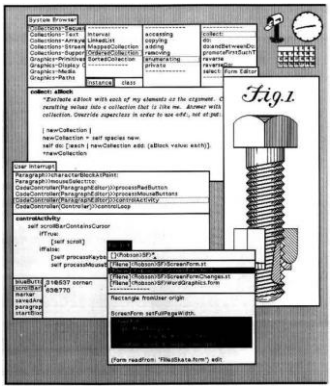
Code by Numbers



Is your computer running slow? Time to upgrade? Think again; computer upgrades are not what they used to be. Improvements in computing speed traditionally came from CPU and memory upgrades; most improvements now come from improved algorithms and software developments. Graphical user interfaces demand more power and memory. To understand what is happening, we need to step back a few decades.

In 1983 Apple® introduced LISA, the first *commercially* available computer with a graphical user interface (GUI), the 5Mb hard drive was an optional extra. The inspiration for Lisa came from something Xerox® had been using for almost a decade. The photocopying behemoth had created a computer called: "Alto" to help make their offices run more efficiently. The GUI that Xerox developed (1974) was called Smalltalk. Users had a three-button mouse, desktop clock, calendar, email, spreadsheet and a word-processer. It is fair to say Xerox was creating the paperless office, not exactly something a photocopying company would want widely adopted. Ironically, they also developed the laser printer, something that would become more pervasive through the use of personal computers.

Smalltalk was the first object-oriented language allowing programmers to use these objects within their programs without having to know *how* the object worked. In 1985 Microsoft® introduced Windows®. Applications in Windows ran on a platform called MS.DOS (Microsoft's Disk Operating System). Creating an application to work in Windows meant you could rely on MS.DOS to handle all the background computer management. Windows 1.0 required 256kB of memory and at least two floppy drives or 256kB hard-drive. The processor speed was approximately 5MHz and only 16bit. Fast-forward 35 years, Windows 11 requires a minimum 1 GHz twin core 64bit processor, 4GB RAM and 64GB storage. How did we get to the point? The simple answer is that processors and memory became cheaper allowing developers to add more and more layers to their code. From the 1980's to early 2000's, computer processing speeds increased almost exponentially. For the past 15 years, the same speed increases have not been possible.

This booklet aims to illustrate how mathematics and coding can be used to improve computational speeds without relying on hardware updates. The activities and investigations are also designed to improve understanding of number. The hierarchical nature of the mathematical content and structure of the coding require these activities be completed sequentially. Each activity includes coding instructions and references, visual and instructional support for mathematical content, reflective questions and a detailed investigation. The 10 minutes of coding references should be completed before commencing the corresponding activity. Coding instructions are included in each activity; however, it is also recommended that the code be modified to improve performance. The first activity "Factors that Count" involves writing a program to count the quantity of factors for any given number. The sample code provided is a 'brute force' approach; this code can be modified to achieve the same result much, much faster! The "Euler Totient" activity repeatedly requires information about the factors of a number; the factor program could be called upon for this purpose, however alternative algorithms can also be used that make the program run many, many times faster.

Why focus on number? Aside from the fact that the mathematical content is accessible, the importance of factors, or the lack of them (prime numbers) is critical to our world's economy thanks to encryption. Many trillions of dollars are digitally transacted every day, the security of these transfers relies on prime numbers and the fact that current algorithms are not particularly efficient at *disassembling* numbers.

© Texas Instruments 2022. You may copy, communicate and modify this material for non-commercial educational purposes provided all acknowledgements associated with this material are maintained.

Author: P. Fox

Table of Contents

© Texas Instruments 2023. You may copy, communicate and modify this material for non-commercial educational purposes provided all acknowledgements associated with this material are maintained.

Factors that Count

Teacher Notes:

A PowerPoint slide show is provided with this activity as an introductory presentation for students to watch. The slides work through the algorithmic process for the determination of the factors for the number: 36. Slides reference mathematical terminology such as: divisor, quotient and remainder, the animations are designed to help students understand these terms.

The presentation does not cover all possible divisors, instead, it leaves students pondering the most efficient algorithm by stopping at a divisor of 9 and posing the comment: “The factors are now starting to repeat themselves”.

A perfect square has been used on purpose in the slide set, it serves as a subtle hint that it may only be necessary to search up to the square-root of the corresponding number. If this efficiency is incorporated, students would need to incorporate a check routine for perfect squares to ensure a double count of the square-root is not erroneously included in the factor count.

Instructions for the simplest program (not the quickest) are provided here so that students may also be assessed on their ability to independently arrive at a more efficient factor searching routine and conditions.

More advanced students may check if the input is odd and therefore start the loop counter with an odd number and use a step size two, therefore skipping divisibility for all even quantities.

The focus of this activity is on counting factors, so the program does not store the actual factors, however, students may choose to investigate ways to store the factors in a list.

TI-Codes Lessons:

Unit 1 – Skill Builder 1

⇩

Unit 4 – Skill Builder 1

Commands:

- Request <input>
- For <counter> EndFor
- If <condition> Then <instruction>
- Disp <output>

Finding and Counting Factors

There are many ways to determine the quantity of factors for a specified number. The most common method is to test the divisibility for all applicable numbers. For example, suppose we want to determine the quantity of factors for 18. We can determine the quotient and remainder for all the numbers from 1 to 18.

Table 1A

Divisor	1	2	3	4	5	6	7	8	9
Quotient	18	9	6	4	3	3	2	2	2
Remainder	0	0	0	2	3	0	4	2	0

Table 1B

Divisor	10	11	12	13	14	15	16	17	18
Quotient	1	1	1	1	1	1	1	1	1
Remainder	8	7	6	5	4	3	2	1	0

Our conclusion is that 18 has six factors since there are six occasions whereby the remainder is equal to zero.

© Texas Instruments 2023. You may copy, communicate and modify this material for non-commercial educational purposes provided all acknowledgements associated with this material are maintained.

Author: P.Fox

This divisibility check for all numbers is exhaustive. You may have ideas about how this process can be made more efficient, however, this method will provide a basis for an algorithm on which to write a simple program to count the quantity of factors for a given number. You can make the necessary improvements and checks once your initial program is complete and functioning.

Question: 1.

Write a description of the steps required to determine the quantity of factors for any whole number: n.

Answer: Student answers will vary, the pseudo-code provides the basis on which the program will be written.

Sample:

- Input number: n
- Set factor count to 0
- Loop from 1 to n
- If n ÷ (loop counter) has no remainder Then increase (factor count)
- End Loop
- Display (factor count)

Instructions:

Start a new document and insert a calculator application.

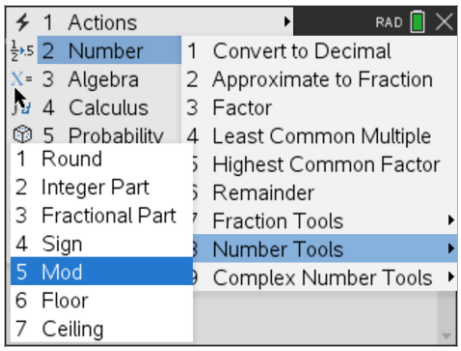
Locate the **mod** command using: **Number** > **Number Tools** > **Mod**

Determine the result of the following calculations:

Mod(18,6)

Mod(18,5)

Mod(18,12)



Question: 2.

