TI in Focus: AP® Calculus

2019 AP® Calculus Exam: AB-5
Volume of a Solid with Known Cross-Section

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Outline

(1) Background; motivation

(2) Derivation

(3) Definition

(4) Examples

Introduction

(1) There are often area/volume questions on the AP Calculus Exam. Many involve a part that asks for the volume of a solid with known cross-sections.

(2) Objective: Understand the formula for the volume of a solid with known cross-sections, and use it to find the volume.

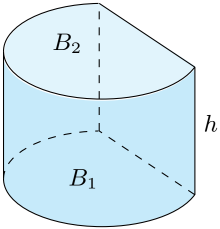
(3) Prerequsites:

   (a) Definition of a definite integral in terms of a Riemann sum.

   (b) The Fundamental Theorem of Calculus

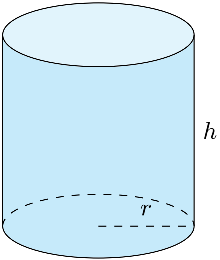
Right Cylinder

- Consider a cylinder bounded below by a plane region B_1: **base**.
- Bounded above by a congruent region B_2 in a parallel plane.
- Cylinder consists of all points on line segments that are perpendicular to the base and join B_1 to B_2.
- Suppose the area of the base is A, and the height is h.
- Volume of the cylinder is defined as $V = A\,h$.



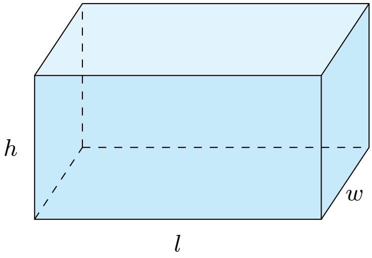
Right Circular Cylinder

- Suppose the base is a circle with radius r.
- The cylinder is a circular cylinder.
- Volume is $V = \pi r^2 h$



Rectangular Box

- Suppose the base is a rectangle with length l and width w.
- The cylinder is a rectangular box (rectangular parallelepiped).
- Volume is $V = lwh$.



A Solid that is not a Cylinder

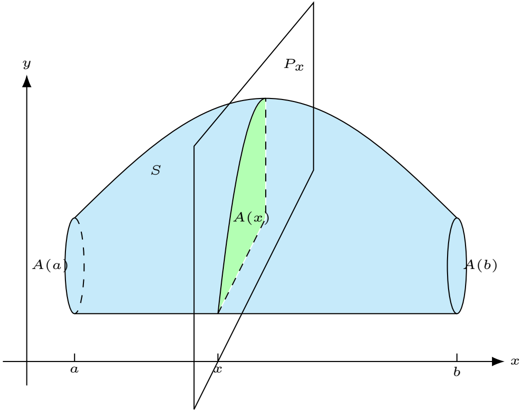
- Cut S into pieces and approximate each piece by a cylinder.
- Estimate the volume of S by adding the volumes of the cylinders.
- Consider the limiting process in which the number of pieces becomes large.
- This allows us to find the exact volume of the solid.

Derivation

(1) Consider the solid S. Pass a plane perpendicular to the x-axis through S.

(2) Consider the intersection of the plane and S, a plane region called a cross-section of S.

(3) Let $A(x)$ be the area of the cross-section of S in the plane P_x perpendicular to the x-axis and passing through the point x, for $a \leq x \leq b$.



Derivation (Continued)

(4) Divide S into n slabs of equal width Δx.
Think of slicing a loaf of bread.

(5) Choose points x_i^* in $[x_{i-1}, x_i]$ and approximate the volume of the ith slice S_i by a cylinder with base $A(x_i^*)$ and height Δx.

(6) Volume of the ith cylinder is $A(x_i^*)\,\Delta x$.

(7) The volume of the ith slice is approximately $A(x_i^*)\,\Delta x$.

(8) Add the volumes of these slices: $V \approx \sum_{i=1}^{n} A(x_i^*)\,\Delta x$

Definition

Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x through x and perpendicular to the x-axis is $A(x)$, where A is a continuous function, then the volume of S is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\,\Delta x = \int_a^b A(x)\,dx$$

A Closer Look

(1) $A(x)$ is the area of a moving cross-section obtained by slicing through S perpendicular to the x-axis.

(2) If S is a cylinder, the cross-sectional area is constant: $A(x) = A$.

$$V = \int_a^b A\,dx = A(b-a)$$

This agrees with the intuitive formula: $V = A\,h$.

