TASK 26

How curvy are my functions?

Name: .... **Class:** .... **Date:** ....

Introduction

Let f be a twice-differentiable function. The curvature k of $y = f(x)$ is defined by $k(x) = \dfrac{|f''(x)|}{\left(1 + \left(f'(x)\right)^2\right)^{\frac{3}{2}}}$. For example, it can be shown that the curvature along any circle is constant, and is dependent on the radius of the circle. In this task you are asked to investigate the curvature k of a variety of functions, where the maximum curvature of f is denoted by k_{max} and the minimum curvature is denoted by k_{min}.

Explorations

Part 1: Curvature of linear and quadratic functions

Consider the general linear function $f(x) = mx + c$ where m and $c \in R$.

a State the curvature of f. Give a brief reason supporting your answer.

Consider $f(x) = x^2$.

b Find an expression for $k(x)$.

c Plot the graphs of $y = f(x)$ and $y = k(x)$ on the same set of axes, indicating the key features of each graph.
Hence determine the coordinates on the graph of f where k_{max} occurs.
State the value of k_{max}.

d Use calculus to verify the results obtained graphically in question **c**.

Consider the general quadratic function $f(x) = ax^2 + bx + c$ where a, b and $c \in R$ with $a \neq 0$.

e Find an expression for $k(x)$.

f Hence, for any quadratic function, determine the coordinates on the graph of f where k_{max} occurs, and determine the value of k_{max}.

g Find $\lim\limits_{x \to \pm\infty} k(x)$ and explain briefly the significance of the result.

h Describe briefly how you would change the maximum curvature k_{max} of an arbitrary quadratic function.

Part 2: Curvature of a trigonometric function

Consider $f(x) = \sin x$.

i Find an expression for $k(x)$.
Plot the graphs of $y = f(x)$ and $y = k(x)$ on the same set of axes for $0 \le x \le 2\pi$.
Determine the exact coordinates on the graph of f at which k_{max} and k_{min} occur, and hence state the value of k_{max} and of k_{min} respectively.

j Classify the points on the graph of f that have maximum and minimum curvature.

Part 3: Curvature of exponential and logarithmic functions

Consider $f(x) = e^x$ and $g(x) = \log_e x$.

k For each function, find an expression for $k(x)$ and hence determine the exact coordinates on the graphs of f and g where k_{max} occurs.
Find the exact value of k_{max} in each case.

l Plot the graphs of $y = f(x)$ and the corresponding $y = k(x)$ on one set of axes, and plot the graphs of $y = g(x)$ and the corresponding $y = k(x)$ together on another separate set of axes.
For each graph label the key features of $y = k(x)$.

m Explain why the functions f and g have the same maximum curvature, i.e. the same k_{max} value.

n Consider $f(x) = a^x$ for $a > 1$.
Use calculus to show that k_{max} occurs at

$$x = -\frac{\log_e\left(2(\log_e a)^2\right)}{2\log_e a}.$$

Hence find the exact value of k_{max}.

Part 4: Curvature of power functions

Consider the family of functions $f_n(x) = x^n$ for $n \ge 2$.

o **i** Plot the graphs of $y = f_3(x)$ and $y = k(x)$ on the same set of axes.

Use calculus to show that k_{max} occurs on the graph of f_3 at $x = \pm\left(\frac{1}{45}\right)^{\frac{1}{4}}$.

Determine the exact coordinates on the graph of f_3 where k_{max} and k_{min} occur. Explain why k_{min} occurs at this point.

ii Plot the graphs of $y = f_4(x)$ and $y = k(x)$ on the same set of axes.

Use calculus to show that k_{max} occurs on the graph of f_4 at $x = \pm\left(\frac{1}{56}\right)^{\frac{1}{6}}$.

Determine the exact coordinates on the graph of f_4 where k_{max} and k_{min} occur. Explain why k_{min} occurs at this point.

p Find an expression for $k(x)$.