Based on your experimentation, what value does the MOD command return?

The mod command returns the remainder upon division.

Question: 3.

If MOD(*a*, *b*) = 0, what does this say about the relationship between *a* and *b*?

If the 'remainder' of *a* ÷ *b* = 0 then *b* must be a factor of *a*.

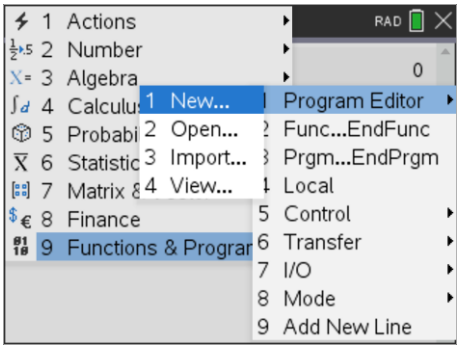
Writing a Program

Create a new program by selecting:

Functions & Programs > **Program Editor** > **New**

Call the program: FactorCount

Note that 'FactorCount' is one word as program names cannot contain spaces.



The Programming application is launched on the same page as the Calculator Application. The page-layout in the document menu can be used to give each application its own page.
Short-cut: [**Ctrl**] + [**6**]. Page 1.1 = Calculator Application. Page 1.2 = Program Application.

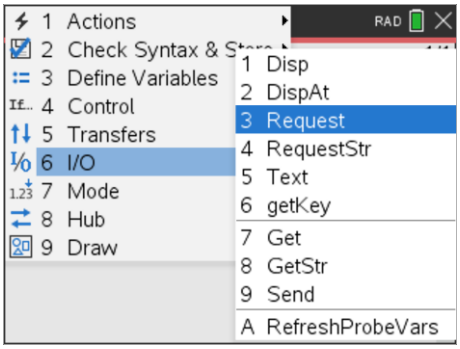
© Texas Instruments 2023. You may copy, communicate and modify this material for non-commercial educational purposes provided all acknowledgements associated with this material are maintained.

Author: P.Fox

The first task is to request a number from the program user.

Use the I/O (input / output) menu to access the **Request** command. The request command can include a text prompt followed by a variable to store the number.

Request "Enter a number",n



- Quotation marks: " " can be entered by pressing [**Ctrl**] + [**x**] (multiplication sign)
- The comma is located in the bottom left corner of the keyboard.

A counter will be used to 'count' the quantity of factors. The counter must be set to zero before the counting process begins.

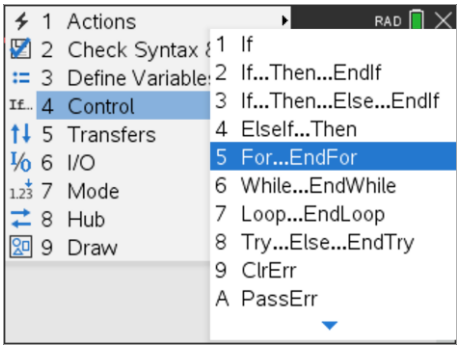
c : = 0

Start a **For** loop by selecting:

Control > **For** … **EndFor**

The loop will start at 1 and finish at n and use ***i*** to track the number of times the loop has been executed.

For *i* , 1, n



An **IF** statement will be used to check if the user's number has a factor each time the program executes the loop.

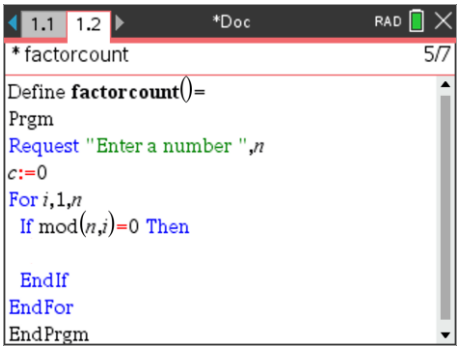
The IF command can be selected by:

Control > **If** … **Then** … **EndIf**

Between the IF and THEN statement insert the command:

mod(n, *i*) = 0

Note that 'mod' can be typed directly from the keyboard or accessed through the catalogue.



Move the cursor into the empty line between **THEN** and **EndIf**. This line of code is only executed if the condition: **MOD**(n, i) = 0 is TRUE.

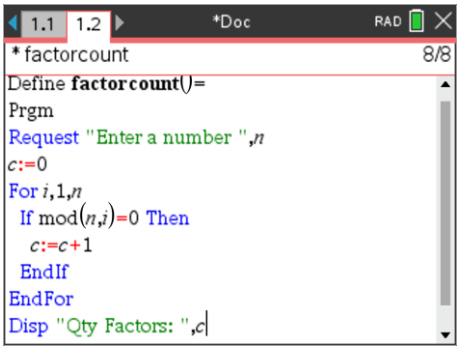
Insert the command:

c := c + 1

Create another line between **EndFor** and **EndPrgm**

From the I/O menu select **Disp** and type the command:

Disp "Qty Factors: ", c



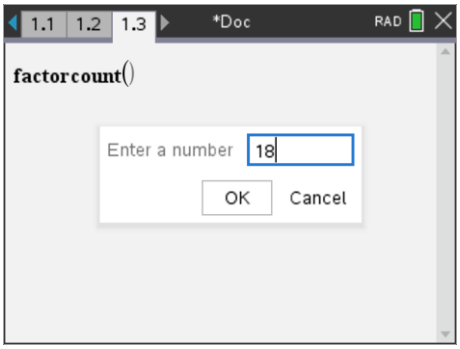
© Texas Instruments 2023. You may copy, communicate and modify this material for non-commercial educational purposes provided all acknowledgements associated with this material are maintained.

Author: P.Fox

Save the program and launch it by pressing [**Ctrl**] + [**R**]. A new Calculator application will be created and the program name automatically pasted.
Press [**Enter**] to launch the program.

Start by checking the factor count for 18.

The table at the start of this activity indicates the program should identify 6 factors.



Question: 4.

Determine the quantity of factors for each of the following numbers:

a. 24 8 factors … {1, 2, 3, 4, 6, 8, 12, 24}

b. 36 9 factors … {1, 2, 3, 4, 6, 9, 12, 18, 36}

c. 37 2 factors (prime numbers have exactly 2 factors) … {1, 37}

d. 144 15 factors … {1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 36, 48, 72, 144}

Check each of your answers by writing down all the factors. (Factors listed above)

Question: 5.

Determine the quantity of factors for each of the following numbers. Identify a specific characteristic about the quantity of factors and use this to classify the numbers into two groups, explain your classification.

