

## TI in Focus: AP<sup>®</sup> Calculus

2020 Mock AP<sup>®</sup> Calculus Exam

AB-2: Solutions, Concepts, and Scoring Guidelines

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## AB 2

$t$	0	2	6	8	10	12
$y'(t)$	4	8	-2	3	-1	-5

The vertical position of a particle moving along the  $y$ -axis is modeled by a twice-differentiable function  $y(t)$  where  $t$  is measured in seconds and  $y(t)$  is measured in meters.

Selected values of  $y'(t)$ , the derivative of  $y(t)$ , over the interval  $0 \leq t \leq 12$  seconds are shown in the table above.

The position of the particle at time  $t = 12$  is  $y(12) = -3$ .

- (g) Explain why there must be at least three times  $t$  in the interval  $0 < t < 12$  such that  $y'(t) = 0$ .

## Key Concepts

### Theorem

If  $f$  is differentiable at  $a$ , then  $f$  is continuous at  $a$ .

### Intermediate Value Theorem

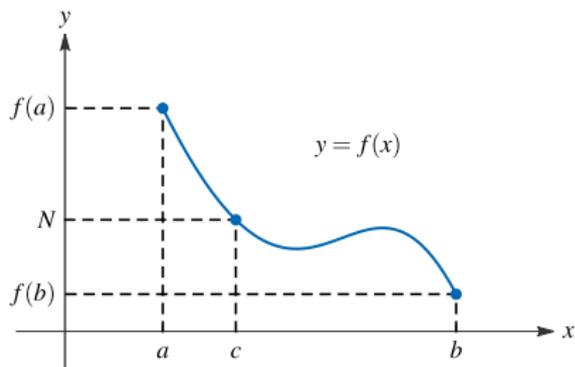
Suppose that  $f$  is continuous on the closed interval  $[a, b]$  and let  $N$  be any number between  $f(a)$  and  $f(b)$ , where  $f(a) \neq f(b)$ . Then there exists a number  $c$  in  $(a, b)$  such that  $f(c) = N$ .

### A Closer Look

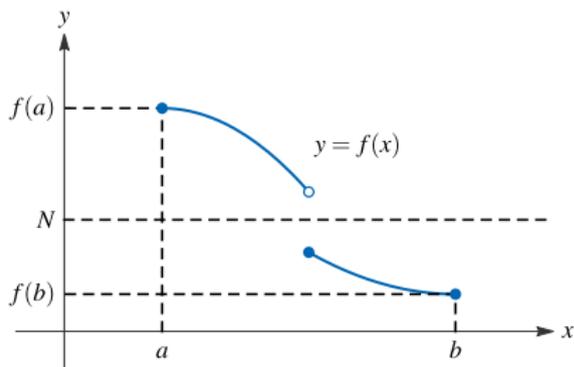
1. Interpretation:  $f$  takes on every value between  $f(a)$  and  $f(b)$ .
2. There is *at least* one  $c$ . There may be more than one.
3. This is an existence theorem.

## A Closer Look

4. The conclusion of the IVT is not necessarily true if the function is discontinuous anywhere in the closed interval.



$f$  is continuous on the closed interval  $[a, b]$ .  $f$  takes on every value between  $f(a)$  and  $f(b)$ .



$f$  is discontinuous. There is no number  $c$  in  $(a, b)$  such that  $f(c) = N$ .

## Solution

$y'$  is differentiable. Therefore  $y'$  is continuous on the interval  $[0, 12]$ .

$$y'(2) = 8 > 0 > -2 = y'(6)$$

There exists a value  $c_1$  in  $(2, 6)$  such that  $y'(c_1) = 0$  by the IVT.

$$y'(6) = -2 < 0 < 3 = y'(8)$$

There exists a value  $c_2$  in  $(6, 8)$  such that  $y'(c_2) = 0$  by the IVT.

$$y'(8) = 3 > 0 > -1 = y'(10)$$

There exists a value  $c_3$  in  $(8, 10)$  such that  $y'(c_3) = 0$  by the IVT.

Therefore, there are at least three times  $t$  in the interval  $(0, 12)$  such that  $y'(t) = 0$ .

## Scoring Guidelines

(g)  $y$  is twice differentiable  $\Rightarrow y'$  is differentiable  
 $\Rightarrow y'$  is continuous over the interval  $[0, 12]$ .

$y'(t)$  changes from positive to negative on the interval  $[2, 6]$ .

$y'(t)$  changes from negative to positive on the interval  $[6, 8]$ .

$y'(t)$  changes from positive to negative on the interval  $[8, 10]$ .

By the Intermediate Value Theorem, there must be times  $a$ ,  $b$ , and  $c$ , such that  $2 < a < 6 < b < 8 < c < 10$ , and  $y'(a) = y'(b) = y'(c) = 0$

2:  $\left\{ \begin{array}{l} 1 : \text{conditions} \\ 1 : \text{conclusion using the Intermediate} \\ \quad \text{Value Theorem} \end{array} \right.$

## Scoring Notes

1: conditions

- The student must establish that  $y'$  is a continuous function.  
Common response: differentiable implies continuous.
- The student must convey that differentiability *implies* continuity.  
Stating  $y'$  is differentiable *and* continuous, does not earn the first point.
- If continuity is mentioned but not established: eligible for the second point.
- Citing the MVT does not earn the second point.