Hence show that k_{max} occurs on the graph of f_n at $x = \pm\left(\dfrac{n-2}{2n^3 - n^2}\right)^{\frac{1}{2n-2}}$ for $n \geq 2$ and that k_{min} occurs at the origin for $n > 2$. Discuss k_{min} in the case of $n = 2$.

q Plot the graph of $x = \pm\left(\dfrac{n-2}{2n^3 - n^2}\right)^{\frac{1}{2n-2}}$ for $n \geq 2$.

Use the graph to describe where k_{max} occurs on the graph of f_n as the value of n varies.

Challenge

Part 1

Find the exact coordinates on the graph of f where k_{max} occurs and the exact value of k_{max} for the following families of functions.

- $f(x) = a\cosh\left(\dfrac{x}{a}\right)$ where a is a non-zero real number and $\cosh\left(\dfrac{x}{a}\right) = \dfrac{e^{\frac{x}{a}} + e^{-\frac{x}{a}}}{2}$.
- $f(x) = \dfrac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ (standard normal distribution probability density function)

Part 2

For a parametric curve given by $x = f(t)$ and $y = g(t)$, the curvature k is given by

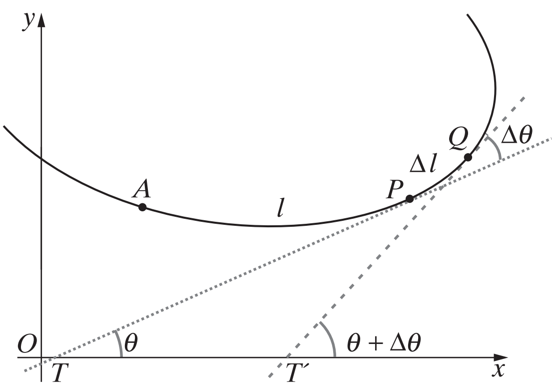
$$\frac{|f'(t)g''(t) - f''(t)g'(t)|}{\left[(f'(t))^2 + (g'(t))^2\right]^{\frac{3}{2}}}$$

Show that the curvature of the circle with radius r and centre $(0, 0)$ is constant and determine its value.

Investigate the maximum curvature k_{max} and the minimum curvature k_{min} of an ellipse with equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with $0 < a < b$ using parametric equations $f(t) = a\cos t$ and $g(t) = b\sin t$ for $0 \leq t \leq 2\pi$.

Teacher notes

- The curvature of a function $y = f(x)$ is defined as the rate of change of direction of a curve with respect to a change in its length. Curvature measures how fast a curve is changing direction at a given point. This amounts to finding the rate at which the tangent line is rotating as the point of tangency P travels along the curve.



- Let θ be the angle that the tangent at P makes with the x-axis and let $\theta + \Delta\theta$ be the angle made by the tangent at a neighbouring point Q. The total curvature of PQ is $\Delta\theta$. If the point P and its tangent move along the curve to Q, the total curvature $(\Delta\theta)$ measures the total change in direction of the tangent which is the total change in direction of the arc PQ.

  If Δl represents the length of the arc of the curve PQ, then the ratio $\dfrac{\Delta\theta}{\Delta l}$ measures the average change in direction per unit length of arc, i.e. the average curvature of the arc PQ. The curvature at a point is given by the limiting value of the average curvature as Q approaches P along the curve, i.e. the curvature at a point is $\lim\limits_{\Delta l \to 0}\left(\dfrac{\Delta\theta}{\Delta l}\right) = \dfrac{d\theta}{dl}$.