29 (2), 84 (12), 104 (8), 87 (4), 22 (4), 37 (2), 101 (2), 97 (2), 45 (6), 43 (2), 133 (4), 153 (6), 173 (2), 107 (2).
The numbers: 29, 37, 101, 97, 43 and 173 all have exactly two factors and form the group of 'prime' numbers.
Teacher Notes: Referring to prime numbers as having exactly two factors removes any potential ambiguity with regards to whether or not 1 is prime.

Question: 6.

Determine the quantity of factors for each of the following numbers. Identify a specific characteristic about the quantity of factors and use this to classify the numbers into two groups, explain your classification.

28 (6), 30 (8), 90 (12), 45 (6), 50 (6), 60 (12), 120 (16), 72 (12), 25 (3), 49 (3), 81 (5), 144 (15), 441 (9), 82 (4), 24 (8), 720 (30)
This classification is harder than the previous one … students need to pick the 'odd' ones out. If students can see that 25, 49, 81, 144 and 441 all have an odd number of factors they should also identify that these numbers are perfect squares. Perfect squares are the only numbers that have an odd number of factors since each contains a 'repeated' factor.

Question: 7.

The FactorCount program works, but it could be more efficient. Use a stop watch to time how long the program takes to count the number of factors for: 10,000; 20,000 and 30,000. Use these times to predict how long the program will take to count the factors for 40,000. Test your answer!
If you are satisfied with your prediction, how long would it take to find factors for the following number:

[250 digits!]

Note: This number is associated with RSA Encryption.

Counting factors for 10,000 takes approximately 7 to 8 seconds. [TI-Nspire CX II series]
Counting factors for 20,000 takes approximately 14 to 15 seconds.
Counting factors for 30,000 takes approximately 21 to 22 seconds.
Students should see that each 10,000 numbers take approximately 7 seconds. Assuming that the larger numbers do not take any longer, 40,000 should take approximately 28 to 29 seconds.

Timed result: ≈ 28.9 seconds.

© Texas Instruments 2023. You may copy, communicate and modify this material for non-commercial educational purposes provided all acknowledgements associated with this material are maintained.

Author: P.Fox

A simple improvement to the algorithm (working to square-root of n) allows the program to count the factors in just under 1 second!

Even using the relatively simple improvement to the algorithm, the really big number containing 250 digits, would take 10^{238} years.
This time estimation doesn't allow for additional routines required to handle the quantity of digits in the number. This time factor is why super-computers are required to work on such large numbers and helps explain why these numbers are used in the public key encryption process.

Teacher Notes

An extension to this activity is titled: Fast Facts. This activity looks at using an improved algorithm to count the quantity of factors. Two relatively simple improvements are made to the algorithm. Setting the FOR loop to the square-root of the number provides a massive improvement, checking if the number is even and adjusting the corresponding STEP size in the FOR loop provides further efficiency. Counting factors for say 100,000,001 under this revised system reduces the counting time to less than 5 seconds instead of something close to 20 hours!

Investigation

Why are factors important? To answer this question, consider the opposite situation, the *absence* of factors. Billions of dollars are moved around electronically every day, to do this securely, the electronic transfers must be encrypted. The most common encryption method (RSA) is built around very large prime numbers, numbers with an *absence* of factors! All encryption methods are essentially built on numbers, so being able to 'assemble' and 'disassemble' numbers is extremely important.

Teacher Notes

To help build relevance to this investigation, consider engaging students in a discussion about the Enigma Machine, a powerful encryption tool created and used by the Germans during World War II. The Imitation Game [movie] is a wonderful way to show students just how important mathematics and mathematicians are to the world. The movie looks at Alan Turing and a team of mathematicians as they build a computing device to help solve the enigma code. While the Enigma machine was not based on prime numbers, it helps illustrate a long history, pre-dating computers, pertaining to the importance of encryption.

In the 21st century, encryption requirements have become ubiquitous. RSA encryption, invented in 1977 by Ron **R**ivest, Adi **S**hamir and Leonard **A**dleman is based on prime numbers. The story reads like a Hollywood movie! Initially the encryption process was supposed to be reserved only for military communications, however Ron, Adi and Leonard were so confident their system could not be hacked, they released it to the world, even explaining how it works! Their encryption system is still in use today!

Imitation Game Move Trailer

Interview with Ron Rivest

© Texas Instruments 2023. You may copy, communicate and modify this material for non-commercial educational purposes provided all acknowledgements associated with this material are maintained.

Your task is to find a rule that determines the quantity of factors for any whole number, given the prime factorisation of that number.

A few clues are provided along the way to help you on your factor sleuth journey. Document your search findings and conclusions using the clues and your constructed program to help expedite your investigation.

Clue 1:

Determine the prime factorisation and corresponding quantity of factors for the following numbers: 36; 100; 441; 3025 & 48841.

Clue 2:

Determine the prime factorisation and corresponding quantity of factors for the following numbers: 24; 250; 1029; 6655 and 198911.

Clue 3:

Based on the first two clues, generate some numbers that you believe have exactly 8 factors. Test your numbers and comment on the results.

Clue 4:

Determine the prime factorisation and corresponding quantity of factors for the following numbers: 2000; 64827; 107811; 668168 and 1585615607. [Note: For this last number you will need a fast algorithm!]

Clue 5:

Create some numbers of the form: $m^2 \times n^5$ where m and n are both prime. Determine the quantity of factors for each of your numbers. [Note: You may want to choose relatively small prime numbers for m and n.]

Continue the exploration, tabulate your results and record your thoughts, hypotheses, tests and reflections as you go. Documenting findings is an important part of the investigative process. Detectives may have many suspects in their initial investigations, however as more clues surface they develop hypotheses. Detectives test each hypothesis, review what they already know or go in search of more clues. Some investigations end up as Cold Cases, however it is critical that detailed documentation of all aspects of their investigation are retained in the event the investigation is re-opened. Some crimes remain unsolved despite having significant suspects, in mathematics these are often called 'conjectures', a theory that seems to work but has never been proven.

Answers: Students should record their findings in a table. The clues prompt students to focus on the exponents rather than the bases. For example: $2^2 \times 3^2 = 36$ and $2^2 \times 5^2 = 100$ both have the same quantity of factors. They have the same indices but different bases, similarly with the other examples of 441, 3025 and 48841.

If students tabulate the indices and quantity of factors they should start to see the connection: {2, 2, 9}; {2, 3, 12}; {1, 2, 6} … adding one to each exponent and then calculating the product results in the quantity of factors.

Students should go beyond numbers with two prime factors and also check that the algorithm works for prime numbers.

© Texas Instruments 2023. You may copy, communicate and modify this material for non-commercial educational purposes provided all acknowledgements associated with this material are maintained.

Author: P.Fox

Euclid's Algorithm

Teacher Notes:

A PowerPoint slide show is provided with this activity as an introductory presentation for students to watch and help them understand how the algorithm works. The slides use 'lengths' to help explain why the algorithm works.

Following the 'lengths' examples, numbers are included to see how Euclid's algorithm translates to working with numbers

The calculator contains a command for the "Highest Common Factor" or "Greatest Common Divisor", it is however good for students to understand how the 'magic' happens.

