TI in Focus: AP® Calculus
2020 Mock AP® Calculus Exam
BC-1: Solutions, Concepts, and Scoring Guidelines

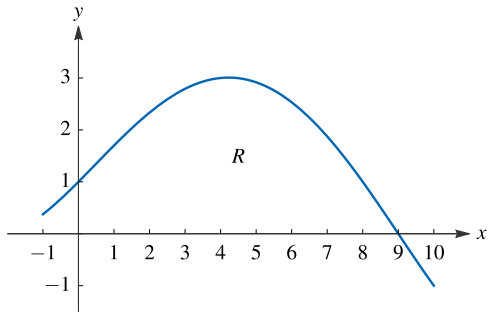Stephen Kokoska
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Former AP® Calculus Chief Reader

BC 1

The graph of g', the derivative of a twice-differentiable function g is shown in the figure. The graph has exactly one horizontal tangent line in the interval -1 to 10, at $x = 4.2$.

Graph of g'

R is the region in the first quadrant bounded by the graph of g' and the x-axis from $x = 0$ to $x = 9$. It is known that $g(0) = -7$, $g(9) = 12$, and $\int_0^9 g(x)\,dx = 27.6$.
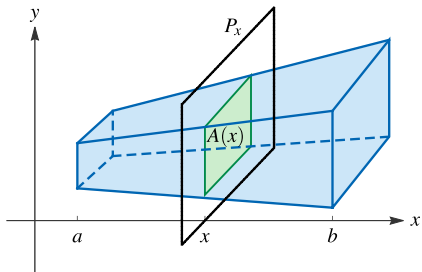
(i) The region R is the base of a solid. For this solid, at each x the cross section perpendicular to the x-axis is a right triangle with height x and base in the region R. The volume of the solid is given by $\int_0^9 A(x)\,dx$. Write an expression for $A(x)$.

Key Concepts

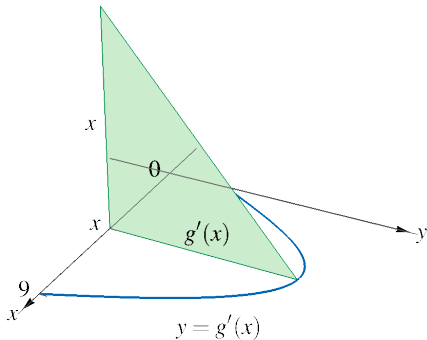### Definition of Volume

Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x through x and perpendicular to the x-axis is $A(x)$, where A is a continuous function, then the volume of S is $V = \int_a^b A(x)\,dx$

Solution

Consider a cross section perpendicular to the x-axis.

$$A(x) = \frac{1}{2} \cdot g'(x) \cdot x = \frac{1}{2} x g'(x)$$

Scoring Guidelines

(i) $A(x)$ represents the area of a right triangle at each x. | 1 : answer

$$A(x) = \frac{1}{2}xg'(x)$$

Scoring Notes

- If the student writes the integral with the correct integrand, they must equate this to the integral with $A(x)$ as the integrand.

- Therefore, $\displaystyle\int_0^9 \frac{1}{2}xg'(x)\,dx = \int_0^9 A(x)\,dx$ earns the point.

- Another variable, or function, may be introduced.

  Example: $R(x) = g'(x) \quad\Rightarrow\quad A(x) = \frac{1}{2}xR(x)$

- Common error: $A(x) = \displaystyle\int_0^9 \frac{1}{2}xg'(x)\,dx$

(j) Find the volume of the solid described in part (h). Show the computations that lead to your answer.

Key Concepts

Integration by Parts

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$ Product Rule

$$\int [f(x)g'(x) + g(x)f'(x)]\, dx = f(x)g(x)$$ Indefinite integral notation

$$\int f(x)g'(x)\, dx + \int g(x)f'(x)\, dx = f(x)g(x)$$ Property of integrals

$$\int f(x)g'(x)\, dx = f(x)g(x) - \int g(x)f'(x)\, dx$$ Rearrange terms

The last expression is called the **formula for integration by parts**.