- To derive a formula for curvature, we start with $\tan\theta = \dfrac{dy}{dx}$, i.e. $\theta = \tan^{-1}\dfrac{dy}{dx}$.

$$\frac{d\theta}{dx} = \frac{\dfrac{d^2y}{dx^2}}{1 + \left(\dfrac{dy}{dx}\right)^2} \qquad \left(\text{if } \theta = \tan^{-1} f(x) \text{ then } \frac{d\theta}{dx} = \frac{f'(x)}{1 + \left(f(x)\right)^2}\right)$$

$$\frac{dl}{dx} = \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{2}} \qquad \left(\text{obtained from } l = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx\right)$$

Hence, $\dfrac{d\theta}{dl} = \dfrac{\dfrac{d^2y}{dx^2}}{\left(1 + \left(\dfrac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}$ (using $\dfrac{d\theta}{dl} = \dfrac{d\theta}{dx} \times \dfrac{dx}{dl}$)

- The curvature of a function $y = f(x)$ is defined by the formula $k(x) = \dfrac{|f''(x)|}{\left(1 + (f'(x))^2\right)^{\frac{3}{2}}}$. Encourage students to define or store their function as $f(x)$, for example, and then define the curvature function. With the curvature function defined, it can be accessed throughout the task.
- Students can use their CAS to answer question parts that contain the phrase 'use calculus'. When this phrase is specified, students should show a derivative function in their working.
- When using a CAS in question **o** part **i** and question **p**, students may find that the signum function appears as part of the derivative for k. If this is the case the signum function will need to be explained to students, i.e. signum (x), usually abbreviated to sign(x), is defined by
$$\text{sign}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$
- In question **o** part **i**, students may need to employ by-hand techniques to show that $\pm\dfrac{5^{\frac{3}{4}}\sqrt{3}}{15} = \pm\left(\dfrac{1}{45}\right)^{\frac{1}{4}}$ and in question **o** part **ii**, they may need to employ by-hand techniques to show that $\pm\dfrac{7^{\frac{5}{6}}\sqrt{2}}{14} = \pm\left(\dfrac{1}{56}\right)^{\frac{1}{6}}$.

  In both cases, students will need to apply their knowledge of index laws to demonstrate these results.
- In question **p**, students using a CAS may need to extract $(2n^3 - n^2)x^{2n} = (n-2)x^2$ from the output obtained from solving $k'(x) = 0$ for x. Students may need to obtain the result $x = \pm\left(\dfrac{n-2}{2n^3 - n^2}\right)^{\frac{1}{2n-2}}$ without their CAS due to potential auto-simplification effects.
- Teachers may choose to set students one of the two challenges.

Curriculum links

first and second derivatives
functions—circular
functions—exponential
functions—linear
functions—logarithmic
functions—power
functions—quadratic
index laws
limits
optimisation
parameters

Answers

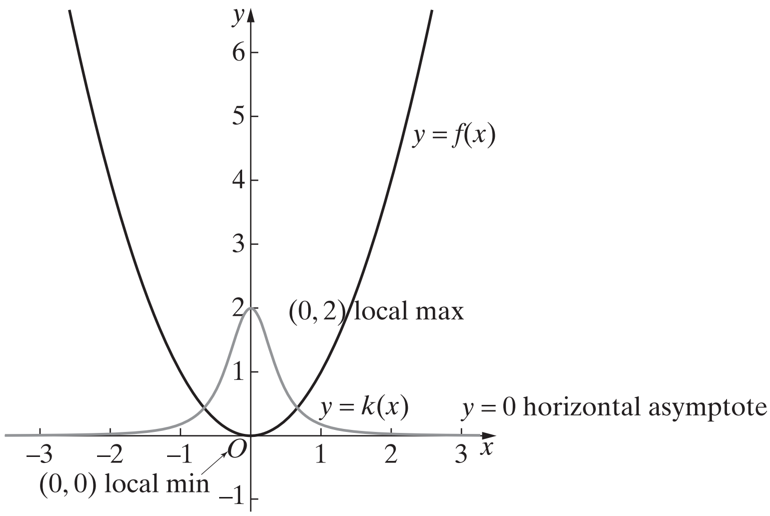
Explorations

a The curvature k is zero.