The GCD command only compares 2 numbers, in the later stages of this activity, students extend their program to 3 numbers and then an entire list!

TI-Codes Lessons:

Unit 1 – Skill Builder 1

⇩

Unit 4 – Skill Builder 2

Commands:

- Request <input>
- While <condition> EndWhile
- If <condition> Then <instruction> Else <instruction>
- Disp <output>

Highest Common Factors

The Highest Common Factor (HCF) or Greatest Common Divisor (GCD) of two numbers is useful for many reasons. The process is valuable when working with fractions, solving packaging problems, developing traffic light sequences and encrypting content for digital communications. Developed more than 2000 years ago, Euclid's algorithm is still the most efficient process used to determine the Highest Common Factor of two numbers.

Euclid's Algorithm:

LINE #1: IF A = 0 THEN GCD(A,B) = B since GCD(0,B) = B

LINE #2: IF B = 0 THEN GCD(A,B) = A since GCD(A,0) = A

LINE #3: A = B x Q + R … where Q is the quotient and R is the remainder

LINE #4: GCD(B,R) = GCD(A,B), now find GCD(B,R)

This algorithm will make more sense when some numbers are used for A and B. Suppose we want to find the highest common factor of (A) 1260 and (B) 385. As neither A = 0 or B = 0 we progress to LINE #3.

1260 = 385 x 3 + 105 [We say that 105 is the remainder when 1260 is divided by 385]

According to LINE #4 of Euclid's algorithm: GCD(1260,385) = GCD(385,105)

We apply the algorithm again. Since 385 ≠ 0 and 105 ≠ 0 we proceed to LINE #3.

385 = 105 x 3 + 70 [We say that 70 is the remainder when 385 is divided by 105]

According to LINE #4 of Euclid's algorithm: GCD(1260,385) = GCD(385,105) = GCD(105,70).

We apply the algorithm again. Since 105 ≠ 0 and 70 ≠ 0, we proceed to LINE #3

105 = 70 x 1 + 35 [We can say that 35 is the remainder when 105 is divided by 70]

We are getting close! According to LINE #4 of Euclid's algorithm: GCD(1260,385) = … = GCD(70,35)

© Texas Instruments 2023. You may copy, communicate and modify this material for non-commercial educational purposes provided all acknowledgements associated with this material are maintained.

Author: P.Fox

Applying the algorithm one more time, as 70 ≠ 0 and 35 ≠ 0, we proceed to LINE #3.

70 = 2 x 35 + 0. [This time the remainder is 0!]

Now we can apply LINE #1 or LINE #2 since we have GCD(35,0) = 35.

Our conclusion is that the Highest Common Factor or Greatest Common Divisor of 1260 and 385 is 35.

Question: 1.

Use Euclid's algorithm to identify the highest common factor of: 3850 and 3234.
Answer: 154

Writing a Program

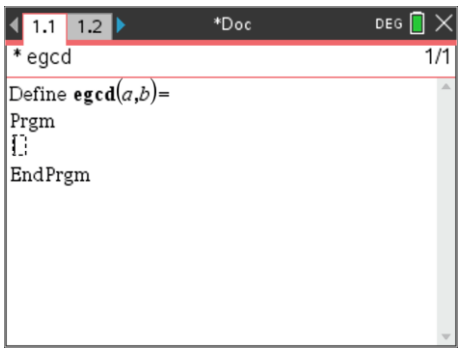
Instructions:

Start a new document; insert a new program.

Add Program Editor > New

Call the program: EGCD

Edit the program definition to include "a" and "b". (See opposite)

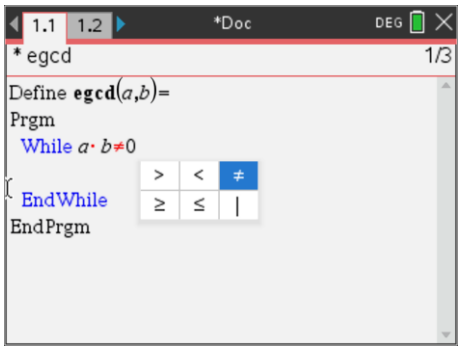


Euclid's algorithm ceases when either a = 0 or b = 0, an easy way to check this is: a x b = 0. The "null factor law" states that if the product of two numbers is zero, then one or both of the numbers must be zero.

The algorithm should continue to run while a x b ≠ 0.

Menu > Control > While … EndWhile

The "not equals" sign can be accessed from the inequality flyout menu.



Modular arithmetic returns the remainder when a ÷ b (where a > b) so an If … Then … Else … statement can be used to process Line #3 of Euclid's algorithm.

Menu > Control > If...Then...Else... EndIf

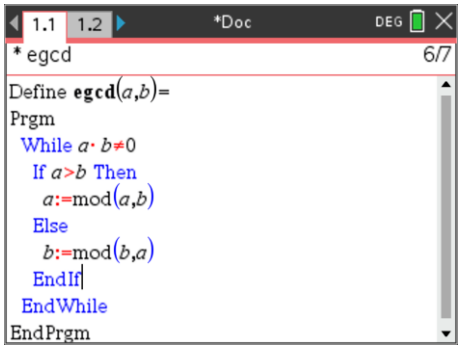
The mod() command can be typed directly or accessed from the catalogue. Enter the corresponding modular arithmetic calculations, note carefully the respective orders for *a* and *b*.

That's the entire algorithm! The only thing remaining is to display the results. You can use a display command such as:

Disp *a*,*b*

This command can be placed in the loop, between EndIf and EndWhile to monitor the progress of the calculations.

Another display command should be used to display the highest common factor at the end of the program.



© Texas Instruments 2023. You may copy, communicate and modify this material for non-commercial educational purposes provided all acknowledgements associated with this material are maintained.

Author: P.Fox

Question: 2.

Determine the highest common factor of: 1914 and 7293 (by hand) using Euclid's algorithm and use your results to check the program.

Answer: 33

Question: 3.

Test your program on some smaller numbers where you know the highest common factor. Record your test results.

Answer: Answers will vary depending on what numbers student chose to explore and test.

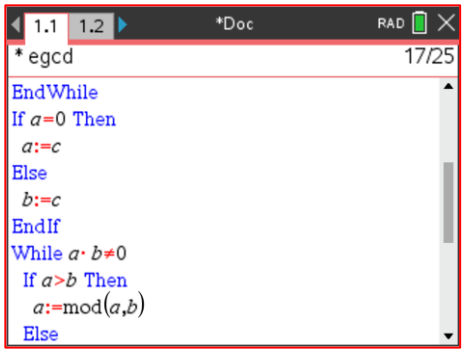
Question: 4.

The **Number** menu in the Calculator Application contains a command to determine the highest common factor of **two** numbers. Edit your program to find the highest common factor of three numbers.
Example: EGCD(a,b,c)

Test and evaluate your program.

