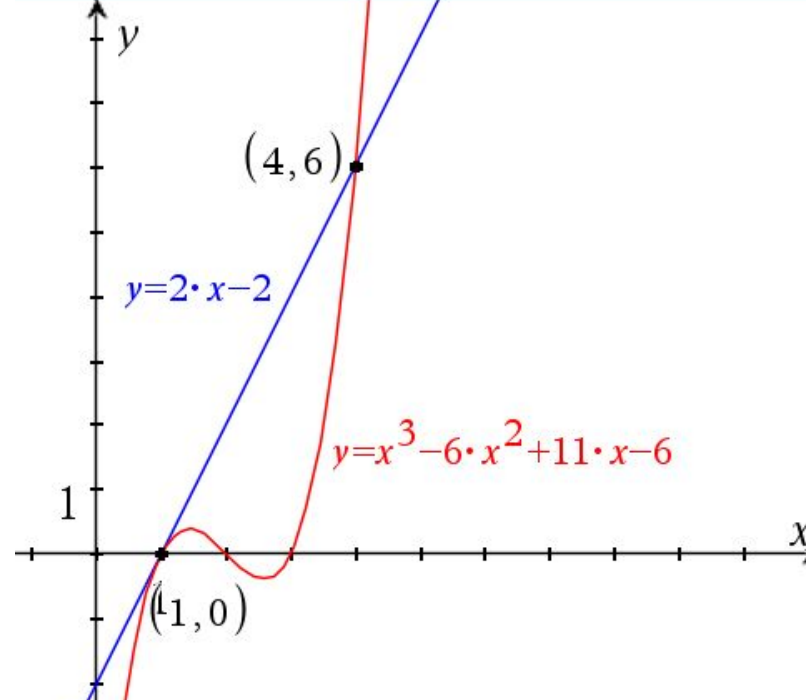


$\text{♩} = 52$

Cubic  
Function

Linear  
Function

A musical score for two staves, 'Cubic Function' and 'Linear Function', in 4/4 time. The tempo is marked as  $\text{♩} = 52$ . The 'Cubic Function' staff features a melody with a red slur spanning from the second measure to the end of the piece. The 'Linear Function' staff features a complex, fast-paced accompaniment. Both staves have green rectangular boxes highlighting the first measure and the final measure. The final measure of both staves ends with a double bar line and a repeat sign.



$\text{♩} = 52$

Cubic Function

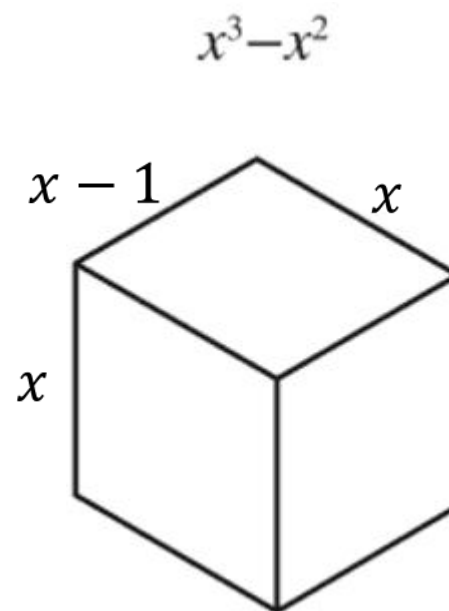
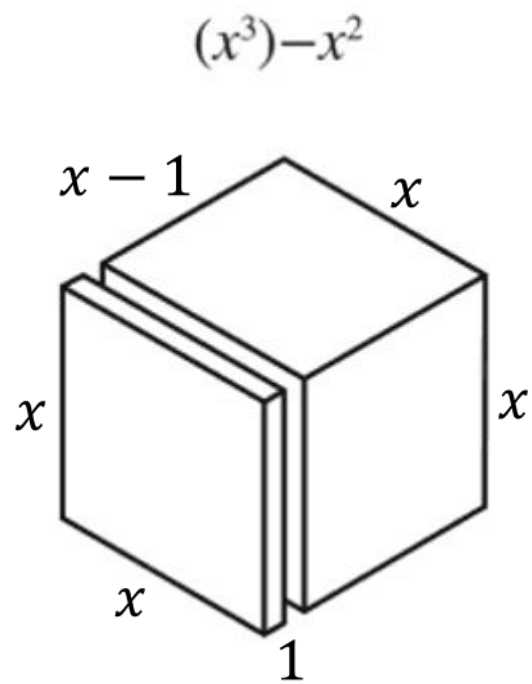
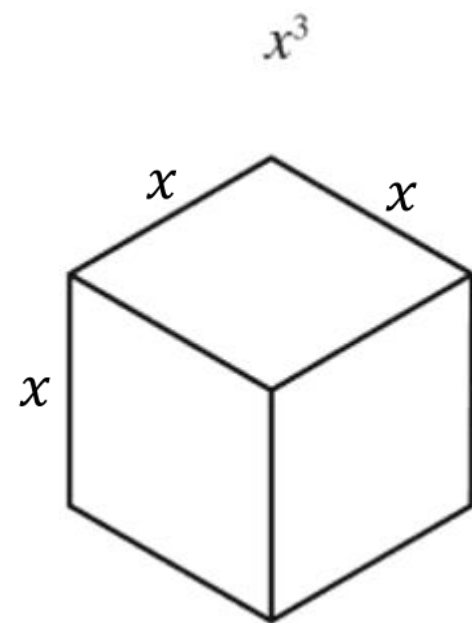
Linear Function

The image shows two staves of musical notation. The top staff is labeled 'Cubic Function' and the bottom staff is labeled 'Linear Function'. Both staves are in 4/4 time. The top staff contains a melody with various notes and rests, including a long note with a slur. The bottom staff contains a more complex melody with many notes. Two green rectangular boxes highlight specific sections of the music: one on the top staff and one on the bottom staff. A red curved line connects the two boxes, suggesting a relationship or comparison between the two functions.

