Guitar Fret Measurement Investigation: Teacher Notes

(Suggested time: 1 to 2 class periods)

Overview

This lesson involves measuring guitar fret distances and discovering which family of functions would be most appropriate for analyzing the data set.

Data Collection

Bring sample guitars, ukuleles, or other stringed fretted instruments to class or use pictures of guitars for students to use to collect the measurements from the bridge to the nut, then the bridge to each fret. Use the data collection spreadsheet to collect the data.

Using the Graphing Calculator as a Teaching Tool

- Enter the fret number into L1, and the fret lengths in L2.
- Ask the students what to do next (rather than immediately graphing a scatter plot, wait for a student to suggest it).
  - Turn off all plots and clear out any equations in Y_1, Y_2, etc, so that they don't interfere with the scatter plot. Then, set up Plot 1 as a scatter plot using Lists L1 and L2.
  - "ZoomStat/Window" graph-fitting strategy: This approach can be used to fit a scatter plot to an appropriate first-quadrant window, with both axes visible. Use ZoomStat to automatically fit the scatter plot to the graph screen, then use Window to adjust the settings in order to focus on the first quadrant and see both the *x*- and *y*-axes. For example, settings in the range of X:[-1,25] by 1's and Y:[-5,30] by 5's should allow you to analyze an acoustic or electric guitar measured in inches. Adjust accordingly when measuring in centimeters, or if using data for a bass guitar, or ukulele.
- Ask: "Does the data appear to follow a pattern?" (Wait for students to respond …) At first glance, it may appear to be approximately linear. We (teachers) know that it is not linear, but the students may assume it is. The linear analysis that follows is precisely intended to be a part of the discovery process of the lesson. So, ask the students what to do (linear regression using L1 and L2, and storing the equation in Y_1). Suggestion: choose the LinReg(a+bx) option rather than the traditional slope-intercept form (see page 4 for a detailed explanation for this choice), and make sure the stat diagnostics are turned on (found under the [mode] key). After doing the regression, note that both R and R^2 will be extremely close to 1, suggesting that a linear regression fits the data extraordinarily well. Press graph to confirm.
- Pause and Discuss: Is a linear model *really* appropriate here? Why or why not? Positive: Since the regression has such a high correlation, it can certainly be used to make reasonable estimates *within* the range of data (interpolation). However, ask the students if they see any issues with the line *outside* of the data set (extrapolation). Pause to let them think about the graph. After they have had a chance to share their thinking, be sure that they note how the line extends below the *x*-axis (you cannot have a negative fret length!). Also, note how the line helps us see that the data set seems to follow a "curved" shape (the point on the far left is above the line, then the next few points "curve" below the line, and the far right point is back above the line). Basically, the line "exposes" the curvature of the points, and it appears that the data set is not

linear after all, even though R and R^2 are both so high! **R and R^2 are valuable and important, but they are not the whole story**. So, what kind of data set is it? Let's continue.