Answer: There are various ways students may edit their program to achieve the desired result. Students should also consider efficiency.

Teacher Notes: Sample Program



A simple addition to the original program is shown here. Following the original While loop, (EndWhile at top of screen) an IF statement is used test which variable, a or b, is equal to zero, the corresponding variable is then replaced with c.

If a = 0 **Then** a:=c **Else** b:=c

Copy the original While loop and paste below EndIf.

Conceptually, copying and pasting the original loop leads students to an understanding that the original loop is repeated and therefore extending the program to a list of numbers involves an over-arching loop.

Students should provide a table of numbers that they have tested. Students should also think about how they generate the numbers to be tested. For example:

a = 29 x 35 = 1015 b = 41 x 35 = 1435 c = 97 x 35 = 3395

Generating the numbers as products with primes ensures the highest common factor of the three numbers is 35 in the above scenario.

© Texas Instruments 2023. You may copy, communicate and modify this material for non-commercial educational purposes provided all acknowledgements associated with this material are maintained.

Author: P.Fox

Question: 5.

Edit your program to test for the highest common factor of an entire list of numbers.
Note: The program can be defined as egcd(data) where data is a list of numbers: {#, #, # ...}. The **dim** command can be used to determine the dimensions (quantity of numbers) entered into the list.

Answer: The sample program shown opposite includes the original *While* loop and is bracketed by a *For* loop that repeats Euclid's algorithm. Each pair of numbers combines the previous highest common factor with the next number in the list. Note that the list is sorted at the start, this may create an efficiency, specifically if the highest common factor is relatively small. List P is used to track the ongoing HCF.

Testing {2030,2870,3115,7147,21399} for the HCF with the SortD included: p = {21399,7,7,7,7}. The highest common factor is identified early. The algorithm is executed 7 times.

Testing the same set of numbers without the Sort command results with p = {2030,70,35,7,7} and the algorithm is executed 10 times.

```
egcd
Define egcd(data)=
Prgm
d:=dim(data)
p:=data
SortD p
For i,1,d-1
  a:=p[i]
  b:=p[i+1]
While a· b≠0
  Disp a,b
  If a>b Then
    a:=mod(a,b)
  Else
    b:=mod(b,a)
  EndIf
EndWhile
If a=0 Then
  p[i+1]:=b
  n:=b
Else
  p[i+1]:=a
  n:=a
EndIf
EndFor
Disp "Highest Common Factor",n
EndPrgm
```

Investigation

The prime factorisation of a number can be used to efficiently find the highest common factor of any two or more numbers. Use your program to find the highest common factor for each list of numbers (below). Write the original numbers and the highest common factor in terms of their prime factorisation. Try some of your own lists, then write a description of how you can use the prime factorisation to determine the highest common factor of any two or more numbers.

List 1: 1260, 1410, 2040, 4290 & 9570

List 2: 220, 1400, 1700, 30940 & 154700

List 3: 2964, 3588, 8892, 10764 & 409032

List 4: 399, 441, 1911, 3381, 5733 & 835107

© Texas Instruments 2023. You may copy, communicate and modify this material for non-commercial educational purposes provided all acknowledgements associated with this material are maintained.

Author: P.Fox

Euler Totient Function

Teacher Notes:

A PowerPoint slide show is provided with this activity as an introductory presentation for students to watch and help them understand how the algorithm works. The slides work progressively through the number from 1 to n, capturing any numbers that are co-prime with the original number n.

There is no calculator command for the Euler Totient function, however there is a short cut approach using the prime factorisation of a number. Once students have completed their program, they use the prime factor approach and compare it to their program.

TI-Codes Lessons:

Unit 1 – Skill Builder 1

⇩

Unit 4 – Skill Builder 1

Commands:

- Request <input>
- For <counter> Endfor
- If <condition> Then <instruction> Else <instruction>
- Disp <output>

Introduction

The Euler Totient Function for a whole number '*n*' counts the quantity of numbers that are co-prime up to the number *n*. To help understand this definition, consider the number 12.

We need to check which numbers: {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12} have a factor in common with 12, these numbers are discarded leaving us with the numbers that are co-prime. This is summarised in the table below.

Whole Numbers < n	1	2	3	4	5	6	7	8	9	10	11	12
Highest Common Factor	1	2	3	4	1	6	1	4	3	2	1	12

There are 4 numbers where the highest common factor is 1, these numbers are co-prime with 12: {1, 5, 7, 11}. The Euler Totient function for 12 is therefore equal to 4, this can be written as: $\varphi(12) = 4$.

Here is another example for the number 9.

Whole Numbers < n	1	2	3	4	5	6	7	8	9
Highest Common Factor	1	1	3	1	1	3	1	1	9

The Euler Totient function for 9 is therefore equal to 6, this can be written as: $\varphi(9) = 6$.

Question: 1.

Create some pseudo-code for the Euler Totient function.

Answer: (Sample)

```
Request input
Reset Counter = 0
Loop from 1 to n
    If GCD(n,1) = 1 Then < increase counter >
End Loop
```

© Texas Instruments 2023. You may copy, communicate and modify this material for non-commercial educational purposes provided all acknowledgements associated with this material are maintained.

Writing a Program

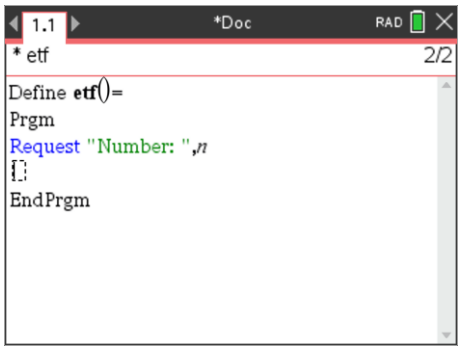
Instructions:

Start a new document; insert a new program.

Add Program Editor > New

Call the program: ETF

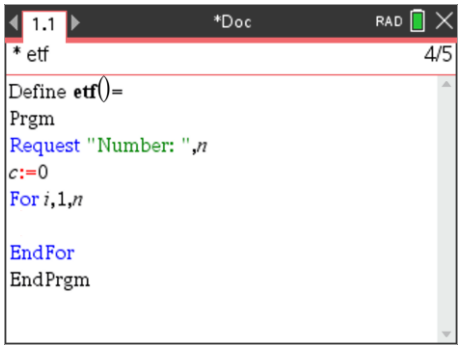
Use the Request command (I/O) to insert value.



A counter is required to track the quantity of numbers that are co-prime with n.

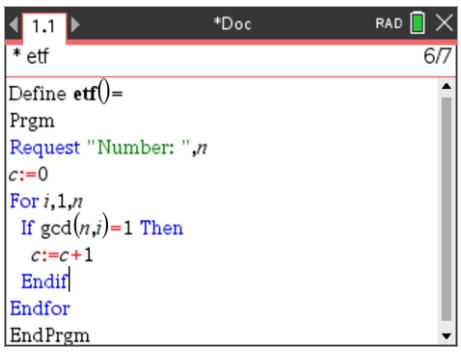
Initialise a counter: c := 0; then insert a **For** loop.