10

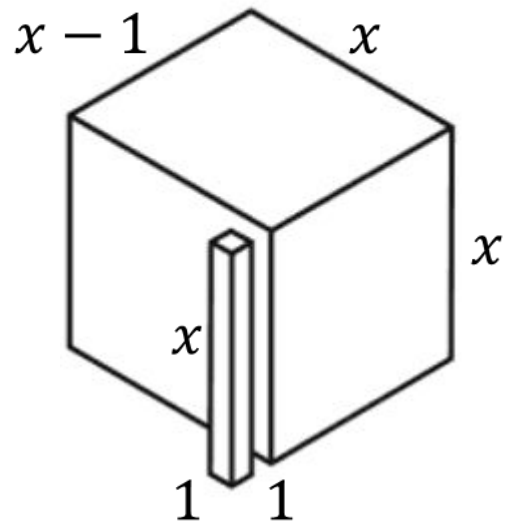
Wordless

$$x^3 - 6x^2 + 9x - 4$$

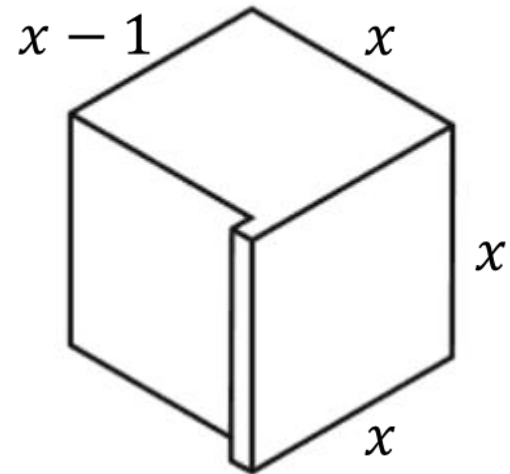


$$x^3 - 6x^2 + 9x - 4$$

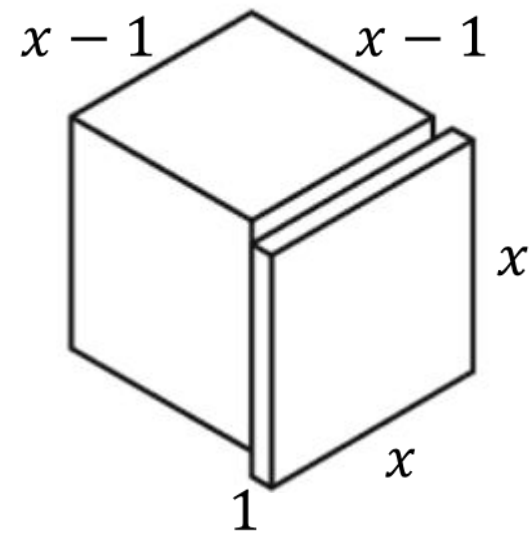
$$(x^3 - x^2) + x$$



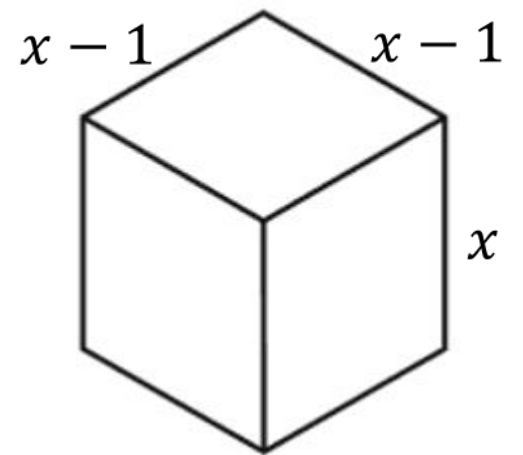
$$x^3 - x^2 + x$$



$$(x^3 - x^2 + x) - x^2$$

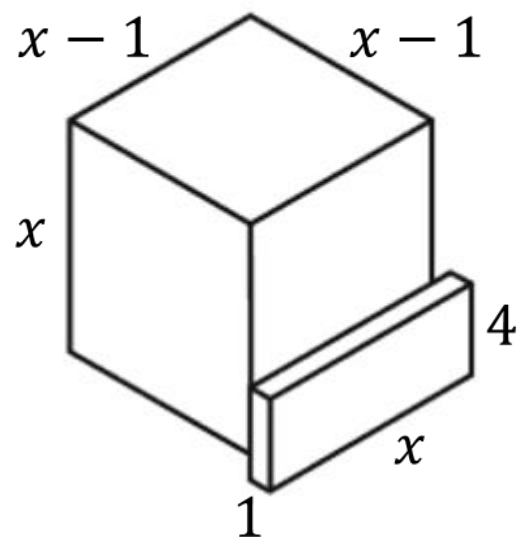


$$x^3 - 2x^2 + x$$

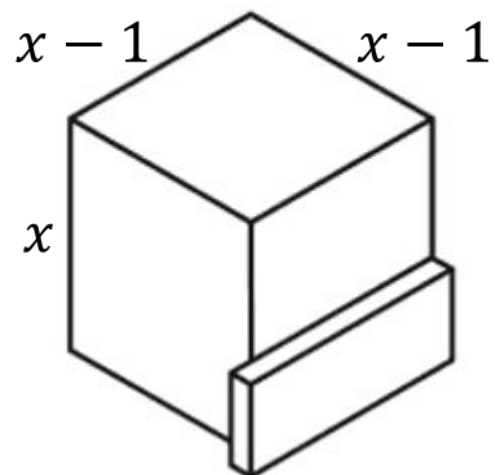


$$x^3 - 6x^2 + 9x - 4$$

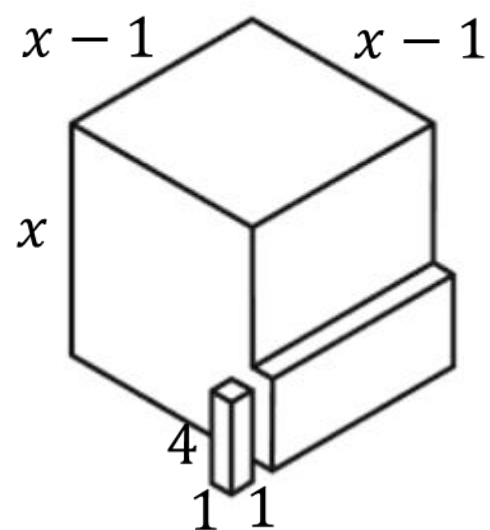
$$(x^3 - 2x^2 + x) + 4x$$



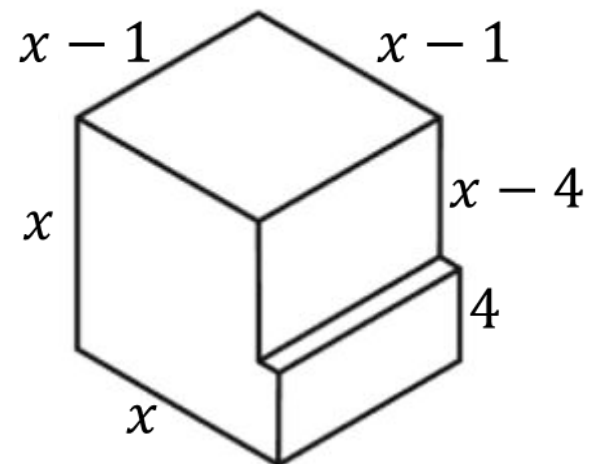
$$x^3 - 2x^2 + 5x$$



$$(x^3 - 2x^2 + 5x) - 4$$

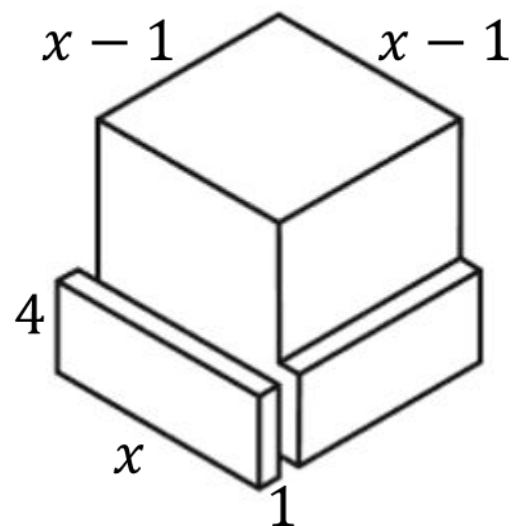


$$x^3 - 2x^2 + 5x - 4$$

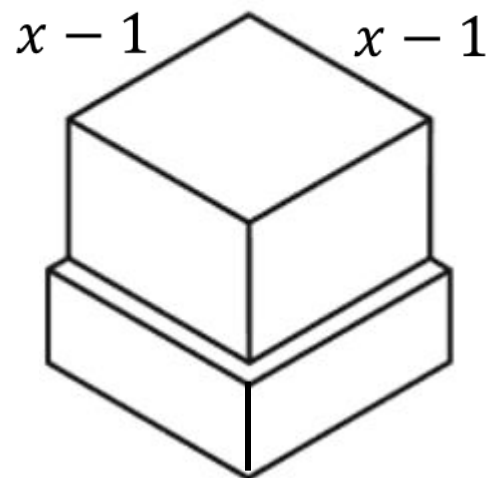


$$x^3 - 6x^2 + 9x - 4$$

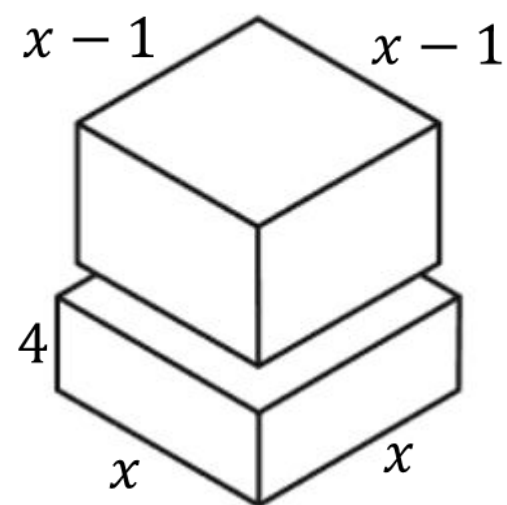
$$(x^3 - 2x^2 + 5x - 4) + 4x$$



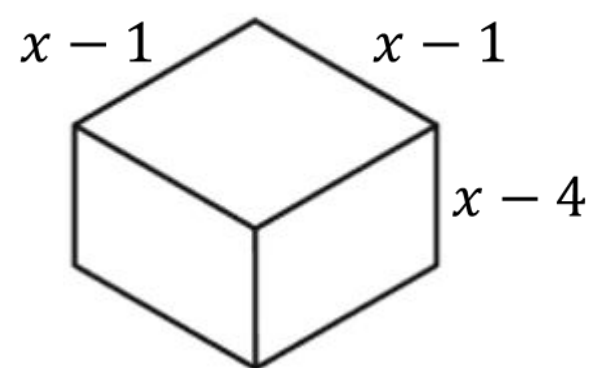
$$x^3 - 2x^2 + 9x - 4$$



$$(x^3 - 2x^2 + 9x - 4) - 4x^2$$



$$x^3 - 6x^2 + 9x - 4$$



68

Word Problem

We want to make a box from a  $6'' \times 6''$  piece of paper by cutting out four equal squares from the corners and folding up the resulting flaps. If the volume of the box is to be  $16 \text{ in}^3$ , how many inches should we cut out of each corner?

*Proof:*

