TI in Focus: AP® Calculus

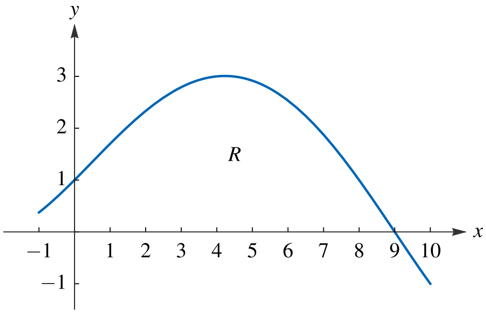
2020 Mock AP® Calculus Exam
BC-1: Solutions, Concepts, and Scoring Guidelines

Stephen Kokoska
Professor, Bloomsburg University
Former AP® Calculus Chief Reader

Texas Instruments

BC 1

The graph of g', the derivative of a twice-differentiable function g is shown in the figure. The graph has exactly one horizontal tangent line in the interval -1 to 10, at $x = 4.2$.



Graph of g'

R is the region in the first quadrant bounded by the graph of g' and the x-axis from $x = 0$ to $x = 9$. It is known that $g(0) = -7$, $g(9) = 12$, and $\int_0^9 g(x)\,dx = 27.6$.

(f) Evaluate $\int_0^9 \left[\frac{1}{2}g(x) - \sqrt{x}\right] dx$. Show the computations that lead to your answer.

Key Concepts

Properties of the Definite Integral

1. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx,$ where c is any constant.

2. $\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

Antidifferentiation Rule

If $f(x) = x^n$, $n \neq -1$, then an antiderivative of f is $F(x) = \frac{x^{n+1}}{n+1}$.

Solution

$$\int_0^9 \left[\frac{1}{2}g(x) - \sqrt{x}\right] dx = \int_0^9 \frac{1}{2}g(x)\,dx - \int_0^9 \sqrt{x}\,dx$$ Property of Definite Integrals

$$= \frac{1}{2}\int_0^9 g(x)\,dx - \int_0^9 x^{1/2}\,dx$$ Property of Definite Integrals

$$= \frac{1}{2}(27.6) - \left[\frac{2}{3}x^{3/2}\right]_0^9$$ Given; antidifferentiation rule

$$= 13.8 - \frac{2}{3}[(9^{3/2} - 0^{3/2}]$$ FTC

$$= 13.8 - 18 = -4.2$$ Simplify

Scoring Guidelines

(f) $$\int_0^9 \left[\frac{1}{2}g(x) - \sqrt{x}\right] dx = \frac{1}{2}\int_0^9 g(x)\,dx - \int_0^9 \sqrt{x}\,dx$$
$$= \frac{1}{2}(27.6) - \left[\frac{2}{3}x^{3/2}\right]_0^9$$
$$= 13.8 - \frac{2}{3}(27)$$
$$= 13.8 - 18 = -4.2$$

$$3: \begin{cases} 1 : \text{properties of definite integrals} \\ 1 : \text{antiderivative of } \sqrt{x} \\ 1 : \text{answer} \end{cases}$$

Scoring Notes

- The first point requires demonstration of both the difference of integrals and the multiplication by a constant.
- The second point is for correct use of the antidifferentiation rule.
- Minimal work: $13.8 - \frac{2}{3}(27)$

(g) Evaluate $\lim_{x\to 0} \dfrac{x\cos x}{g(x)+2x-1}$. Show the computations that lead to your answer.

Key Concepts

L'Hospital's Rule

Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval that contains a (except possibly at a). Suppose that

$$\lim_{x\to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x\to a} g(x) = 0$$

or that

$$\lim_{x\to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x\to a} g(x) = \pm\infty$$

That is, if the limit $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ is in an indeterminate form of type $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$.

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

A Closer Look

1. L'Hospital's Rule says that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives, provided that the given conditions are met.