Just like the examples, all the numbers from 1 to n need to be checked.



If the highest common factor (greatest common divisor) between n and the loop counter is equal to 1, **then** increase the counter value.

Once the loop has finished, add a 'display' command to show the value of the Euler Totient function.



Question: 2.

Check that your program produces the same results for the two worked examples, then try several others (by hand) and compare results.

Answer: The program returns the correct values for all numbers.

Question: 3.

Explore the Euler Totient function for prime numbers, what do you notice?

Answer: The Euler Totient function for a prime number 'n', returns the value n – 1.

Question: 4.

Determine the fraction: $\frac{n}{\varphi(n)}$ for the following values of n: 30, 60 and 90, comment on the results.

Answer: $\frac{30}{\varphi(30)} = \frac{30}{8} = 3.75$, $\frac{60}{\varphi(60)} = \frac{60}{16} = 3.75$ and $\frac{90}{\varphi(90)} = \frac{90}{24} = 3.75$

Other values for n with the same fraction (ratio) include: 120, 150, 180, 240, 270, 300 but 210 and 330 have very different results.

© Texas Instruments 2023. You may copy, communicate and modify this material for non-commercial educational purposes provided all acknowledgements associated with this material are maintained.

Author: P.Fox

Teacher Notes: Students may sample a selection of multiples of 30 (as above) and jump to a conclusion too quickly if they miss 210 and 330. The clue lies in the prime factorisation of the multiples of 30.
$30 = 2 \times 3 \times 5$; $60 = 2^2 \times 3 \times 5$; $90 = 2 \times 3^2 \times 5$; $120 = 2^3 \times 3 \times 5$; $150 = 2 \times 3 \times 5^2$; $180 = 2^2 \times 3^2 \times 5$; however, $210 = 2 \times 3 \times 5 \times 7$ which results in prime factorisation involving 4 prime factors, specifically a departure from: $2^a \times 3^b \times 5^c$. Students should establish that for the Euler Totient function, the bases that are important not the exponents.

Question: 5.

The number 100 can be expressed as: $2^2 \times 5^2$. Compare the Euler Totient value for 100 with the following calculation:

$$100 \times \left(1-\frac{1}{2}\right)\left(1-\frac{1}{5}\right)$$

Answer: $\varphi(100) = 40$. $100 \times ½ \times 4/5 = 40$. The results are the same.

Question: 6.

The number 1125 can be expressed as: $3^2 \times 5^3$. Compare the Euler Totient value for 1125 with the following calculation:

$$1125 \times \left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)$$

Answer: $\varphi(1125) = 600$. $1125 \times 2/3 \times 4/5 = 600$. The results are the same!

Question: 7.

Use the previous to questions to explore the prime factorisation approach to the Euler Totient function with the Euler Totient value determined by your program.

Answer: Results will vary, depending on the values that students chose, however in each case the answers will be the same.

Question: 8.

How does the prime factorisation approach to calculating the Euler Totient function explain your results to Question 4?

Answer: The prime factorisation for 30, 60 and 90 are of the form: $2^a \times 3^b \times 5^c$. The prime factorisation approach for calculating the Euler Totient function can be considered as two parts, the first part being the original number, the second, a combination of the prime factors (bases only). By dividing out the original number, we are only left with a calculation involving the prime factors, ignoring duplicity.

Question: 9.

Why does the 'short cut' approach to the Euler Totient function work?

Answer: Each prime factor removes the corresponding fraction of remaining number.
Example, if 2 is a prime factor of some number 'n', then ½ of the numbers up to (and including n) will have a factor in common (2). Similarly, if 3 is a prime factor of 'n', then 1/3 of the remaining numbers will also have a factor in common with 'n'. The co-primes will be the complement of these calculations.

© Texas Instruments 2023. You may copy, communicate and modify this material for non-commercial educational purposes provided all acknowledgements associated with this material are maintained.

Author: P.Fox

Investigation

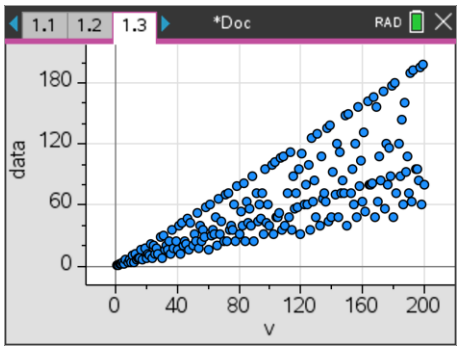
Re-write the Euler Totient function program to determine the Euler Totient function for a range of numbers, graph the results and explore any patterns.

Answer: Sample program (shown opposite), generates all the Euler Totient values from 1 to n. To generate a scatterplot, the whole numbers from 1 to n are stored in variable v.

The two "dispat" commands show the progressive results on screen as the program is running. These commands can be removed to expedite the program.

```
"etf" stored successfully
Define etf()=
Prgm
data:={}
Request "Number: ",n
v:=seq(x,x,1,n)
For j,1,n
c:=0
For i,1,j
  If gcd(j,i)=1 Then
    c:=c+1
  EndIf
EndFor
DispAt 1,"Euler Totient Value: "
DispAt 2,c
data:=augment(data,{c})
EndFor
Disp data
EndPrgm
```

Sample results: (1, 200)



There is a clear line of data above which there are no points. The points along this line are the prime numbers. φ(p) = p – 1, where p is prime.

There is also a 'line' of points assembled around $y = \frac{1}{2}(x-2)$. This collection of points corresponds to prime factorisations of the form: 2 x p, where p is prime.

With the exception of φ(1) = 1 and φ(2) = 1, φ(n) is even. Why? This can be seen from the prime factorisation calculation method for the Euler Totient function. Consider *n* as even and then *n* as odd.

Another set of points of particular interest are those at the bottom of the graph:

1, 2, 3, 4, 6, 8, 10, 12, 14, 18, 20, 24, 30, 36, 42, 48, 60

The highly composite numbers are a subset of these numbers.

Number:	2	3	4	6	8	10	12	14
Euler Totient:	1	2	2	2	4	4	4	6
Prime Factorisation:	2	3	2^2	2×3	2^3	2×5	$2^2\times3$	2×7
Number:	18	20	24	30	36	42	48	60
Euler Totient:	6	8	8	8	12	12	16	16
Prime Factorisation:	2×3^2	$2^2\times5$	$2^3\times3$	$2\times3\times5$	$2^2\times3^2$	$2\times3\times7$	$2^4\times3$	$2^2\times3\times5$

Expressing each calculation using the prime factorisation method helps show why these number fall along the bottom of the graph.

© Texas Instruments 2023. You may copy, communicate and modify this material for non-commercial educational purposes provided all acknowledgements associated with this material are maintained.

Author: P.Fox

Highly Composite Numbers

Teacher Notes:

A PowerPoint slide show is provided with this activity as an introductory presentation for students to watch and help them understand highly composite numbers.