Let  $r$  be the number of inches cut from each corner. Since the volume = length  $\times$  width  $\times$  height, and we want the volume to be  $16 \text{ in}^3$ , we need to solve the equation:

$$(6 - 2r)(6 - 2r)r = 16$$

$$2(3 - r)2(3 - r)r = 16$$

$$(3 - r)(3 - r)r = 4$$

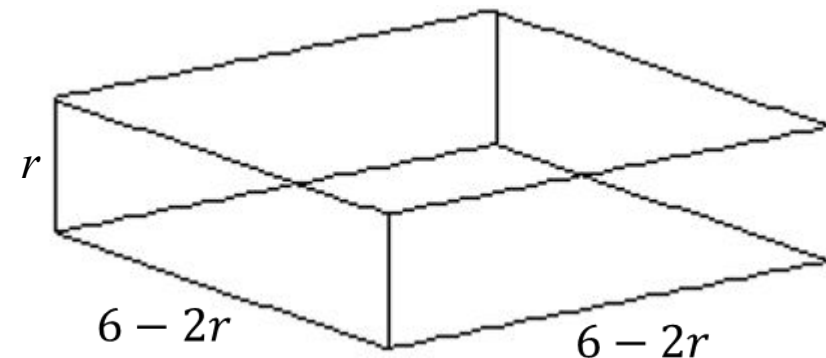
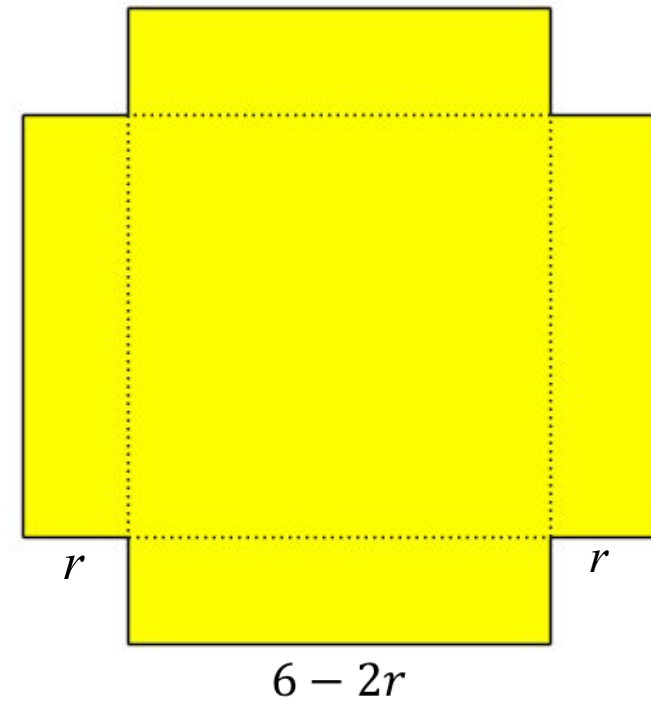
$$(9 - 6r + r^2)r = 4$$

$$r^3 - 6r^2 + 9r = 4$$

$$r^3 - 6r^2 + 9r - 4 = 0$$

$$(r - 1)(r - 1)(r - 4) = 0$$

$$r = 1, r \neq 4$$



11

Student Exam

Instructions: Write your solution on this page. Clearly show all appropriate algebraic work. The use of any communications device is strictly prohibited.

Solve:  $x^3 - 6x^2 + 11x - 6 = 2x - 2$

Please refer to my scratch paper for the solution.

$$x^3 - 6x^2 + 11x - 6 = 2x - 2$$

13

Reductio ad  
Absurdum

**Theorem.** *If  $x^3 - 6x^2 + 11x - 6 = 2x - 2$ , then  $x = 1$  or  $x = 4$ .*

*Proof:*

Suppose there is a third solution  $x$  such that  $x \neq 1$  and  $x \neq 4$ . This means that  $x - 1 \neq 0$  and  $x - 4 \neq 0$ .

$$x^3 - 6x^2 + 11x - 6 = 2x - 2$$

$$(x - 1)(x^2 - 5x + 6) = 2(x - 1)$$

$$x^2 - 5x + 6 = 2$$

$$x^2 - 5x + 4 = 0$$

$$(x - 1)(x - 4) = 0$$

$$x - 1 = 0$$

$$1 = 0$$

Based on our assumption that there was a third solution, we reached a conclusion which is absurd.