- **Teaching moment:** Since the *x*-values are increasing by 1's, we can use L3 to determine whether the data is linear or not by calculating first differences of the fret lengths using the command L3=ΔList(L2). The ΔList command is found in the List operations under [2nd], [Stat], OPS, 7:ΔList. Observe that the **first differences are <u>not</u> constant**. This violates the change characteristic of linear functions (**constant rate**), so the data is not linear. So what path *does* the data follow?
    - **Review/Reinforcement:** If necessary, explore this concept further by creating a table of values for $\boldsymbol{y = 2x}$, for $x = 0, 1, 2, 3, 4$. Note how all of the first differences are constant. Note also that the pedagogy of exploring first differences depends on the *x*-values changing at a constant rate – best seen when the changing by 1's.
    - **Statistics Connection (optional):** The non-linearity of the data can also be confirmed by viewing a **residual plot** of the data. The **residuals** are the differences between each actual *y*-value of a data point and its predicted value from the regression model. During all regression calculations, the residuals are calculated automatically and stored in a list named RESID. A residual plot is a graph that shows the residuals on the vertical axis and the independent variable on the horizontal axis. If the points in a residual plot are randomly dispersed around the horizontal axis, the proposed regression model that generated the residuals is appropriate for analyzing the data; otherwise, a different model is more appropriate. **Key concept**: **Appropriate regression models generate random residual plots**. Inappropriate regression models contain some kind of pattern in their residual plots (indicating that the data is "misaligning" with the model according to some systematic fluctuation pattern). In this case, the residual plot for the linear model shows a clear pattern (curved, not randomly dispersed), which indicates that a linear model is not appropriate, and we must find a different model.
- **Teaching moment:** Since we know that the data is not linear, what non-linear models might we consider? First, could the curve be quadratic? **In quadratic functions the 2nd differences are constant.** Review/reinforce by creating a second table of values, this time for $\boldsymbol{y = x^2}$, for $x = 0, 1, 2, 3, 4$. Demonstrate how the *second* differences (i.e., the differences of the first differences) are constant. So, we should calculate the 2nd differences of the data set to determine whether or not it follows an approximately quadratic path. Using L4, i.e., L4=ΔList(L3), observe that the 2nd differences are not constant, so the data is not quadratic, either. So, what path *does* it follow?
- **Teaching moment**: This data set must be changing by a different kind of change. The fret lengths seem to be decreasing somewhat steadily, but what is math behind the pattern? **We need to explore another kind of change**. Introduce exponential change by using a table of values for $\boldsymbol{y = 2^x}$, for $x = 0, 1, 2, 3, 4$ including the 1st and 2nd differences. Note how the both the 1st and 2nd differences are not constant, but rather, the exponential pattern keeps getting reproduced in the differences! In exponential functions the term-by-term change is not related to addition or subtraction. Go back to the table for $y = 2^x$. Observe that as you *divide backwards* term by term, the quotient (or ratio) is constant. Summary: In exponential functions terms change by a **constant ratio** (called the **common ratio**), whereas in linear functions terms change

by a **constant rate** (constant first differences), and in quadratic functions terms change by constant second differences (cubic, third differences, etc.).

- **Calculus connection (optional)**: This activity is quite rich on many levels. It also connects to calculus in some very powerful ways. As we have seen, in linear functions, first differences are constant (hence, $\frac{\Delta y}{\Delta x}$ is constant). This helps students understand why the first derivative of a linear term is constant. Similarly, since the second differences of a quadratic function are constant, students can understanding why the second derivative of a quadratic term is constant. Finally, recall that prior to investigating consecutive *ratios* for $y = 2^x$, we saw in passing that the exponential pattern was reproduced in the *differences* (a similarly interesting change occurs with $y = e^x$). Recall that the derivative is defined by *differences* (i.e., $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$), and in exponential functions, the differences keep reproducing an exponential pattern! This helps students understand why the derivative of e^x is e^x. Likewise, the derivative of 2^x is $\ln(2) \cdot 2^x$ (also exponential). Very cool!

- **Back to the Calculator:** So, how do we use the calculator to explore the concept of consecutive ratios? Look back again at the data set (i.e., [stat], [enter]). Unfortunately, there is no consecutive ratio command similar to ΔList, but we can still explore consecutive ratios in the following way. Clear out L3 and L4. We will only use L3, but we don't need the unnecessary "clutter" left in L4. Move to the first cell in L3, and do the actual calculation right in the cell (make sure that students divide "backwards"). Just type the division and press [enter]. Note that this is not a spreadsheet – you do not need to first put an equal sign as you would in Excel. Move to the next cell and calculate the next ratio, which should be very close to the first ratio. It is not exact, because this is real life data, but it is very close. Continue to find the remaining ratios.

> **Tech Tip:** The number of ratios will be one less than the number of frets. Ex: if your data includes 20 frets, students will have to do 19 ratio calculations. This can be done with the sequence command as follows: L3=seq(L2(N+1)/L2(N),N,1,19).

All of the ratios are very close in value, which indicates that that **the data is changing by a nearly constant ratio**. Therefore, an exponential model can be used to analyze the data. So how do we do that (without just doing an exponential regression)?