Integration by Parts

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

Let $u = f(x) \quad \Rightarrow \quad du = f'(x)\,dx$

Let $v = g(x) \quad \Rightarrow \quad dv = g'(x)\,dx$

$$\int u\,dv = uv - \int v\,du$$

A Closer Look

1. Objective: write a complicated integral in terms of another, hopefully easier, integral. The choices for u and dv are important.

2. Select dv so that it can be integrated easily.

3. In general, choose u so that $\int v\,du$ is simpler than $\int u\,dv$.

Solution

$$V = \int_0^9 A(x)\,dx = \frac{1}{2}\int_0^9 xg'(x)\,dx$$

Use integration by parts.

$$u = x \qquad\qquad dv = g'(x)\,dx$$

$$du = dx \qquad\qquad v = \int g'(x)\,dx = g(x)$$

$$V = \frac{1}{2}\left(\Big[x \cdot g(x)\Big]_0^9 - \int_0^9 g(x)\,dx\right) \qquad\qquad \text{Integration by parts formula}$$

$$= \frac{1}{2}\left([9 \cdot g(9) - 0 \cdot g(0)] - 27.6\right) \qquad\qquad \text{FTC; given value}$$

$$= \frac{1}{2}(9 \cdot 12 - 27.6) = \frac{1}{2} \cdot 80.4 = 40.2 \qquad\qquad \text{Given values; simplify}$$

TEXAS INSTRUMENTS

Scoring Guidelines

(j) $V = \int_0^9 A(x)\,dx = \dfrac{1}{2}\int_0^9 xg'(x)\,dx$

Use integration by parts.

$$u = x \qquad\qquad dv = g'(x)\,dx$$

$$du = dx \qquad\qquad v = \int g'(x)\,dx = g(x)$$

$$V = \frac{1}{2}\left(\Big[x\cdot g(x)\Big]_0^9 - \int_0^9 g(x)\,dx\right)$$

$$= \frac{1}{2}\left([9\cdot g(9) - 0\cdot g(0)] - 27.6\right)$$

$$= \frac{1}{2}(9\cdot 12 - 27.6) = 40.2$$

$2 : \begin{cases} 1 : \text{integration by parts} \\ 1 : \text{answer} \end{cases}$

Scoring Notes

- No evidence of integration by parts: no points earned.

- First point: indefinite integration by parts.
  Bounds are associated with the answer point.
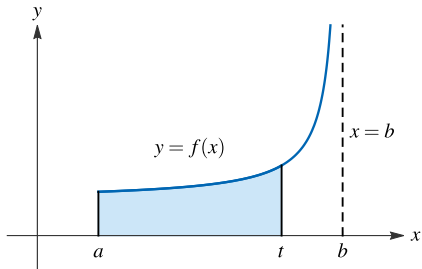
- Minimal work for first point: $9\cdot 12 - 0 - 27.6$

(k) Find the value of $\displaystyle\int_0^9 \frac{g''(x)}{g'(x)}\, dx$ or show that it does not exist.

Key Concepts

Discontinuous Integrand

Suppose f is a positive continuous function defined on a finite interval $[a, b)$.

The graph of f has a vertical asymptote at b.

$$\int_a^b f(x)\, dx = \lim_{t \to b^-} \int_a^t f(x)\, dx$$

Improper Integral of Type 2

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\, dx = \lim_{t \to b^-} \int_a^t f(x)\, dx$$

if this limit exists as a finite number.

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\, dx = \lim_{t \to a^+} \int_t^b f(x)\, dx$$

if this limit exists as a finite number.