As a linear function never changes direction, the curvature of a linear function must be zero for all values of x.

b $\dfrac{2}{\left(1+4x^2\right)^{\frac{3}{2}}}$

c



The graph of $y = f(x)$ has a local minimum turning point at $(0, 0)$.

The graph of $y = k(x)$ has a local maximum turning point at $(0, 2)$ and a horizontal asymptote with equation $y = 0$.

k_{max} occurs at $(0, 0)$ where $k_{\text{max}} = 2$

d Refer to the Solutions.

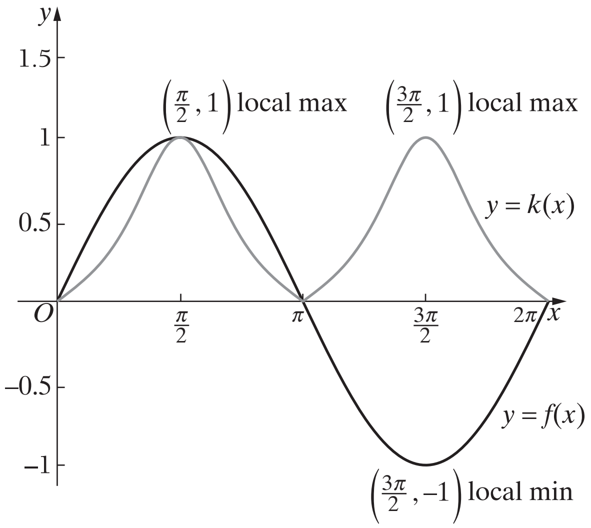
e $\dfrac{|2a|}{\left(1+(2ax+b)^2\right)^{\frac{3}{2}}}$ or $\dfrac{2|a|}{\left(4a^2x^2+4abx+b^2+1\right)^{\frac{3}{2}}}$

f k_{max} occurs at the turning point of the quadratic $\left(-\dfrac{b}{2a},\ c-\dfrac{b^2}{4a}\right)$ where $k_{\text{max}} = 2|a|$.

g $\lim\limits_{x\to\pm\infty} k(x) = 0$ and so the curvature of a quadratic approaches zero as $x \to \pm\infty$.

h To change the maximum curvature, you only need to change the value of the leading coefficient a. The parameters b and c have no effect on a quadratic function's maximum curvature.

i $\dfrac{|\sin x|}{\left(\cos^2 x + 1\right)^{\frac{3}{2}}}$



On the graph of $y = f(x)$, k_{max} occurs at $\left(\dfrac{\pi}{2}, 1\right)$ and $\left(\dfrac{3\pi}{2}, -1\right)$. At these points, $k_{max} = 1$.

On the graph of $y = f(x)$, k_{min} occurs at $(0, 0)$, $(\pi, 0)$ and $(2\pi, 0)$. At these points, $k_{min} = 0$.

j k_{max} occurs at the turning points of f, and k_{min} occurs at the non-stationary points of inflection of f.

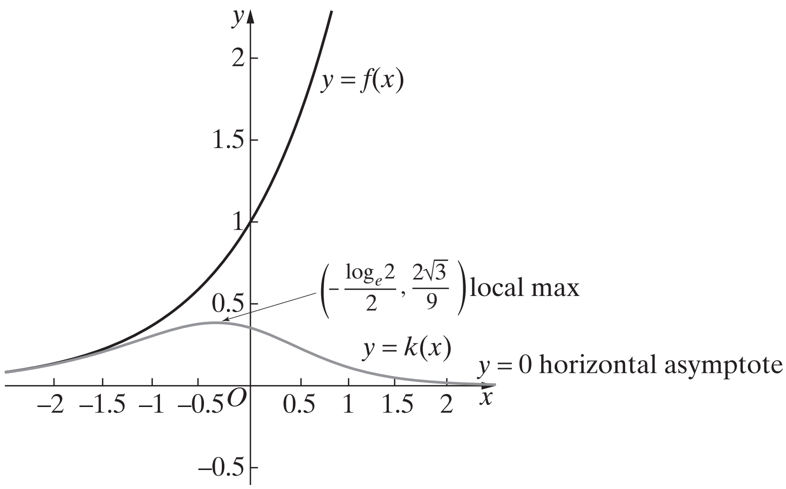
k For $f(x) = e^x$, $k(x) = \dfrac{e^x}{\left(e^{2x} + 1\right)^{\frac{3}{2}}}$ and $k'(x) = \dfrac{-e^x\left(2e^{2x} - 1\right)}{\left(e^{2x} + 1\right)^{\frac{5}{2}}}$