1: conclusion using the Intermediate Value Theorem

- To earn the second point: must explicitly convey appropriate inequalities (3) involving 0, either mathematically or in words.  
Examples:  $8 > 0 > -2$  or  $y'(2) > 0 > y'(6)$
- Second point requires a definitive conclusion.

- (h) Explain why there must be at least two times  $t$  in the interval  $0 < t < 12$  such that  $y''(t) = 0$ .

## Key Concepts

### Theorem

If  $f$  is differentiable at  $a$ , then  $f$  is continuous at  $a$ .

### The Mean Value Theorem

Let  $f$  be a function that satisfies the following hypotheses:

- (1)  $f$  is continuous on the closed interval  $[a, b]$ .
- (2)  $f$  is differentiable on the open interval  $(a, b)$ .

Then there is a number  $c$  in  $(a, b)$  such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or equivalently,

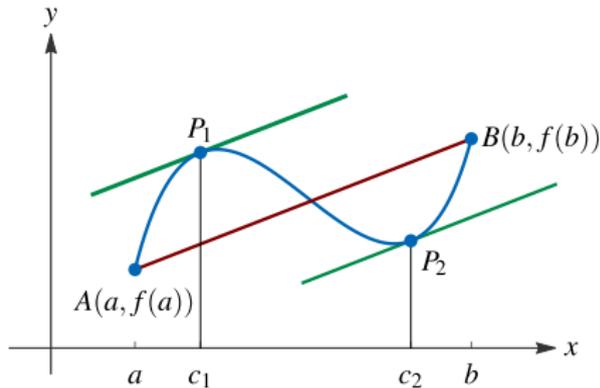
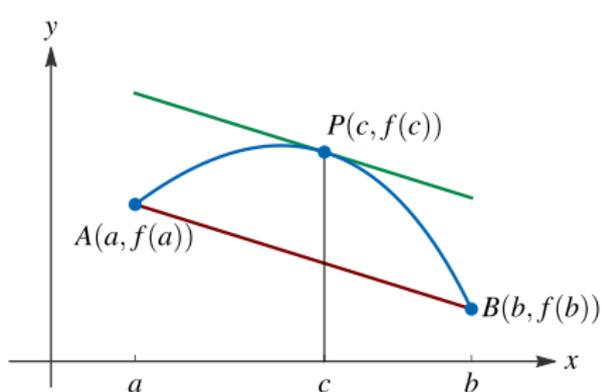
$$f(b) - f(a) = f'(c)(b - a)$$

## A Closer Look

1. The Mean Value Theorem is an existence theorem.
2. Rolle's Theorem is a special case of the MVT. If  $f(a) = f(b)$  then

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{0}{b - a} = 0$$

3. Geometric interpretation:



## Solution

$y'$  is differentiable. Therefore  $y'$  is continuous on the interval  $[0, 12]$ .

Using the result from part (g):

There exist three times  $t = a$ ,  $t = b$ , and  $t = c$  such that

$$0 < a < b < c < 12 \quad \text{and} \quad y'(a) = y'(b) = y'(c) = 0$$

$$\frac{y'(b) - y'(a)}{b - a} = \frac{0}{b - a} = 0 \quad \text{and} \quad \frac{y'(c) - y'(b)}{c - b} = \frac{0}{c - b} = 0$$

There must be a time  $t = t_1$  such that  $a < t_1 < b$  and  $y''(t_1) = 0$  by the MVT.

There must be a time  $t = t_2$  such that  $b < t_2 < c$  and  $y''(t_2) = 0$  by the MVT.

Therefore, there are at least two times  $t$  in the interval  $(0, 12)$  such that  $y''(t) = 0$ .

## Scoring Guidelines

- (h)  $y$  is twice differentiable  $\Rightarrow y'$  is differentiable  
 $\Rightarrow y'$  is continuous over the interval  $[0, 12]$ .

From part (g): there are times  $t = a$ ,  $t = b$ , and  $t = c$   
such that  $0 < a < b < c < 12$  and  
 $y'(a) = y'(b) = y'(c) = 0$ .

$$\frac{y'(b) - y'(a)}{b - a} = 0 \quad \text{and} \quad \frac{y'(c) - y'(b)}{c - b} = 0$$

By the Mean Value Theorem (or Rolle's Theorem) there  
must be a time  $t = t_1$  such that  $a < t_1 < b$  and  
 $y''(t_1) = 0$  and a time  $t = t_2$  such that  $b < t_2 < c$  and  
 $y''(t_2) = 0$

- 2:  $\left\{ \begin{array}{l} 1 : \text{conditions} \\ 1 : \text{conclusion using the Mean Value} \\ \quad \text{Theorem} \end{array} \right.$

## Scoring Notes

### 1: conditions

- The student must establish that  $y'$  is a continuous function. The most common, and acceptable, response is *differentiable implies continuous*.
- The student must convey that differentiability implies continuity. Stating  $y'$  is differentiable *and* continuous, does not earn the first point.
- The response that mentions continuity without establishing continuity is still eligible for the second point.

### 1: conclusion using the Mean Value Theorem

- The second point requires a definitive conclusion.
- Can earn the second point without stating MVT.

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