So our assumption that there was a third solution other than 1 and 4 must be false. Hence  $x = 1$  or  $x = 4$ .

14

Contrapositive

A *conditional* statement and its *contrapositive* are logically equivalent.

$$\text{i.e. } p \rightarrow q \equiv \sim q \rightarrow \sim p$$

or (*if  $p$ , then  $q$*   $\equiv$  *if  $\sim q$ , then  $\sim p$* )

**Theorem:** *If  $x^3 - 6x^2 + 11x - 6 = 2x - 2$ , then  $x = 1$  or  $x = 4$ .*

Contrapositive: *If  $x - 1 \neq 0$  and  $x - 4 \neq 0$ , then  $x^3 - 6x^2 + 11x - 6 \neq 2x - 2$*

*Proof:*

Assume the hypothesis of the contrapositive:  $x - 1 \neq 0$  and  $x - 4 \neq 0$

Then

$$(x - 1)(x - 1)(x - 4) \neq 0$$

$$x^3 - 6x^2 + 9x - 4 \neq 0$$

$$x^3 - 6x^2 + (11x - 2x) - (6 - 2) \neq 0$$

$$(x^3 - 6x^2 + 11x - 6) - (2x - 2) \neq 0$$

$$x^3 - 6x^2 + 11x - 6 \neq 2x - 2$$

Based on our assumption, we reached the conclusion stated in the contrapositive, which proves that the contrapositive statement is true, which also means that it's logically equivalent statement, the original conditional statement must be true, so the roots must be 1 and 4.

2

Two-Column

Prove: The function  $x^3 - 6x^2 + 11x - 6 = 2x - 2$   
where  $x$  is a real number, has the solutions  $x = 1$  or  $4$

Statement	Reason
1. $x^3 - 6x^2 + 11x - 6 = 2x - 2$	Given
2. $x^3 - 6x^2 + 9x - 4 = 0$	Addition Property
3. $x^3 - (1 + 5)x^2 + (5 + 4)x - 4 = 0$	Addition
4. $x^3 - x^2 - 5x^2 + 5x + 4x - 4 = 0$	Distributive Property
5. $x^2(x - 1) - 5x(x - 1) + 4(x - 1) = 0$	Factor (Distributive Property)
6. $(x^2 - 5x + 4)(x - 1) = 0$	Distributive Property
7. $[(x - 4)(x - 1)](x - 1) = 0$	Factor
8. $x = 1 \text{ or } x = 4$	QED

4

Elementary

Proposition: If  $x$  is a real number and  $x^3 - 6x^2 + 11x - 6 = 2x - 2$  then  $x = 1$  or  $x = 4$

$$x^3 - 6x^2 + 11x - 6 = 2x - 2$$

$$x^3 - 6x^2 + 9x - 4 = 0$$

$$(x^3 - 6x^2 + 5x) + (4x - 4) = 0$$

$$(x^2 - 5x)(x - 1) + 4(x - 1) = 0$$

$$(x^2 - 5x + 4)(x - 1) = 0$$

$$[(x - 4)(x - 1)](x - 1) = 0$$

Thus  $x = 1$  or  $x = 4$

# 5 Puzzle

Suppose that among four consecutive integers, the product of the first three equals twice the third.

What's the fourth integer?

Let the 4<sup>th</sup> integer =  $x$ .

Then the 1<sup>st</sup> integer =  $x - 3$ , the 2<sup>nd</sup> integer =  $x - 2$ , and the 3<sup>rd</sup> integer =  $x - 1$

$$(x - 3)(x - 2)(x - 1) = 2(x - 1)$$

$$(x - 3)(x - 2)(x - 1) - 2(x - 1) = 0$$

$$(x - 1)[(x - 3)(x - 2) - 2] = 0$$

$$(x - 1)(x^2 - 5x + 6 - 2) = 0$$

$$(x - 1)(x^2 - 5x + 4) = 0$$

$$(x - 1)(x - 1)(x - 4) = 0$$

$$x = 1 \text{ or } x = 4$$

If the 4<sup>th</sup> integer is 1, then the four consecutive integers are  $-2, -1, 0, 1$ , and  $(-2)(-1)(0) = 2(0)$ . ✓

If the 4<sup>th</sup> integer is 4, then the four consecutive integers are  $1, 2, 3, 4$ , and  $(1)(2)(3) = 2(3)$ . ✓

30

Formulaic

$$x^3 - 6x^2 + 11x - 6 = 2x - 2$$

$$x^3 - 6x^2 + 9x - 4 = 0$$

The root of a cubic polynomial of the form  $ax^3 + bx^2 + cx + d$  is given by Cardano's formula:

$$x = \frac{-b}{3a} + \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}}$$

$$+ \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}}$$

$$x = \frac{-(-6)}{3(1)} + \sqrt[3]{\left(\frac{-(-6)^3}{27(1)^3} + \frac{(-6)(9)}{6(1)^2} - \frac{(-4)}{2(1)}\right) + \sqrt{\left(\frac{-(-6)^3}{27(1)^3} + \frac{(-6)(9)}{6(1)^2} - \frac{(-4)}{2(1)}\right)^2 + \left(\frac{(9)}{3(1)} - \frac{(-6)^2}{9(1)^2}\right)^3}}$$

$$+ \sqrt[3]{\left(\frac{-(-6)^3}{27(1)^3} + \frac{(-6)(9)}{6(1)^2} - \frac{(-4)}{2(1)}\right) - \sqrt{\left(\frac{-(-6)^3}{27(1)^3} + \frac{(-6)(9)}{6(1)^2} - \frac{(-4)}{2(1)}\right)^2 + \left(\frac{(9)}{3(1)} - \frac{(-6)^2}{9(1)^2}\right)^3}}$$

$$= 2 + \sqrt[3]{1} + \sqrt[3]{1}$$

$$= 2 + 1 + 1$$

$$= 4$$

31

Counterexample

Cardano's formula returns the following equation:

$$x = 2 + \sqrt[3]{1} + \sqrt[3]{1}$$

However, 1 is not the only number whose cube equals 1. There are additional solutions resulting from different *complex* cube roots. The other two complex cube roots of 1 are

$$\frac{-1 + i\sqrt{3}}{2} \text{ and } \frac{-1 - i\sqrt{3}}{2}$$

Combining these two complex cube roots of unity yields a second solution

$$x = 2 + \sqrt[3]{1} + \sqrt[3]{1} = 2 + \frac{-1 + i\sqrt{3}}{2} + \frac{-1 - i\sqrt{3}}{2} = 1$$

And a third solution

$$x = 2 + \sqrt[3]{1} + \sqrt[3]{1} = 2 + \frac{-1 - i\sqrt{3}}{2} + \frac{-1 + i\sqrt{3}}{2} = 1$$

Here is a complete formula for each root  $x_n$  of the general cubic equation, where  $n = 0, 1, 2$ :

$$x_n = \frac{-b}{3a} + \left(\frac{-1 + i\sqrt{3}}{2}\right)^n \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} \\ + \left(\frac{-1 + i\sqrt{3}}{2}\right)^{2n} \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}}$$

23

Symmetry

**Theorem.** If  $x^3 - 6x^2 + 11x - 6 = 2x - 2$ , then  $x = 1$  or  $x = 4$ .

*Proof:*

$$x^3 - 6x^2 + 11x - 6 = 2x - 2$$

$$x^3 - 6x^2 + 9x - 4 = 0$$

The solutions of the equation are the roots of the cubic curve  $y = x^3 - 6x^2 + 9x - 4$ .