   It is very important to verify the conditions regarding the limits of f and g before using L'Hospital's Rule. That means, you can only apply this rule if the limit is in an indeterminate form of type $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$.

2. L'Hospital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity. That means $x \to a$ can be replaced by any of the symbols $x \to a^+$, $x \to a^-$, $x \to \infty$, or $x \to -\infty$.

3. This is an amazing technique, but many limits may be determined using other, often simpler, methods.

4. L'Hospital's Rule may be used repeatedly to evaluate a limit in an appropriate indeterminate form.

5. There are other indeterminate forms:

   $0 \cdot \infty, \quad \infty - \infty, \quad 0^0, \quad \infty^0, \quad 1^\infty$

Key Concepts

The Product Rule

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\,\frac{d}{dx}[g(x)] + \frac{d}{dx}[f(x)]\,g(x)$$

A Closer Look

1. In words: the derivative of the product of two functions is the first function times the derivative of the second function plus the derivative of the first function times the second function.
2. Using prime notation: $(fg)' = fg' + f'g$
3. The product rule can be extended:
   $(fgh)' = fgh' + fg'h + f'gh$

Solution

$$\begin{aligned}\lim_{x\to 0}(x\cos x) &= \lim_{x\to 0} x \cdot \lim_{x\to 0}\cos x \\ &= 0 \cdot 1 = 0\end{aligned}$$

$$\begin{aligned}\lim_{x\to 0}[g(x)+2x-1] &= \lim_{x\to 0} g(x) + 2\lim_{x\to 0} x - \lim_{x\to 0} 1 \\ &= g(0) + 2\cdot 0 - 1 = 1 + 0 - 1 = 0\end{aligned}$$

The limit $\displaystyle\lim_{x\to 0}\frac{x\cos x}{g(x)+2x-1}$ is in an indeterminate form of type $\dfrac{0}{0}$.

We can use L'Hospital's Rule.

Solution

$$\lim_{x\to 0} \frac{x\cos x}{g(x)+2x-1} = \lim_{x\to 0} \frac{x\cdot(-\sin x)+1\cdot\cos x}{g'(x)+2}$$ L'Hospital's Rule; Product Rule

$$= \frac{0\cdot(-\sin 0)+1\cdot\cos 0}{g'(0)+2}$$ Continuity

$$= \frac{0+1}{1+2} = \frac{1}{3}$$ Simplify

Scoring Guidelines

(g) $\lim_{x\to 0}(x\cos x) = 0$

$\lim_{x\to 0}(g(x) + 2x - 1) = 0$

Therefore the limit $\lim_{x\to 0} \dfrac{x\cos x}{g(x) + 2x - 1}$ is in the indeterminate form $\dfrac{0}{0}$ and L'Hospital's Rule can be applied.

$$\lim_{x\to 0} \frac{x\cos x}{g(x) + 2x - 1} = \lim_{x\to 0} \frac{x\cdot(-\sin x) + 1\cdot\cos x}{g'(x) + 2}$$

$$= \frac{0\cdot(-\sin 0) + 1\cdot\cos 0}{g'(0) + 2} = \frac{1}{3}$$

$$3 : \begin{cases} 1 : \text{conditions for L'Hospital's Rule} \\ 1 : \text{applies L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$$

Scoring Notes

- The first point is earned for conveying that both the numerator and the denominator each approach 0.
- The response does not need to reference continuity or differentiability.
- Any notation in which limit $= \frac{0}{0}$ does not earn the first point.

  This can appear in *disguise*.
- Other presentations are OK: $\to \frac{0}{0}$, $=$ indeterminate form $\frac{0}{0}$.
- The second point is earned by presenting a limit of a fraction whose numerator and denominator are clearly attempts at derivatives.
- The second point may be earned even if the limit notation is missing from some expressions.

  But the second point cannot be earned unless the response conveys a limit.
- The third point is earned for the correct answer $\dfrac{1}{3}$.