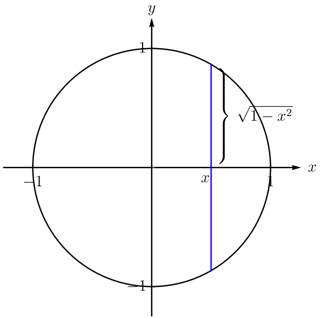
Example 1 Fair and Square

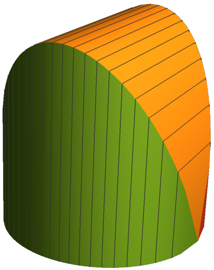
The base of a solid S is a circle of radius 1. For this solid, the cross-sections perpendicular to the x-axis are squares. Find the volume of this solid.

Solution

Position the circle in the xy-plane, centered at the origin.

The plane P_x intersects the solid in a square with side $2\sqrt{1-x^2}$.





Solution (Continued)

Cross-sectional area: $A(x) = \left[2\sqrt{1-x^2}\right]^2 = 4(1-x^2)$

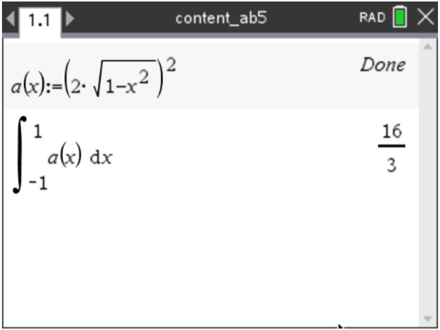
$$V = \int_{-1}^{1} A(x)\,dx = \int_{-1}^{1} 4(1-x^2)\,dx$$ Definition of volume

$$= 8\int_0^1 (1-x^2)\,dx$$ Integrand is an even function

$$= 8\left[x - \frac{x^3}{3}\right]_0^1$$ Basic antidifferentiation rules

$$= 8\left[\left(1-\frac{1}{3}\right) - (0)\right] = \frac{16}{3}$$ FTC; simplify

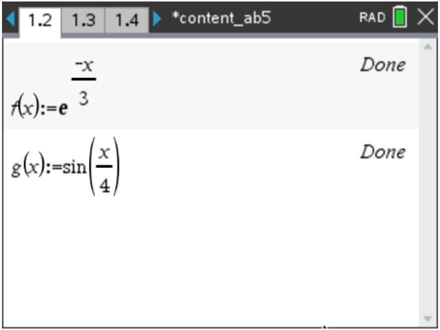
Technology Solution

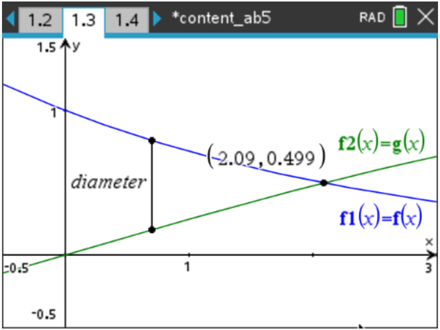


Example 2 Semicircles as Cross Sections

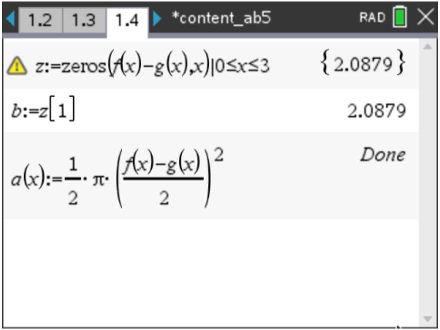
The region R in the plane, bounded above by the graph of $f(x) = e^{-x/3}$, below by the graph of $g(x) = \sin(x/4)$, and on the left by the y-axis, is the base of a solid. For this solid, the cross-sections perpendicular to the x-axis are semicircles with diameters extending from $y = f(x)$ to $y = g(x)$. Find the volume of the solid.

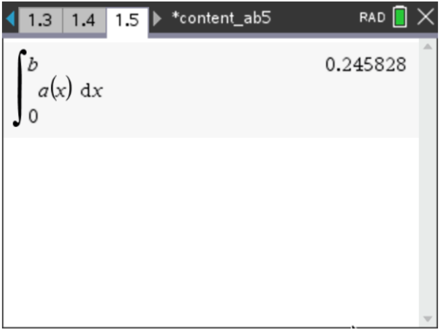
Solution





Solution (Continued)





Example 3 Various Bases and Cross-Sections

(a) The region R in the plane, bounded above by the graph of $f(x) = \sin(2x) + 2$, below by the graph of $g(x) = -\sin(2x) - 2$, on the left by the y-axis, and on the right by the line $x = 2\pi$, is the base of a solid. For this solid, the cross-sections perpendicular to the x-axis are equilateral triangles. Find the volume of the solid.

(b) The region R in the plane, bounded above by the graph of $f(x) = x + 1$ and below by the graph of $g(x) = x^2 - 2x$ is the base of a solid. For this solid, the cross-sections perpendicular to the x-axis are rectangles. The width of the rectangle, in the xy-plane, is twice the height. FInd the volume of the solid.

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