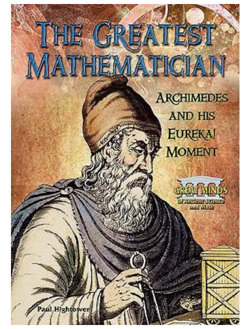
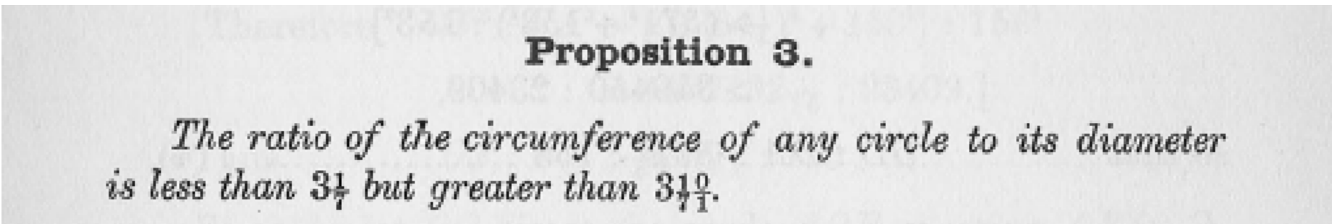
Archimedes' Approximation of Pi



A. Archimedes (c. 287 – c. 212 BC) was a Greek mathematician who used the *"Method of Exhaustion"* to approximate pi. This method involves a circle which is sandwiched between an inscribed polygon and a circumscribed polygon. By increasing the number of sides, the circumference of the circle is more closely approximated. Archimedes began with hexagons (6 sides), then doubled the number of sides to 12, then 24, then 48, then 96 (source: Wikipedia.org). His resulting value was stated in his treatise *"Measurement of a Circle"* and is shown below.

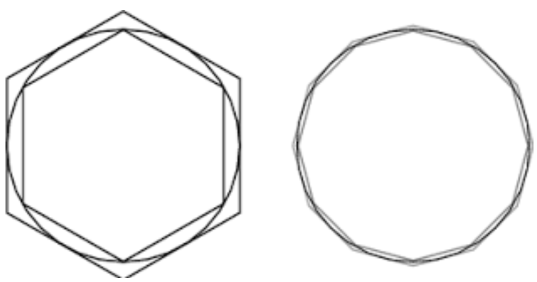


Proposition 3.

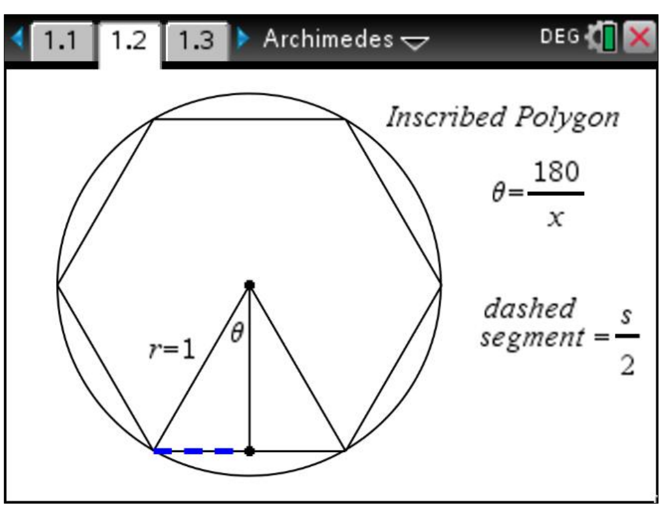
The ratio of the circumference of any circle to its diameter is less than $3\frac{1}{7}$ but greater than $3\frac{10}{71}$.

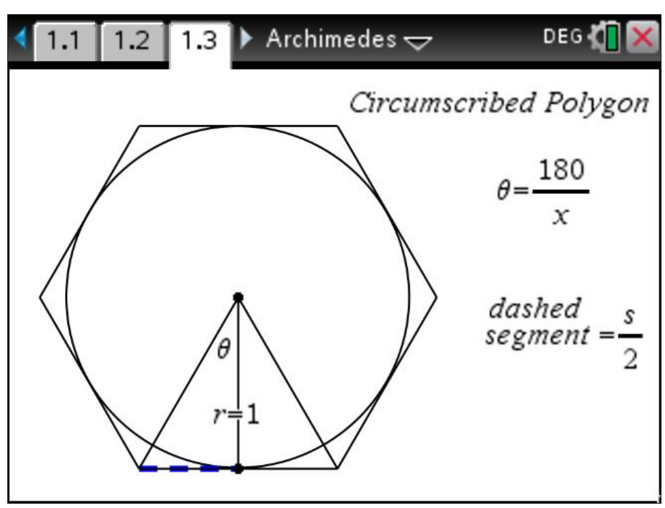
Archimedes worked with rational numbers (fractions), not decimals, and his method involved approximations of radicals as well. His lower and upper bounds for pi in decimal form are 3.14084507… and 3.14285714…