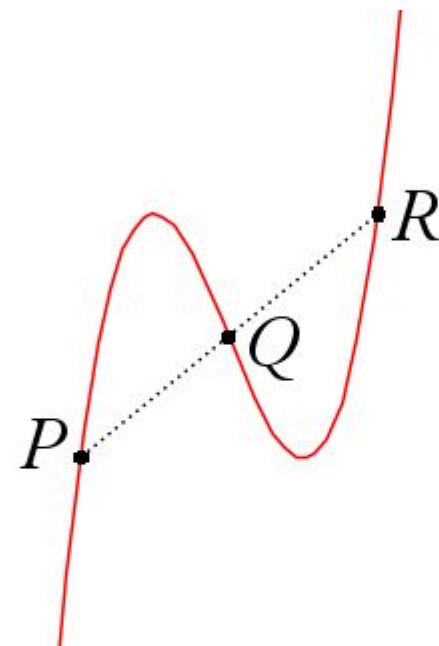
Every cubic curve is symmetric about its point of inflection  $Q$ , so if  $P$  is on the curve, so is its reflection  $R$  thru  $Q$ .

$$\frac{P + R}{2} = Q$$

$$P + R = 2Q$$

$$R = 2Q - P$$

A root  $P$  of the cubic curve may be realized as the reflection  $R = 2Q - P$ .



The point of inflection occurs where the second derivative is equal to zero.

$$y = x^3 - 6x^2 + 9x - 4$$

$$y' = 3x^2 - 12x + 9$$

$$y'' = 6x - 12$$

$$6x - 12 = 0$$

$$x = 2$$

$$y = (2)^3 - 6(2)^2 + 9(2) - 4 = -2$$

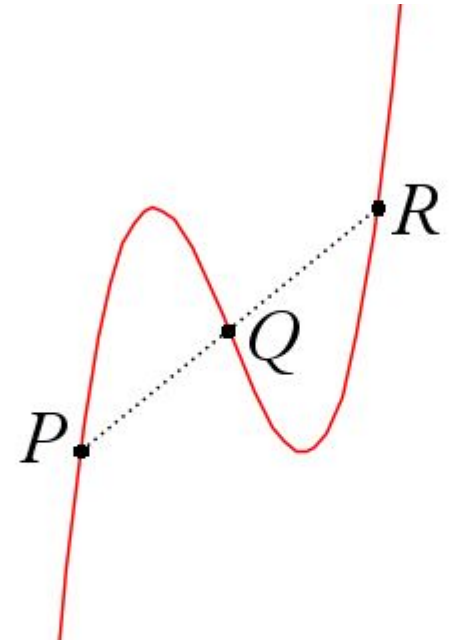
Therefore, the point of inflection is  $Q(2, -2)$ .

The root  $P(x, 0)$  reflected through  $Q(2, -2)$  yields  $R = 2Q - P$ .

$$R = 2(2, -2) - (x, 0)$$

$$R = (4, -4) - (x, 0)$$

$$R = (4 - x, -4)$$



$$R = (4 - x, -4)$$

$$x^3 - 6x^2 + 9x - 4 = y$$

$$(4 - x)^3 - 6(4 - x)^2 + 9(4 - x) - 4 = -4$$

Make the substitution  $4 - x = r$  for the  $x$ -coordinate of  $R$  in the cubic equation:

$$r^3 - 6r^2 + 9r - 4 = -4$$

$$r^3 - 6r^2 + 9r = 0$$

$$r(r^2 - 6r + 9) = 0$$

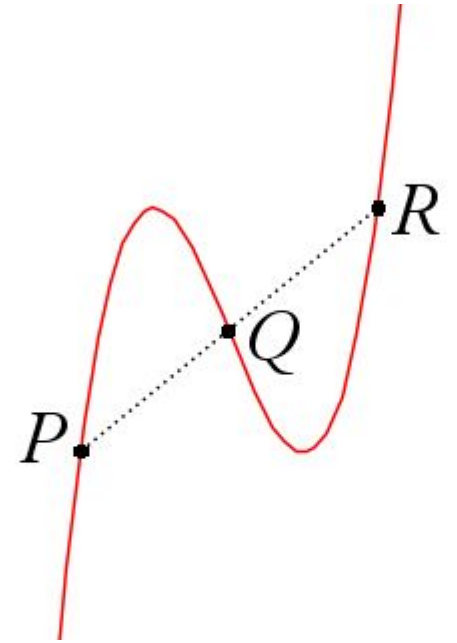
$$r(r - 3)^2 = 0$$

$$r = 0 \text{ or } r = 3$$

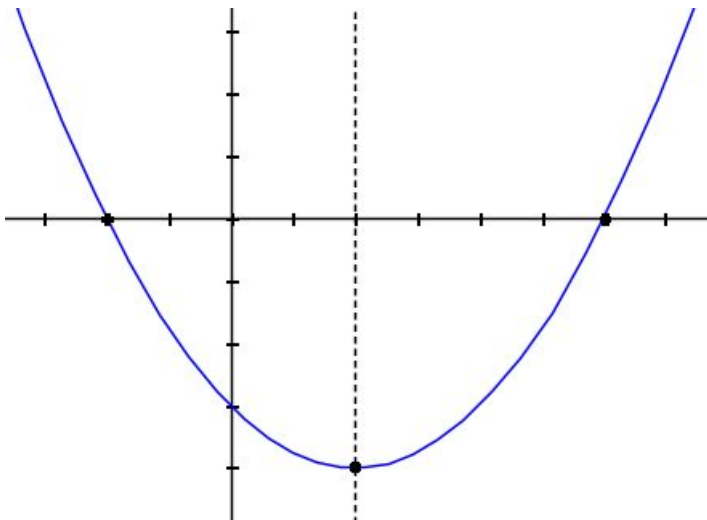
But  $4 - x = r$                        $4 - x = 0 \text{ or } 4 - x = 3$

$$x = 4 \text{ or } x = 1$$

So by *symmetry* the roots of the cubic curve are  $P(4,0)$  or  $P(1,0)$

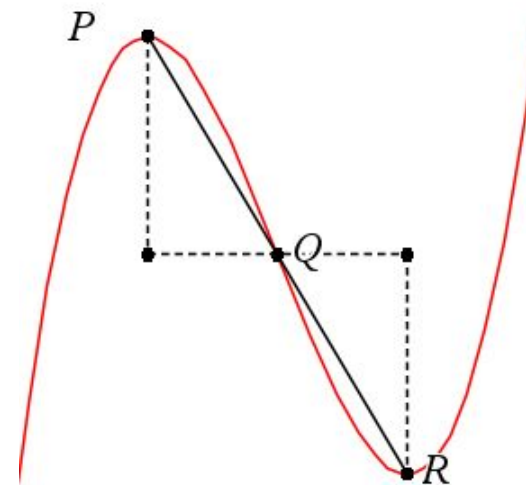


$$\begin{aligned}
 y &= ax^2 + bx + c \\
 y' &= 2ax + b \\
 2ax + b &= 0 \\
 x &= \frac{-b}{2a} \\
 ax^2 + bx + c &= 0 \\
 x &= \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}
 \end{aligned}$$



