TI in Focus: AP® Calculus

2020 Mock AP® Calculus Exam

BC-1: Solutions, Concepts, and Scoring Guidelines

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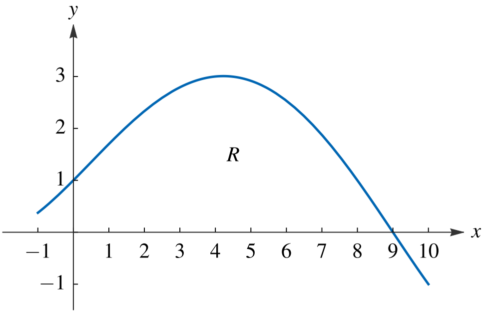
BC 1

The graph of g', the derivative of a twice-differentiable function g is shown in the figure. The graph has exactly one horizontal tangent line in the interval -1 to 10, at $x = 4.2$.



Graph of g'

R is the region in the first quadrant bounded by the graph of g' and the x-axis from $x = 0$ to $x = 9$. It is known that $g(0) = -7$, $g(9) = 12$, and $\int_0^9 g(x)\,dx = 27.6$.

(c) Find the area of the region R.

Key Concepts

Suppose $f(x) \geq 0$ for $a \leq x \leq b$ and f is continuous on $[a, b]$.

The definite integral $\int_a^b f(x)\,dx$ can be interpreted as the area of the region bounded above by the graph of $y = f(x)$, below by the x-axis, and between the lines $x = a$ and $x = b$.

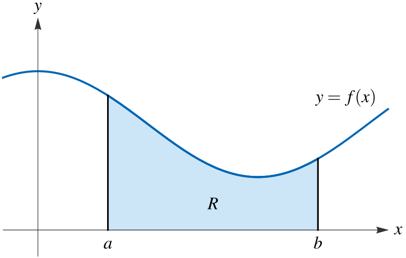
Or simply, the area under the curve $y = f(x)$ from a to b.

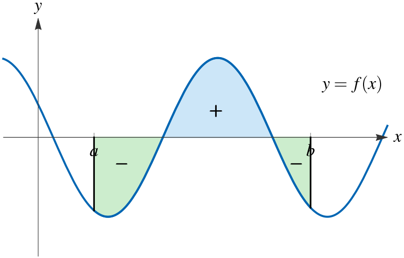
If f takes on both positive and negative values over the interval $[a, b]$, then the definite integral $\int_a^b f(x)\,dx$ can be interpreted as a net area.

That is, $\int_a^b f(x)\,dx = A_1 - A_2$

where A_1 is the area of the region above the x-axis and below the graph of f, and A_2 is the area of the region below the x-axis and above the graph of f.

Illustration:





$\displaystyle\int_a^b f(x)\,dx$ is the area under the curve $y = f(x)$, from a to b.

$\displaystyle\int_a^b f(x)\,dx$ is the net area.

The Fundamental Theorem of Calculus, Part 2

If f is a continuous on $[a, b]$, then

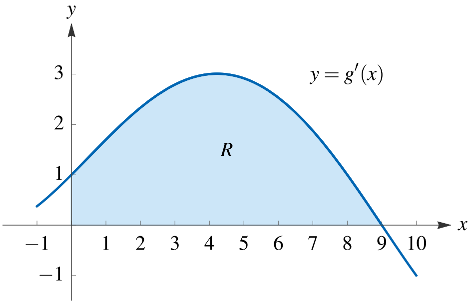
$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

Note:

This theorem says that the value of $\displaystyle\int_a^b f(x)\,dx$ can be obtained by finding an antiderivative of F of the integrand f, then subtract: $F(b) - F(a)$

Solution



$$\text{Area} = \int_0^9 g'(x)\,dx$$ $g'(x) \geq 0$ on $[0, 9]$

$$= \Big[g(x)\Big]_0^9 = g(9) - g(0)$$ Fundamental Theorem of Calculus

$$= 12 - (-7) = 19$$ Use given values

Scoring Guidelines

(c) Area $= \displaystyle\int_0^9 g'(x)\,dx = \Big[g(x)\Big]_0^9$

$= g(9) - g(0) = 12 - (-7) = 19$

$$3: \begin{cases} 1 : \text{definite integral for area} \\ 1 : \text{Fundamental Theorem of} \\ \quad\;\; \text{Calculus} \\ 1 : \text{answer} \end{cases}$$

Scoring Notes

First point:

- First point is earned for the correct presentation of the definite integral that represents the area of the region R.
- Incorrect bounds: does not earn the point.
- Must be a definite integral: $\displaystyle\int g'(x)\,dx$ does not earn the point.

  Still eligible for second and third points.
- Notation issues: $\displaystyle\int_0^9 g'(x) \qquad \int_0^9 g'(t)\,dx$

Scoring Notes

Second point:

- The second point is earned for correctly applying the Fundamental Theorem of Calculus.