Here are diagrams of circles sandwiched between regular hexagons (6 sides) and dodecagons (12 sides). Observe how the 12-sided polygons are closer to the edges of the circle, thus providing improved approximations (source: mathworld.wolfram.com).



B. Consider these diagrams of inscribed and circumscribed polygons. The number of sides is x, the side length is s, and the half-central angle is θ. The radius r = 1.





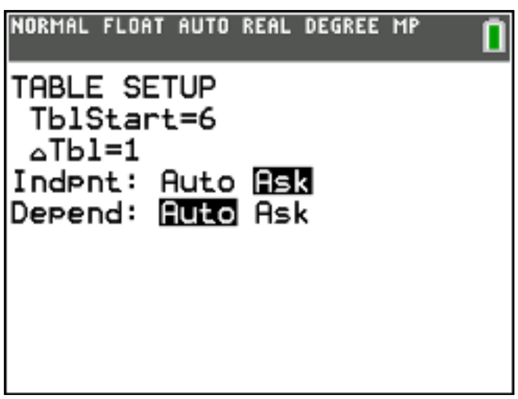
Do some calculations to convince yourself that $\theta = 180/x$ (Or if working in radians, that $\theta = \pi/x$).

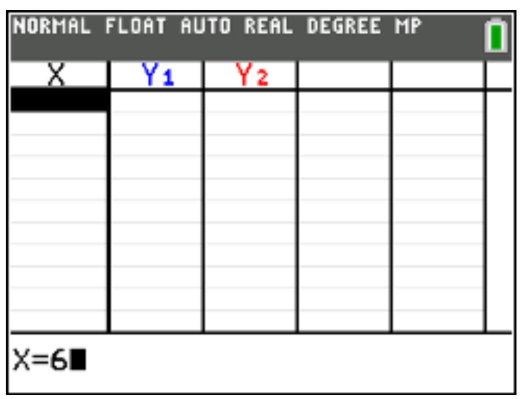
Then use basic trigonometry functions (sin or cos or tan) to find an expression for $s/2$ in terms of θ for each polygon. Then find s and the perimeter P of the regular polygon.

Inscribed Polygon	Circumscribed Polygon
_______ $(\theta) =$ $\frac{s}{2} =$ $s =$ $P =$	_______ $(\theta) =$ $\frac{s}{2} =$ $s =$ $P =$

C. We are using perimeters to approximate the circumference, and $C = 2\pi \bullet r = 2\pi$ since radius = 1.
Divide each perimeter formula by 2 to approximate π and enter these into Y1 and Y2 on a TI=84+ calculator.

Set **MODE** to degree or radian and **TABLE SETUP** as shown. Go to the **TABLE** and enter X = 6, then 12, then 24, then 48, then 96 to replicate Archimedes' perimeter measurements. Highlight the values in the columns for Y1 and Y2 to see how the approximation improves as the number of sides increases.





X	Y1 (Perimeter Inscribed Polygon)/2	Y2 (Perimeter Circumscribed Polygon)/2

B In a regular polygon, the central angle = 360°/x, where x = the number of sides (or $2\pi/x$ for radians). The angle marked θ on these diagrams is half of a central angle, so θ = $180/x$ or π/x .

Perimeter of a regular polygon P = (number of sides)•(side length) so $P = x \bullet s$.

For the inscribed polygon, use sin(θ) to relate x, $s/2$ and θ.
sin(θ) = $(s/2)/r$ therefore sin($180/x$) = $s/2$ and s = 2•sin($180/x$)

P = x•2•sin($180/x$) and dividing by 2 yields Y1 = x•sin($180/x$)

For the circumscribed polygon, use tan(θ) to relate x, $s/2$ and θ.
tan(θ) = $(s/2)/r$ therefore tan($180/x$) = $s/2$ and s = 2•tan($180/x$)

P = x•2•tan($180/x$) and dividing by 2 yields Y2 = x•tan($180/x$)

C The formulas are shown below in both degrees and radians. The results of the table are also shown, note the accuracy found for the lower and upper bounds using a 96-sided polygon.