The line of symmetry of a parabola goes through the vertex, and the zeros are equally spaced on either side of the axis of symmetry.

$$\begin{aligned}
 y &= ax^3 + bx^2 + cx + d \\
 y' &= 3ax^2 + 2bx + c \\
 y'' &= 6ax + 2b \\
 6ax + 2b &= 0 \\
 x &= \frac{-b}{3a} \\
 3ax^2 + 2bx + c &= 0 \\
 x &= \frac{-b}{3a} \pm \frac{\sqrt{b^2 - 3ac}}{3a}
 \end{aligned}$$



The minimum and maximum values of a cubic function are equally spaced on either side of the point of inflection.

63

# Rational Zeros Theorem

# Rational Zeros Theorem

## Demonstration

# Slide Rule

Solve  $x^3 - 6x^2 + 11x - 6 = 2x - 2$

Write in standard form:  $x^3 - 6x^2 + 9x - 4 = 0$

Substitute  $y + 2$  for  $x$  to get

$$(y + 2)^3 - 6(y + 2)^2 + 9(y + 2) - 4 = 0$$

$$y^3 + 6y^2 + 12y + 8 - 6y^2 - 24y - 24 + 9y + 18 - 4 = 0$$

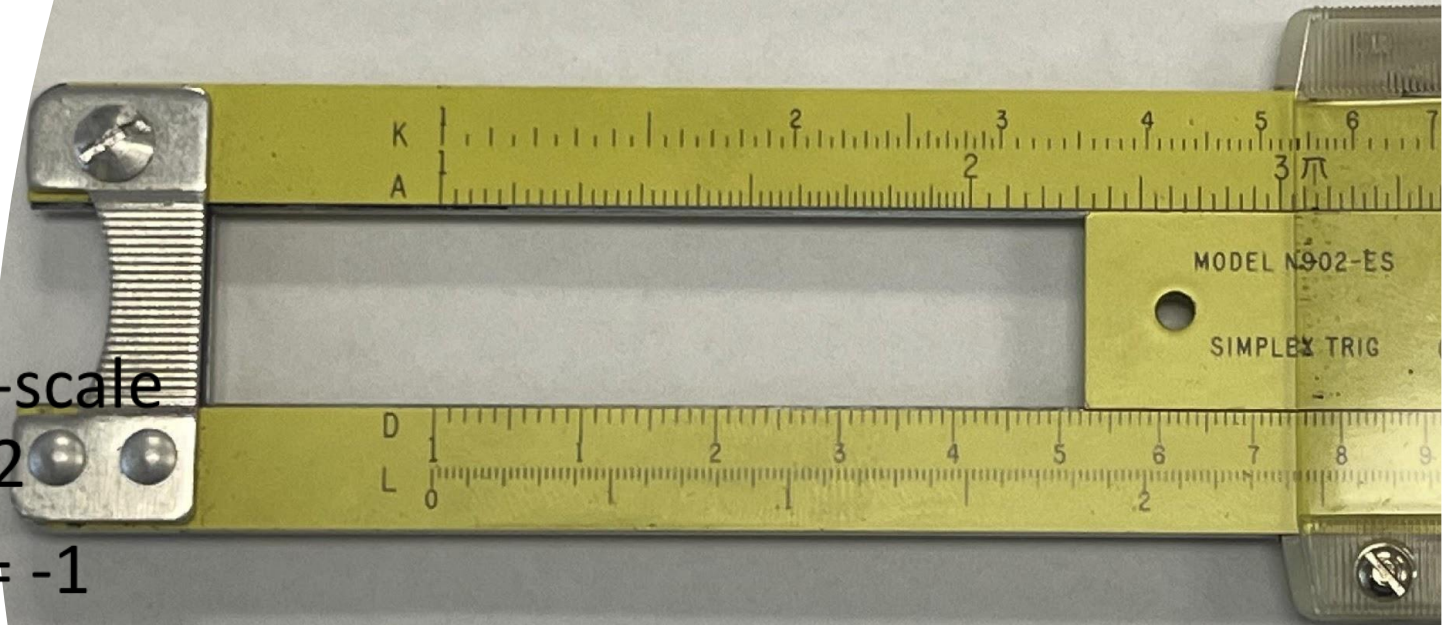
$$y^3 - 3y - 2 = 0$$

# Slide Rule

$$y^3 - 3y - 2 = 0$$

For roots  $y_1, y_2$ , and  $y_3$

- The sum of the roots is 0
- The product of the roots is 2
- Difference of the A-scale and C-scale of 3 yields D-scale of 2, so  $y_1 = 2$
- Then  $y_2 y_3 = 1$ , so  $y_2 = -1$  and  $y_3 = -1$



15

Matrices

# Matrices

We wish to show that if  $\lambda$  is an eigenvalue of the linear operator of the matrix A

$$\begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix} \quad \text{then } \lambda=1 \text{ or } \lambda=4 .$$

We use the matrix since the Characteristic Polynomial can be found on the Nspire as

$$\text{charPoly}\left(\begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}, x\right) = -x^3 + 6 \cdot x^2 - 9 \cdot x + 4$$

# Matrices (continued)

- A number  $\lambda$  is an eigenvalue of the matrix  $\mathbf{A}$  if there is a non-zero vector  $\mathbf{x}$  such that  $\mathbf{Ax} = \lambda\mathbf{x}$
  - This implies the homogeneous set of simultaneous equations  $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$ , which have a nontrivial solution when the determinant  $(\mathbf{A} - \lambda\mathbf{I})$  vanishes.
  - Expanding the determinant of  $(\mathbf{A} - \lambda\mathbf{I})$  gives:
  - The roots of this equation are  $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 4$
- $$\begin{aligned} 0 &= \begin{vmatrix} 3-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 2 & 0 & 2-\lambda \end{vmatrix} \\ &= (3-\lambda)(1-\lambda)(2-\lambda) - 2(1-\lambda) \\ &= (1-\lambda)^2(4-\lambda) \end{aligned}$$

# Matrices (continued)

- The eigenvalues can also be shown on the Nspire calculator:

$$\text{eigVl}\left(\begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}\right)$$

$$\{4., 1., 1.\}$$

33

Calculus

Define  $\mathfrak{R} \rightarrow \mathfrak{R}$  to be the function  $f(x) = x^3 - 6x^2 + 9x - 4$ .