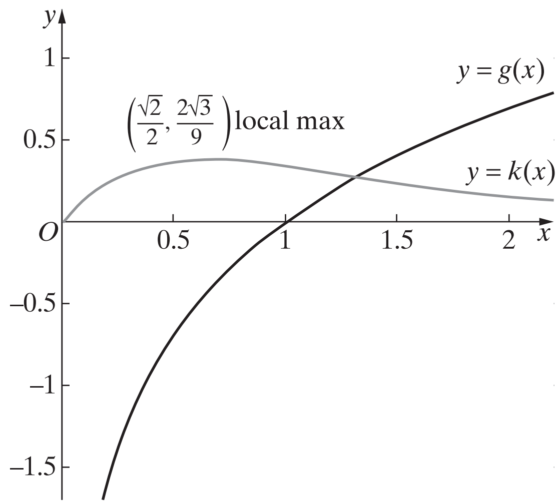
On the graph of $y = f(x)$, k_{max} occurs at $\left(-\dfrac{\log_e 2}{2}, \dfrac{\sqrt{2}}{2}\right)$. At this point, $k_{max} = \dfrac{2\sqrt{3}}{9}$.

For $g(x) = \log_e x$ with $x > 0$, $k(x) = \dfrac{x}{\left(x^2 + 1\right)^{\frac{3}{2}}}$ and $k'(x) = \dfrac{1 - 2x^2}{\left(x^2 + 1\right)^{\frac{5}{2}}}$.

On the graph of $y = g(x)$, k_{max} occurs at $\left(\dfrac{\sqrt{2}}{2}, -\dfrac{\log_e 2}{2}\right)$. At this point, $k_{max} = \dfrac{2\sqrt{3}}{9}$.

l

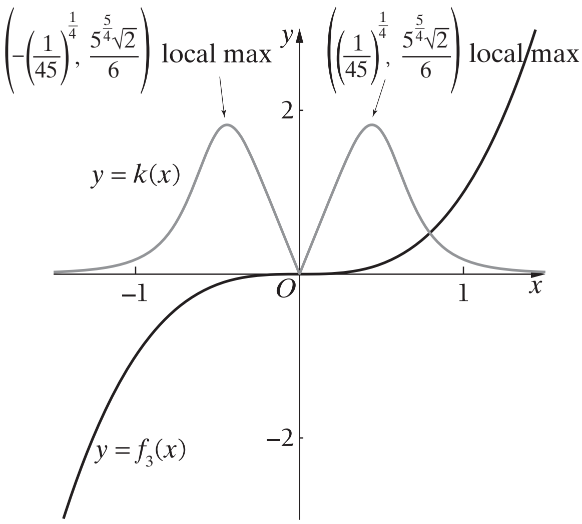




m Both functions have the same k_{max} value because they are inverse functions, i.e. they are reflections of each other about the line $y = x$.