Students may wish to call upon previous programs that count factors.

TI-Codes Lessons:

Unit 1 – Skill Builder 1

⇩

Unit 4 – Skill Builder 2

Commands:

- Request <input>
- While <condition> EndWhile
- If <condition> Then <instruction> Else <instruction>
- Disp <output>

Introduction

A highly composite number has more factors than any of its predecessors. Think of it as competition along the number line. The difficulty in locating highly composite numbers is that you must already know the previous highly composite number in order to identify how many factors the next number must have in order to qualify. Any search for highly composite number therefore generally starts at 1.

Whilst 1 only has one factor, there are no predecessors, so by default, 1 is the first highly composite number. Naturally 2 is the next highly composite number having two factors. The next is 4 with three factors then 6 with four factors. With one, two, three and four factors already checked, it would be easy to assume that the next highly composite number would have five factors, however 12 is the next highly composite number with six factors.

Question: 1.

Write a description of a program that will determine the Highly Composite number up to some value n.
Note: The quantity of factors for any number can be references as 'factor_count'.

Answer: Answers will vary, however students must use a 'record holder' to track the current highly composite number.

Sample:

```
Record:= 0
Input <Number> n
Loop start = 1, finish = n
        If factor_count(loop_counter) > record Then
                Increase record
                Store loop_counter
End Loop
Display  Highly Composite numbers <stored_loop counters>
```

© Texas Instruments 2023. You may copy, communicate and modify this material for non-commercial educational purposes provided all acknowledgements associated with this material are maintained.

Author: P.Fox

Teacher Notes:

Notice how referencing the "factor count" program simplifies the entire program. In programming languages this is often referred to as a sub-routine. In educational neuroscience this is referred to as 'chunking', putting procedures or a collection of procedures into bite size pieces making them easier to digest. In mathematics this might be referring to "solving simultaneously" as one step in a much larger problem. Simultaneous equation would have been taught as a topic unto itself, however, if students understand what 'solving simultaneously' means, they are able to refer to it as a single step in a much bigger problem.

Writing a Program

Instructions:

Start a new document and insert a new program.

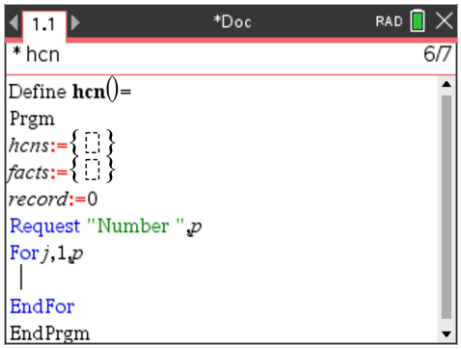
Add Program Editor > New

Call the program: HCN

There are a few variables to be set up for this program. It seems appropriate to record the highly composite numbers and as an added check, record the quantity of factors for each.

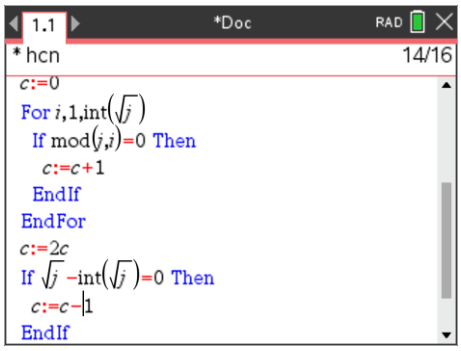
hcns:={ }	[List of Highly Composite Numbers]
facts:={ }	[List for the quantity of factors]
record:=0	Track most recent HCN.

Request the highest number to be searched and start a For loop.

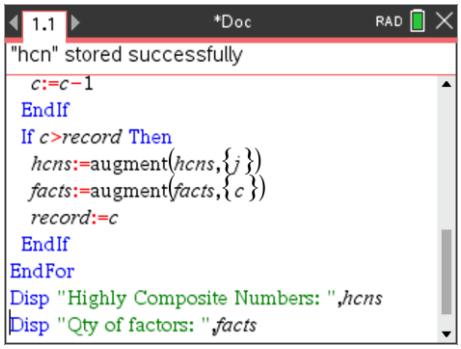


If the factor counting program is in the same document or available from the public library, it can be called up on here to count factors. It is very important however that variables are not duplicated. The factor counting program will also need to be very efficient as it is called upon many times!

The sample code shown opposite is a relatively fast factor counting program that can be placed inside the previous loop.



If the most recent factor count is higher than the previous record then the list of highly composite numbers (hcns) needs to be updated, so too the record for the quantity of factors (facts) and finally, the record itself needs to be updated.



Question: 2.

Run your program and check that the first five highly composite numbers are: 1, 2, 4, 6, 12; then determine all the highly composite number from 1 to 100.

Answer: Highly Composite Numbers: 1, 2, 4, 6, 12, 24, 36, 48 & 60.
Quantity of factors for each: 1, 2, 3, 4, 6, 8, 9, 10, 12.

© Texas Instruments 2023. You may copy, communicate and modify this material for non-commercial educational purposes provided all acknowledgements associated with this material are maintained.

Author: P.Fox

Question: 3.

Determine all the highly composite numbers from 1 to 1000 and their corresponding quantity of factors.

Answer:

HCNs	1	2	4	6	12	24	36	48	60
Qty Factors	1	2	3	4	6	8	9	10	12
HCNs	120	180	240	360	720	840			
Qty Factors	16	18	20	24	30	32			

Note: Students may be surprised that 144 is not a highly composite number given that $144 = 12^2$.

Question: 4.

Express each of the Highly Composite Number in the previous question as a product of its prime factors.

Answer:

HCNs	1	2	4	6	12	24	36	48
Prime Factorisation	1	2	2^2	2×3	$2^2\times3$	$2^3\times3$	$2^2\times3^2$	$2^4\times3$
HCNs	60	120	180	240	360	720	840	
Prime Factorisation	$2^2\times3\times5$	$2^3\times3\times5$	$2^2\times3^2\times5$	$2^4\times3\times5$	$2^3\times3^2\times5$	$2^4\times3^2\times5$	$2^3\times3\times5\times7$	

Question: 5.

Study the prime factorisations closely. Suggest a possible prime factorisation for the next highly composite number, the corresponding number and quantity of factors.
Note: You may have more than one educated guess.

Answer: Based on the previous prime factorisations... $2^3\times3$ went to $2^2\times3^2$, $2^3\times3\times5$ went to $2^2\times3^2\times5$, so it is likely that $2^3\times3\times5\times7$ will transition to: $2^2\times3^2\times5\times7$ (1260) which has 36 factors. The current calculator program validates this answer (prediction).

Investigation

To continue exploring Highly Composite Numbers, a more efficient program (or new program) is required, one that no longer starts at 1, rather one that starts at some previously identified Highly Composite Number and uses information gleaned from the first sixteen highly composite numbers.