If  $f(x) = 0$ , then  $x = 1$  or  $x = 4$ .

Construct the Taylor series of  $f$  about the first of the two purported roots,  $x = 1$ .

Derivatives of  $f$ :

$$f'(x) = 3x^2 - 12x + 9$$

$$f''(x) = 6x - 12$$

$$f'''(x) = 6$$

$$f^{(n)} = 0 \text{ for } n \geq 4$$

$$\begin{aligned}
 \bullet f(x) &= f(1) + \frac{f'(1)}{1!} (x-1) + \frac{f''(1)}{2!} (x-1)^2 + \frac{f'''(1)}{3!} (x-1)^3 + \dots \\
 &= 0 + 0 - 3(x-1)^2 + (x-1)^3 + 0 \\
 &= (x-1)^2(-3 + x - 1) \\
 &= (x-1)^2(x-4)
 \end{aligned}$$

Since  $f(x) = 0$ , the roots of  $f$  are 1 and 4.

60

Geometric

**Theorem.** If  $x^3 - 6x^2 + 11x - 6 = 2x - 2$ , then  $x = 1$  or  $x = 4$ .

*Proof:*

$$x^3 - 6x^2 + 9x - 4 = 0$$

Since  $\triangle PBQ \sim \triangle OAP$

$$\frac{QB}{6-x} = \frac{x}{1}$$

$$QB = x(6-x)$$

Since  $\triangle OAP \sim \triangle QCR$

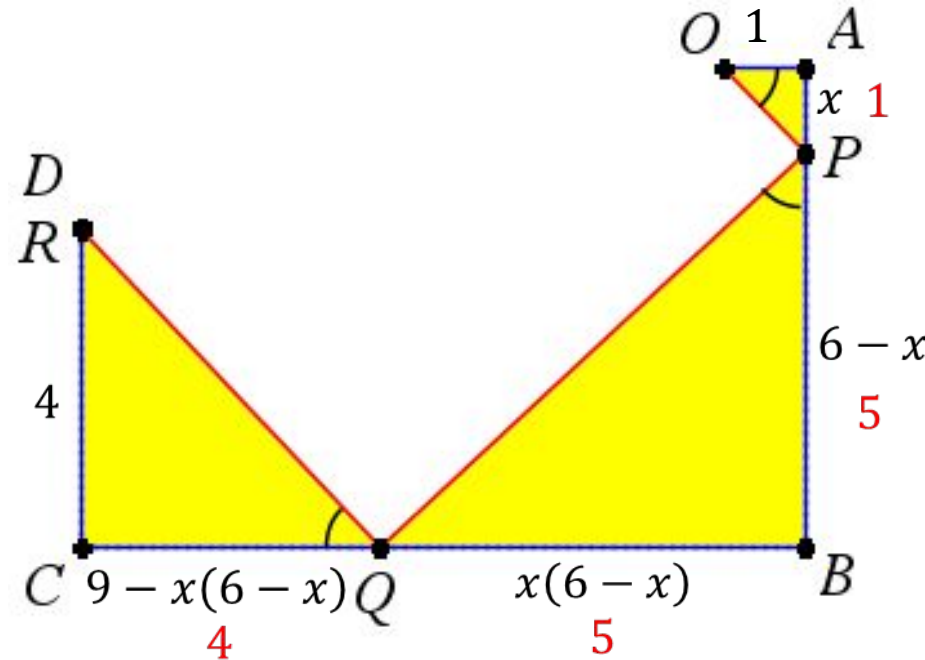
$$\frac{x}{1} = \frac{4}{9-x(6-x)}$$

$$x(9-x(6-x)) = 4$$

$$-4 + x(9-x(6-x)) = 0$$

Let

$$p(x) = -4 + x(9-x(6-x))$$



$$p(1) = -4 + 1(9 - 1(6 - 1))$$

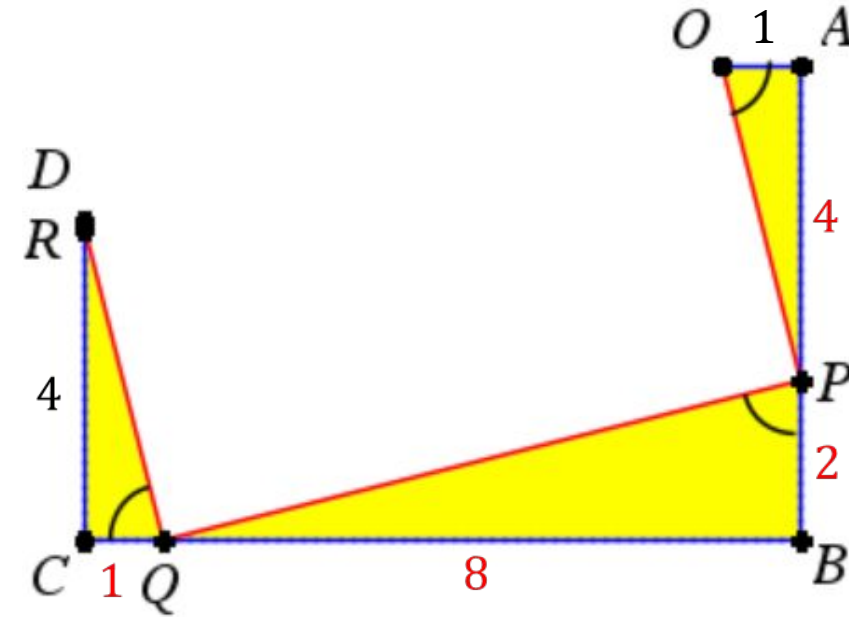
$$p(1) = -4 + 1(9 - 1(5))$$

$$p(1) = -4 + 1(9 - 5)$$

$$p(1) = -4 + 1(4)$$

$$p(1) = -4 + 4$$

$$p(1) = 0$$



$$p(4) = -4 + 4(9 - 4(6 - 4))$$

$$p(4) = -4 + 4(9 - 4(2))$$

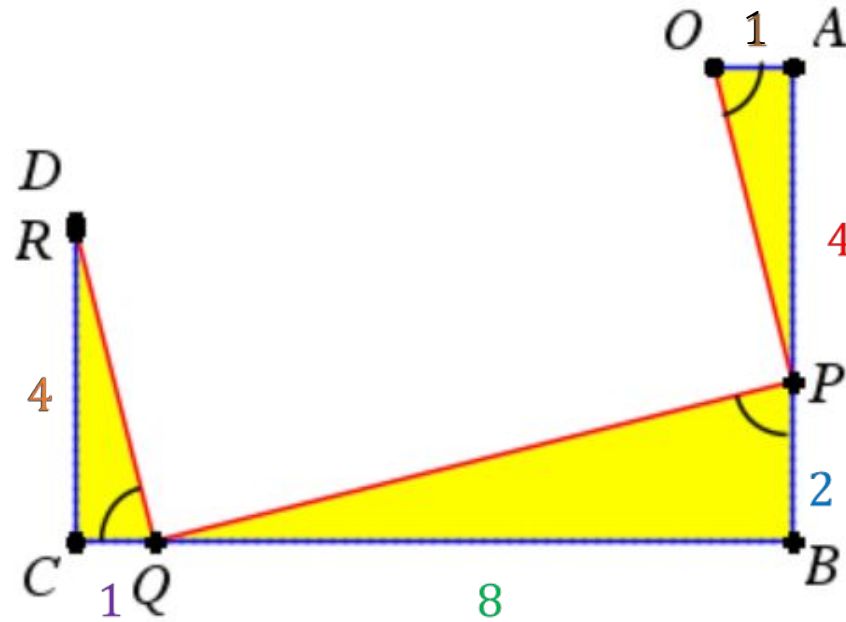
$$p(4) = -4 + 4(9 - 8)$$

$$p(4) = -4 + 4(1)$$

$$p(4) = -4 + 4$$

$$p(4) = 0$$

# Comments on Lill's Method



$$\begin{aligned}
 p(x) &= -4 + x(9 - x(6 - x)) \\
 p(x) &= -4 + x(9 - x(-x + 6)) \\
 p(x) &= -4 + x(9 + x(x - 6)) \\
 \boxed{p(x) &= ((x - 6)x + 9)x - 4}
 \end{aligned}$$

Horner's Form