n $k_{max} = \dfrac{2\sqrt{3}\log_e a}{9}$

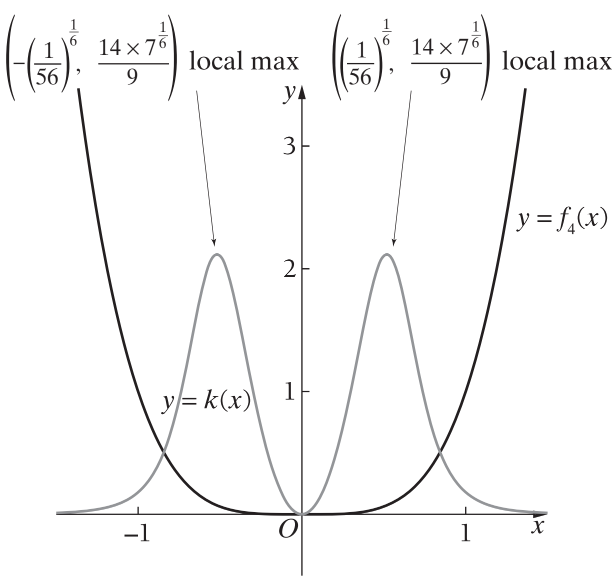
o **i** $k(x) = \dfrac{6|x|}{\left(9x^4 + 1\right)^{\frac{3}{2}}}$



On the graph of $y = f_3(x)$, $k_{\max}$ occurs at $\left(\pm\left(\dfrac{1}{45}\right)^{\frac{1}{4}}, \pm\left(\dfrac{1}{45}\right)^{\frac{3}{4}}\right)$, two points located symmetrically about O. At these points, $k_{\max} = \dfrac{5^{\frac{5}{4}}\sqrt{2}}{6}$.

$k_{\min} = 0$ and occurs at the origin (stationary point of inflection).

ii $k(x) = \dfrac{12x^2}{\left(16x^6 + 1\right)^{\frac{3}{2}}}$



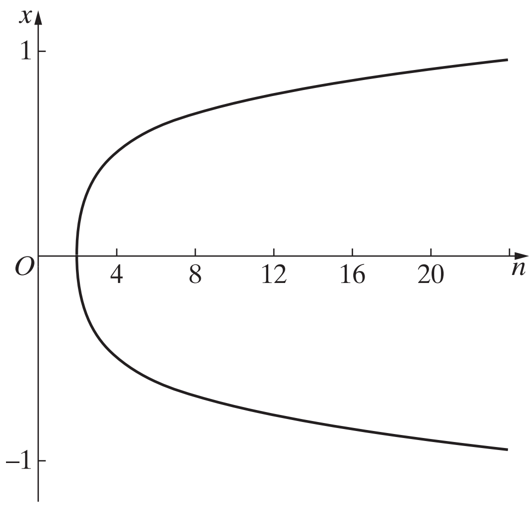
On the graph of $y = f_4(x)$, $k_{\max}$ occurs at $\left(\pm\left(\frac{1}{56}\right)^{\frac{1}{6}}, \left(\frac{1}{56}\right)^{\frac{2}{3}}\right)$. At these points, $k_{\max} = \frac{14 \times 7^{\frac{1}{6}}}{9}$.

$k_{\min} = 0$ and occurs at the origin. Also note that higher even-degree functions become increasingly flat near O. As $x \to \pm\infty$, the curvature of the graph of $y = f_4(x)$ approaches zero.

p $k(x) = \dfrac{|n(n-1)x^{n-2}|}{(n^2x^{2n-2}+1)^{\frac{3}{2}}}$ or equivalently $k(x) = \dfrac{|n(n-1)x^{n+1}|}{(n^2x^{2n}+x^2)^{\frac{3}{2}}}$

For $n = 2$, $\lim\limits_{x\to\pm\infty} k(x) = 0$.

q



As $\lim\limits_{n\to\infty} x \ \pm\left(\dfrac{n-2}{2n^3-n^2}\right)^{\frac{1}{2n-2}} = \pm1$, this shows that as n increases the two critical points at which $k_{\max}$ occurs approach ±1. For $n > 2$, $k_{\min}$ occurs at the origin.

Challenge

Refer to the Solutions.