(h) Let h be the function defined by $h(x) = \int_{x^2}^{0} g(t)\,dt$. Find $h'(3)$. Show the computations that lead to your answer.

Key Concepts

The Fundamental Theorem of Calculus, Part 1

If f is a continuous function on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt, \quad a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

A Closer Look

1. The derivative of a definite integral with respect to its upper limit is the integrand evaluated at the upper limit.
2. Other notation: $\dfrac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$

   Integration and differentiation are inverse operations.

Key Concepts

The Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and $F'(x)$ is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

A Closer Look

1. In words: the derivative of the composition of two functions is the product of the derivative of the *outer* function evaluated at the *inner* function and of the derivative of the inner function.
2. Other notation: $(f \circ g)' = f'(g(x)) \cdot g'(x)$
3. And more notation: $\dfrac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$
4. Given $(f \circ g)(x)$, here is a procedure for finding $(f \circ g)'(x)$ using the Chain Rule.
   (a) Identify $f(x)$ and $g(x)$.
   (b) Find $f'(x)$ and $g'(x)$.
   (c) Write the final answer as $f'(g(x))\, g'(x)$.
5. Chain Rule: differentiate the outer function f [at the inner function $g(x)$] and then we multiply by the derivative of the inner function.

$$\frac{d}{dx} \underbrace{f}_{\text{outer function}} \underbrace{(g(x))}_{\text{evaluated at inner function}} = \underbrace{f'}_{\text{derivative of outer function}} \underbrace{(g(x))}_{\text{evaluated at inner function}} \cdot \underbrace{g'(x)}_{\text{derivative of inner function}}$$

Solution

$$h'(x) = \frac{d}{dx}\left[\int_{x^2}^{0} g(t)\,dt\right] \qquad \text{Derivative of } h$$

$$= -\frac{d}{dx}\left[\int_{0}^{x^2} g(t)\,dt\right] \qquad \text{Property of definite integrals}$$

$$= -\frac{d}{dx}\left[\int_{0}^{u} g(t)\,dt\right] \qquad \text{Let } u = x^2$$

$$= -\frac{d}{du}\left[\int_{0}^{u} g(t)\,dt\right]\frac{du}{dx} \qquad \text{Chain Rule}$$

$$= -g(u)\cdot\frac{du}{dx} \qquad \text{FTC Part 1}$$

$$= -g(x^2)\cdot(2x) = -2xg(x^2) \qquad \text{Use } u = x^2$$

$$h'(3) = -2\cdot 3\cdot g(9) = -6\cdot 12 = -72$$

Scoring Guidelines

(h) $h'(x) = \dfrac{d}{dx}\left[\int_{x^2}^{0} g(t)\,dx\right]$

$$= -\frac{d}{dx}\left[\int_{0}^{x^2} g(t)\,dt\right]$$

$$= -g(x^2)\cdot(2x) = -2xg(x^2)$$

$$h'(3) = -2\cdot 3\cdot g(9) = -6\cdot 12 = -72$$

$$3:\begin{cases} 1: \text{Fundamental Theorem of Calculus} \\ 1: \text{Chain Rule} \\ 1: \text{answer} \end{cases}$$

Scoring Notes

FTC and Chain Rule Points

- These two points may be awarded in either order, or simultaneously.
- Need to see $g(x^2)$ for the FTC point.
  Need to see $2x$ for the Chain Rule point.

  If $g(0)$ is part of the derivative of h or included in the calculation of $h'(3)$: FTC point not earned.
- Examples

$h'(x) = g(x^2) \cdot 2x$	Earns the FTC point and the Chain Rule point
$g(x^2)$	Earns the FTC point but not the Chain Rule point
$g(x^2) - g(0)$	Does not earn either point
$g(x^2) \cdot 2x - g(0)$	Earns the Chain Rule point but not the FTC point

Third Point

- Eligibility: must earn the first two points.
- Must be our answer.
- Alternate version of FTC: $g(x^2) \cdot 2x - g(0) \cdot 0$

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