- **Teaching moment:** Given that slope-intercept form is deeply embedded throughout mathematics, recall from page 2 that we used the $y = a + bx$ form of a linear regression, or "intercept-slope" form. Of course, either form is correct, but there are two considerations for using the alternate form in this lesson (practical and pedagogical). First, in practical terms, when solving real-world problems, many students find it more intuitive to think in terms of "start value plus change" rather than in terms of slope first, then *y*-intercept. For example, in the classic T-shirt problem, if there is a one-time set up fee of $200 and a cost of $5 per shirt, a cost function of $y = 200 + 5x$ is more natural than $y = 5x + 200$ for many students, because we are "starting" with a set-up fee and then repeatedly adding the cost of each shirt. The second pedagogical point to consider is that the "intercept-slope" form provides a cohesive connection

between linear and exponential functions in that the similarity of the equation forms allows for a seamless transition from linear to exponential. The models transition from "start value *plus* change" to "start value *times* change." The equation form transitions from $y = \mathbf{a} + \mathbf{b}x$ to $y = \mathbf{a} * \mathbf{b}^x$, where **a** is the *y*-intercept, or "start value," in both cases, and **b** is the constant rate in linear models, and the constant rate in linear functions common ratio in exponential models. Since the current data set has a "start value" (string length of Fret 0, e.g., 25.5 for many guitars), and the terms change by an approximately common ratio (not *exactly* constant, but very close!), it is reasonable to use an exponential model to analyze the data. To complete the model, we need to find *one* value that we can use for our common ratio. From the Home Screen use the List operation [2nd], [stat], MATH, 3:mean to find mean(L3). For all guitars this value should be approximately 0.944. Thus, an example of a "manually" calculated exponential regression model for a 25.5 in Fret 0 guitar would be $\boldsymbol{y = 25.5 * 0.944^x}$.

- Store this equation in Y_2, then prepare to be amazed as you graph it!
- Compare and contrast this model to the linear model. Why is it more appropriate? Suggested discussion points: This model more accurately seems to "fit" the data; This model more accurately represents our known "start" value; This model follows a curved path, slowly dies down over time, and levels off to the right (not negative).
- For the final punch line, calculate the exponential regression, store it in Y_3, and then graph it. Prepare to be amazed again as you see how close the calculator's exponential regression is to the class model. Pause, reflect, celebrate, and discuss these findings with your students!

Concept Assessment and Closure/Wrap-up Suggestions

- Ask students to summarize the strengths and weaknesses of a linear model for this data set.
  - *Suggested answer:* High values for both R and R^2 indicate that a linear regression model is a close fit to the data set, but the data set is not linear as seen by the variability of the first differences. Also, the line eventually goes below the *x*-axis, but the fret lengths cannot possibly be negative. Finally, the line helps us see more clearly that the data seems to follow a "curved" path rather than a linear path.
- Ask students to write a convincing argument to explain why a linear model is not appropriate for this data set.
  - *Suggested answer:* The first differences of the *y*-values are not constant, so a linear model is not appropriate. Also, the line extends below the *x*-axis, which does not make sense for this situation. Or, there is a non-random pattern the residual plot of the linear regression (in this case it is "curved") indicating that a linear model is not appropriate.
- Ask students to summarize each of the "change" characteristics of linear, quadratic, and exponential functions.
  - *Suggested answer:* Linear functions change at a constant rate, and exponential functions change by a constant ratio. When *x*-values change by ones, *y*-values of linear functions change by constant first differences, *y*-values of quadratic functions change by constant second differences, *y*-values of exponential functions change by constant ratios.
- Ask students to compare and contrast the similarities and differences of the equation forms of linear and exponential functions using both symbolic representations and associated vocabulary.
  - *Suggested answer:* Linear functions can be modeled as "start value plus change." The equation form of a linear function is $y = a + bx$, where a is the *y*-intercept (or start value), and b is the constant rate of change. Exponential functions can be modeled as "start value times change." The equation form of an exponential function is $y = a * b^x$, where a is the *y*-intercept (or start value), and b is the common ratio.
- Ask students to summarize the calculator tasks they used during this lesson. Have each student explain and demonstrate one of those tasks to another student.
  - *Suggested calculator tasks:*
    - Managing data (e.g., clearing lists, entering data into lists, doing calculations within a cell in a list)
    - Graphing a scatter plot (and/or graphing a residual plot)
    - Using a "ZoomStat/Window" graph-fitting strategy
    - Calculating regression equations (linear and exponential), and storing regression equations in Y_1, Y_2, etc.
    - Turning stat diagnostics on and off
    - Using list calculations to determine first and second differences, e.g., L3=ΔList(L2)
    - Using the sequence command in a list calculation to determine consecutive ratios, e.g., L3= seq(L2(N+1)/L2(N),N,1,19).
    - Finding the mean of a list, e.g., mean(L3)