  Examples:

  - $\int_0^9 g'(x)\,dx = g(x)\Big]_0^9$
  - $g(x)\Big]_0^9$
  - $g(9) - g(0)$

- $\int_0^9 g(x)\,dx = g(x)\Big]_0^9 = g(9) - g(0)$ Does not earn the FTC point.

Scoring Notes

Third point:

- The response must use the values of $g(9)$ and $g(0)$ to earn this point.

- Examples:

$g(9) - g(0) = 12 - (-7)$	1 - 1 - 1
$12 + 7$	0 - 0 - 1
19	0 - 0 - 0
$\int g'(x)\,dx = g(9) - g(0) = 12 + 7 = 19$	0 - 1 - 1

(d) Write an expression that represents the perimeter of the region R. Do not evaluate this expression.

Key Concepts

- The Arc Length Formula

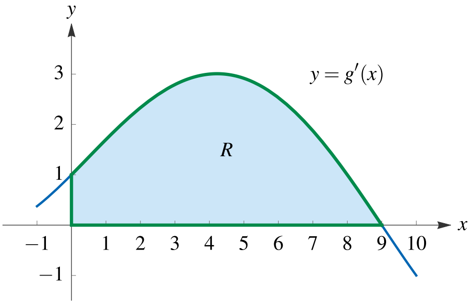
  If f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$, $a \leq x \leq b$, is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

- Using Leibniz notation:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Solution



P $=$ (Length along y-axis) $+$ (Length along x-axis) $+$ (Arc length from 0 to 9)

$$= 1 + 9 + \int_0^9 \sqrt{1 + g''(x)^2}\, dx$$

Scoring Guidelines

(d) $P = 1 + 9 + \int_0^9 \sqrt{1 + g''(x)^2}\,dx$

$2: \begin{cases} 1 : \text{definite integral} \\ 1 : \text{answer} \end{cases}$

Scoring Notes

- The first point is for the correct definite integral.
- The second point is for the correct expression for the perimeter.
- Indefinite integral: 0 points earned.
- Incorrect definite integral: 0 points earned.
- Eligibility: must earn the first point to be eligible for the second point.

(e) Must there exist a value of c, for $0 < c < 9$, such that $g(c) = 0$? Justify your answer.

Key Concepts

The Intermediate Value Theorem

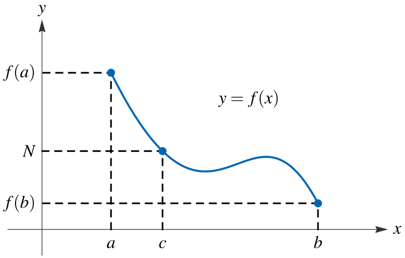
Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

A Closer Look

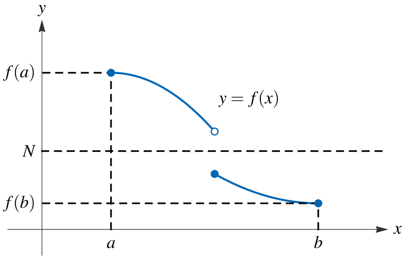
1. Interpretation: f takes on every value between $f(a)$ and $f(b)$.
2. There is *at least* on c. There may be more than one.
3. This is an existence theorem.

A Closer Look

4. The conclusion of the IVT is not necessarily true if the function is discontinuous anywhere in the closed interval.



f is continuous on the closed interval $[a, b]$. f takes on every value between $f(a)$ and $f(b)$.



f is discontinuous. There is no number c in (a, b) such that $f(c) = N$.

Solution

g is differentiable. Therefore g is continuous on the interval $[0, 9]$.

$g(0) = -7 < 0 < 12 = g(9)$

There exists a value c in $(0, 9)$ such that $g(c) = 0$ by the IVT.

Scoring Guidelines

(e) Since g is differentiable, then g is continuous on $0 \le x \le 9$. $g(0) = -7 < 0 < 12 = g(9)$ By the Intermediate Value Theorem, there exists a value of c, for $0 < c < 9$, such that $g(c) = 0$.	$2: \begin{cases} 1 : \text{conditions} \\ 1 : \text{conclusion using the Intermediate Value Theorem} \end{cases}$

Scoring Notes

- The student must establish that g is a continuous function.
  Common response: differentiable implies continuous.
- The student must convey that differentiability *implies* continuity.
  Stating g is differentiable *and* continuous, does not earn the first point.
- If continuity is mentioned but not established: eligible for the second point.
- Citing the MVT does not earn the second point.
- To earn the second point: must explicitly convey an appropriate inequality involving 0.
  Can be presented mathematically or in words.
  Examples: $-7 < 0 < 12$ or $g(0) < 0 < g(9)$
- Second point requires an answer of *yes*, or an equivalent statement.

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