Horner's Method

$$\begin{aligned}
 p(x) &= ((1x - 6)x + 9)x - 4 \\
 p(4) &= ((1 \cdot 4 - 6)4 + 9)4 - 4 \\
 p(4) &= ((4 - 6)4 + 9)4 - 4 \\
 p(4) &= ((-2)4 + 9)4 - 4 \\
 p(4) &= (-8 + 9)4 - 4 \\
 p(4) &= (1)4 - 4 \\
 p(4) &= 4 - 4 \\
 p(4) &= 0
 \end{aligned}$$

Synthetic Division

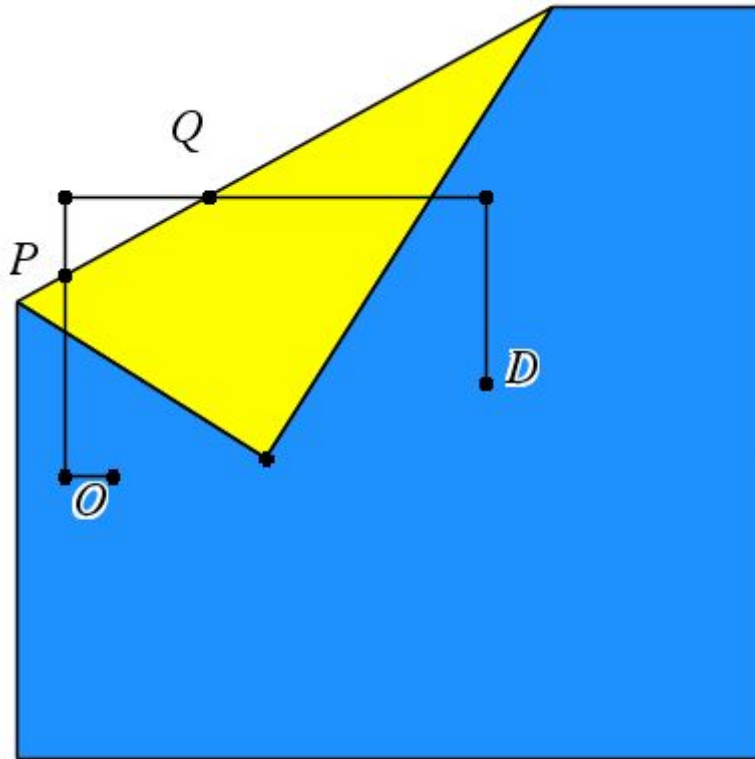
$$\begin{array}{r|rrrr}
 x^3 - 6x^2 + 9x - 4 & & & & \\
 \hline
 x - 4 & & & & 
 \end{array}$$

$$\begin{array}{r|rrrr}
 4 & 1 & -6 & 9 & -4 \\
 & & 4 & -8 & 4 \\
 \hline
 & 1 & -2 & 1 & \boxed{0}
 \end{array}$$

39

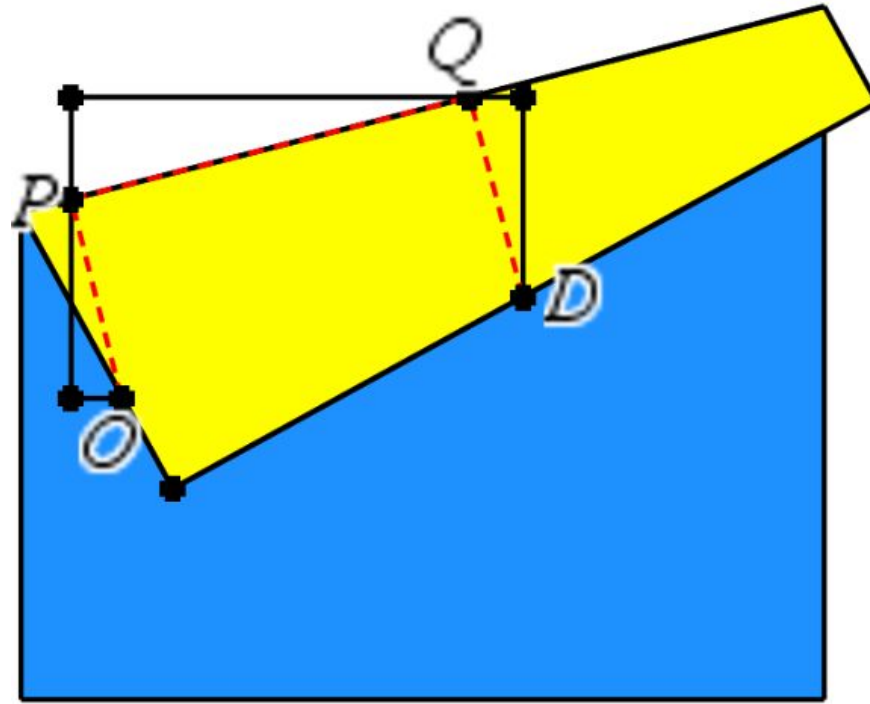
Origami

# Beloch Fold



A single fold which places two distinct points onto two distinct lines simultaneously (or vice-versa).

### $\overleftrightarrow{PQ}$ is a Beloch Fold



$\overleftrightarrow{PQ}$  creates a right-angled path from  $O$  to  $D$ .

94

Authority

# Authority

If  $x^3 - 6x^2 + 11x - 6 = 2x - 2$ , then it follows from Euler that the real number in question must be 1 or 4.

How do we recognize mathematical truth? If a theorem has a short complete proof, we can check it. But if the proof is deep, difficult, and already fills 100 journal pages, if no one has the time and energy to fill in the details, if a “complete” proof would be 100,000 pages long, then we rely on the judgements of the bosses in the field. In mathematics, a theorem is true, or it is not a theorem. But even in mathematics, truth can be political. (from *Notices of the AMS*, Melvyn Nathanson.)

When the authority of reference is oneself, a proof by authority becomes a proof by *intimidation*.