The improper integral $\int_a^b f(x)\, dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$ and both $\int_a^c f(x)\, dx$ and $\int_c^b f(x)\, dx$ are convergent, then we define

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$

Solution

$$\int_0^9 \frac{g''(x)}{g'(x)} \, dx = \lim_{t \to 9^-} \int_0^t \frac{g''(x)}{g'(x)} \, dx \qquad \text{Improper integral}$$

$$= \lim_{t \to 9^-} \left[\ln |g'(x)| \right]_0^t \qquad \text{Basic antidifferentiation formula}$$

$$= \lim_{t \to 9^-} \left[\ln |g'(t)| - \ln |g'(0)| \right] \qquad \text{Use bounds}$$

$$= \lim_{t \to 9^-} \ln |g'(t)| = -\infty \qquad \text{As } u \to 0^+, \ln u \to -\infty$$

$\int_0^9 \dfrac{g''(x)}{g'(x)} \, dx$ diverges

TEXAS INSTRUMENTS

Scoring Guidelines

(k) $\displaystyle\int_0^9 \frac{g''(x)}{g'(x)}\,dx = \lim_{t\to 9^-}\int_0^t \frac{g''(x)}{g'(x)}\,dx$

Let $u = g'(x)$, then $du = g''(x)\,dx$ and $dx = \dfrac{du}{g''(x)}$

$$\int \frac{g''(x)}{g'(x)}\,dx = \int \frac{g''(x)}{u}\cdot\frac{du}{g''(x)} = \int \frac{du}{u}$$

$$= \ln|u| = \ln|g'(x)|$$

$$\lim_{t\to 9^-}\int_0^9 \frac{g''(x)}{g'(x)}\,dx = \lim_{t\to 9^-}\Big[\ln g'(x)\Big]_0^t$$

$$= \lim_{t\to 9^-}[\ln g'(t) - \ln g'(0)]$$

$$= \lim_{t\to 9^-}\ln g'(t) = -\infty$$

Therefore the improper integral does not exist.

$3 : \begin{cases} 1 : \text{improper integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

Scoring Notes

- No evidence of explicit substitution needed.
  Absolute value not necessary here.

- If error in the limits with explicit substitution: cannot earn the answer point.

- If no improper integral: max 0 - 1 - 0

- Limit notation must be used in order to earn the improper integral point.

- Must earn the improper integral point to be eligible for the answer point.

- Can earn the antiderivative point working only with an indefinite integral.

(l) If $g''(0) = 0.7$, find the second degree Taylor polynomial for g about $x = 0$.

Key Concepts

Power Series

If the function f has a power series expansion at a, then it must be of the form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} \, (x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!} \, (x-a) + \frac{f''(a)}{2!} \, (x-a)^2 + \frac{f'''(a)}{3!} \, (x-a)^3 + \cdots$$

This series is called the **Taylor series of the function f at a**
(or **about a** or **centered at a**).

For the special case $a = 0$ the Taylor series becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots$$

This special case is called a **Maclaurin series**.

Solution

$$g(0) = -7, \quad g'(0) = 1, \quad g''(0) = 0.7$$

$$T_2(x) = g(0) + \frac{g'(0)}{1!}x + \frac{g''(0)}{2!}x^2$$

$$= -7 + 1 \cdot x + \frac{0.7}{2}x^2$$

$$= -7 + x + 0.35\,x^2$$

Scoring Guidelines

(I) $g(0) = -7, \quad g'(0) = 1, \quad g''(0) = 0.7$

$$T_2(x) = g(0) + g'(0)x + \frac{g''(0)}{2!}x^2$$

$$= -7 + 1 \cdot x + \frac{0.7}{2}x^2$$

$$= -7 + x + 0.35\,x^2$$

$$2 : \begin{cases} 1 : \text{form of } T_2(x) \\ 1 : \text{answer} \end{cases}$$

Scoring Notes

- First point: correct form of the coefficients and powers of x.

- Only polynomials centered at $x = 0$ are eligible for any points.

- Including terms of degree greater than 2 or adding $+\cdots$ to the Taylor polynomial presented does not earn the second point.

- Simplification error: does not earn the second point.

- We will accept any naming of this polynomial, even $g(x)$.

  However, cannot equate the presented polynomial with $g(0)$ or $T_2(0)$.

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