- Re-write your HCN program so that it can start at any HCN.
- Continue recording HCNs and the corresponding prime factorisations. When and what will be the next prime factor to be included in the prime factorisation?
- Identify any patterns you can find in the prime factorisation that would help in locating subsequent prime factorisations.
- What prior learning are you using to identify the quantity of factors, make predictions and search?

Answer: There is a LOT to explore here, famous mathematicians such as Ramanujan explored HCNs, indeed, the back story makes for interesting reading. Prime factorisation can certainly act as a guide to predicting future HCN's.

© Texas Instruments 2023. You may copy, communicate and modify this material for non-commercial educational purposes provided all acknowledgements associated with this material are maintained.

Author: P.Fox

Current HCN:

$2^2 \times 3^2 \times 5 \times 7 = 1260$ (36 factors)

Based on previous HCN's there are a couple of options for the next HCN:

- $2^4 \times 3 \times 5 \times 7 = 1680$ (40 factors) [Increase exponent of 2, reduce exponent of 3]
- $2^3 \times 3^2 \times 5 \times 7 = 2520$ (48 factors) [Increase exponent of 2]
- $2^2 \times 3^2 \times 5^2 \times 7 = 6300$ (54 factors) [Increase exponent of 5]
- $2^2 \times 3^2 \times 5 \times 7 \times 11 = 13860$ (72 factors) [Introduce another prime factor]

Note: Increasing the exponent of 3 should not be a consideration. The result would produce the same quantity of factors as increasing the exponent of 2, but the numerical result would be greater.

Each option introduces more factors, however the numerical expense of repeating the 5 or introducing the next prime factor are too much (at this stage). The first option multiplies the previous HCN by 4/3. The second option multiplies the previous HCN by 2.

Student's should be confident of their HCN prediction which can be validated by the existing program structure. Further exploration using the existing program structure however will become problematic as the algorithm searches every number.

Current HCN:

$2^4 \times 3 \times 5 \times 7 = 1680$ (40 factors)

The next HCN is slightly less predictable. Using data collected so far, the prime factors 5 and 7 were introduced as similar junctions.

- $2 \times 3 \times 5 \times 7 \times 11 = 2310$ (32 factors) [Decrease all exponents, introduce another prime factor]
- $2^3 \times 3^2 \times 5 \times 7 = 2520$ (48 factors) [Decrease exponent of 2, increase exponent of 3]
- $2^2 \times 3 \times 5 \times 7 \times 11 = 4620$ (48 factors) [Decrease exponent of 3, introduce another prime factor]

Introducing the prime factor (11) is "too expensive" as a trade off with regards to the final calculation versus additional factors, indeed the first option produces less factors than the previous HCN.

Students should be reasonably confident that the next HCN is therefore 2520.

Students may also consider 'reverse engineering' a solution here by consideration of the quantity of factors. The missing options for the quantity of factors are: 41, 42, 43, 44, 45, 46 and 47. Using their understanding of how the quantity of factors can be calculated, HCNs with 41, 43 or 47 factors clearly don't work.

Consider: 42 = 6 x 7 or 2 x 3 x 7, the exponents could be: {5, 6} or {2, 3, 5}. The logical approach would be to place the largest exponents on the smallest bases:

- $2^6 \times 3^5 = 15552$ (42 factors)
- $2^5 \times 3^3 \times 5^2 = 21600$ (42 factors)

Neither of these results are satisfactory.

Consider: 44 = 11 x 4, a number with 44 factors could be produced using exponents of 10 and 3 only.

- $2^{10} \times 3^3 = 27648$.

Consider a number with 45 factors, it must be a perfect square since it has an odd number of factors!

© Texas Instruments 2023. You may copy, communicate and modify this material for non-commercial educational purposes provided all acknowledgements associated with this material are maintained.

Since 45 = 9 x 5 = 3 x 3 x 5, the exponents could be either {8, 4} or {2, 2, 4}, which means the following numbers would be options:

- $2^8 \times 3^4 = 20736$ [$144^2 = 20736$]
- $2^4 \times 3^2 \times 5^2 = 3600$ [$60^2 = 3600$ and 60 is a previous HCN]

In the case of 3600, we note that 2520 has more factors. Why? The prime factorisation of 2520 involves the introduction of the prime factor 7.

Students should quickly realise that a number with 46 factors would require exponents of 22 and 1, the computed result would be much too large! This leads to the conclusion that then next HCN after 1680 must have 48 factors.

Current list of highly composite numbers:

1, 2, 4, 6, 12, 24, 36, 48, 60, 120, 180, 240, 360, 720, 840, 1260, 1680, 2520

Where $2520 = 2^3 \times 3^2 \times 5 \times 7$ (48 factors)

Now the highly composite numbers themselves provide a clue as to how many factors the next highly composite number might contain: 60 (factors).

$60 = 2^2 \times 3 \times 5$

This means the exponents could be:

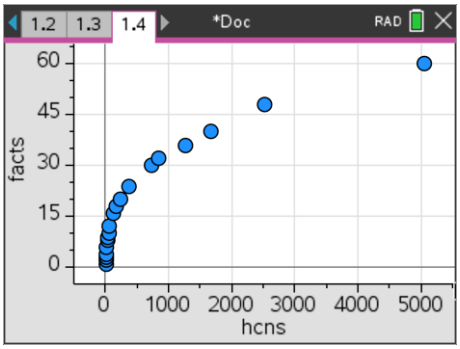
- 1, 1, 2, 4
- 3, 2, 4

Applying these exponents in the appropriate order means the next HCN could be:

- $2^4 \times 3^2 \times 5 \times 7 = 5040$
- $2^4 \times 3^3 \times 5^2 = 10800$

At this point in time it is worth exploring a graph of the HCNs versus the quantity of factors.

The relationship looks almost logarithmic ... but it's not.



Programming

The existing HCN program can be modified by starting the search for the next series of HCNs at the last known value. The search loop should also use an increment of at least 30. For example, if the most recent HCN = 5040, it is not necessary to check 5041, we know from the prime factorisation, the next HCN will have factors of 2, 3 and 5. Once students are confident that 7 will be included in all subsequent HCN's, the step size can be 210 and eventually 210 x 11 = 2310.

Primorial Factorisation

Students may also be encouraged to explore primorial representation. Primorial (Harvey Dubner) is a mixture of prime numbers and factorial.

Example:

Factorial: 5! = 5 x 4 x 3 x 2 x 1 = 120

Primorial: 5# = 5 x 3 x 2 x 1 = 30 (Product of primes less than or equal to 5)

The use of primorial becomes 'obvious' when considering the prime factorisation of a number, particularly highly composite numbers.

Example:

$720720 = 2^4 \times 3^2 \times 5 \times 7 \times 11 \times 13 = 2^2 \times (3 \times 2) \times (13 \times 11 \times 7 \times 5 \times 3 \times 2) = 2^2 \times 3\# \times 13\#$ or $2^2 \times 6 \times 30030$

© Texas Instruments 2023. You may copy, communicate and modify this material for non-commercial educational purposes provided all acknowledgements associated with this material are maintained.

Author: P.Fox