

TI in Focus: AP[®] Calculus

2019 AP[®] Calculus Exam: AB-4/BC-4

Technology Solutions and Problem Extensions

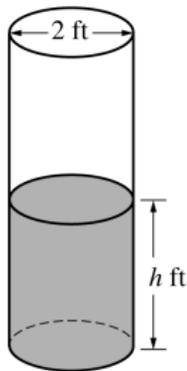
Stephen Kokoska

Professor, Bloomsburg University

Former AP[®] Calculus Chief Reader

Outline

- (1) Free Response Question
- (2) Scoring Guidelines
- (3) Solutions in greater detail
- (4) Solutions using technology
- (5) Problem Extensions



4. A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as shown in the figure above. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)

- (a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.
- (b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.
- (c) At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t .

$$(a) V = \pi r^2 h = \pi(1)^2 h = \pi h$$

$$\left. \frac{dV}{dt} \right|_{h=4} = \pi \left. \frac{dh}{dt} \right|_{h=4} = \pi \left(-\frac{1}{10} \sqrt{4} \right) = -\frac{\pi}{5} \text{ cubic feet per second}$$

$$(b) \frac{d^2 h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt} = -\frac{1}{20\sqrt{h}} \cdot \left(-\frac{1}{10} \sqrt{h} \right) = \frac{1}{200}$$

Because $\frac{d^2 h}{dt^2} = \frac{1}{200} > 0$ for $h > 0$, the rate of change of the height is increasing when the height of the water is 3 feet.

$$2: \begin{cases} 1: \frac{dV}{dt} = \pi \frac{dh}{dt} \\ 1: \text{answer with units} \end{cases}$$

$$3: \begin{cases} 1: \frac{d}{dh} \left(-\frac{1}{10} \sqrt{h} \right) = -\frac{1}{20\sqrt{h}} \\ 1: \frac{d^2 h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt} \\ 1: \text{answer with explanation} \end{cases}$$

$$(c) \frac{dh}{\sqrt{h}} = -\frac{1}{10} dt$$

$$\int \frac{dh}{\sqrt{h}} = \int -\frac{1}{10} dt$$

$$2\sqrt{h} = -\frac{1}{10}t + C$$

$$2\sqrt{5} = -\frac{1}{10} \cdot 0 + C \Rightarrow C = 2\sqrt{5}$$

$$2\sqrt{h} = -\frac{1}{10}t + 2\sqrt{5}$$

$$h(t) = \left(-\frac{1}{20}t + \sqrt{5}\right)^2$$

$$4 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ \quad \text{and uses initial condition} \\ 1 : h(t) \end{cases}$$

Note: 0/4 if no separation of variables

Note: max 2/4 [1-1-0-0] if no constant of integration

Part (a)

Solution

$$V = \pi r^2 h = \pi(1)^2 h = \pi h$$

$$\frac{dV}{dt} = \pi(1) \frac{dh}{dt} = \pi \frac{dh}{dt} = \pi \left(-\frac{1}{10} \sqrt{h} \right)$$

$$\left. \frac{dV}{dt} \right|_{h=4} = \pi \left(-\frac{1}{10} \sqrt{4} \right) = -\frac{\pi}{5} \text{ cubic feet per second}$$

Note:
$$\frac{dV}{dt} = \pi \left(2r \frac{dr}{dt} h + r^2 \frac{dh}{dt} \right)$$

Technology Solution

The screenshot shows a TI-84 Plus calculator interface with the following content:

- Top navigation: 1.1, 1.2, 1.3, *extensio...ab4, RAD, and a close button.
- Row 1: $v(t) := \pi \cdot r^2 \cdot h(t)$ on the left and *Done* on the right.
- Row 2: $\frac{d}{dt}(v(t))$ on the left and $\frac{d}{dt}(h(t)) \cdot r^2 \cdot \pi$ on the right.
- Row 3: $\pi \cdot r^2 \cdot \frac{-1}{10} \cdot \sqrt{h} \mid r=1 \text{ and } h=4$ on the left and $\frac{-\pi}{5}$ on the right.

Part (b)

$$\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$$

Represents the rate of change of the height of the water with respect to time.

$$\frac{d^2h}{dt^2} = -\frac{1}{10} \cdot \frac{1}{2}h^{-1/2} \cdot \frac{dh}{dt}$$

Power Rule and Chain Rule

$$= -\frac{1}{20\sqrt{h}} \cdot \left(-\frac{1}{10}\sqrt{h}\right)$$

Use expression for $\frac{dh}{dt}$

$$= \frac{1}{200}$$

Simplify

Part (b)

Note: $\frac{d^2h}{dt^2} = \frac{1}{200} > 0$ for all $h > 0$.

Therefore, the rate of change of the height is increasing when the height of the water is 3 feet.

The screenshot shows a TI-84 Plus calculator interface. At the top, the window title is '*extensio... ab4'. The mode is set to 'RAD'. The variable 'hprime(t)' is defined as $hprime(t) := \frac{-1}{10} \cdot \sqrt{h(t)}$. The calculator is displaying the derivative of $hprime(t)$ with respect to t , which is $\frac{d}{dt}(hprime(t)) = \frac{-\frac{d}{dt}(h(t))}{20 \cdot \sqrt{h(t)}}$. The final result shown is $\frac{-\frac{1}{10} \cdot \sqrt{h(t)}}{20 \cdot \sqrt{h(t)}}$, which simplifies to $\frac{1}{200}$. A yellow warning icon is visible on the left side of the screen.

Example 1 Height Remodeled

Suppose the rate of change of the height h of water in the barrel with respect to time t is modeled by $\frac{dh}{dt} = 2e^{-\sqrt{h}}$.

When the height of the water is 4 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Justify your answer.

Part (c)

Solve the initial value problem using the method of separation of variables.

$$\frac{dh}{\sqrt{h}} = -\frac{1}{10} dt$$

Separate variables

$$\int \frac{dh}{\sqrt{h}} = \int -\frac{1}{10} dt$$

Integrate both sides

$$\frac{h^{1/2}}{1/2} = 2\sqrt{h} = -\frac{1}{10} t + C$$

Basic antidifferentiation rules

$$2\sqrt{5} = -\frac{1}{10} \cdot 0 + C \implies C = 2\sqrt{5}$$

Use initial condition

$$2\sqrt{h} = -\frac{1}{10} t + 2\sqrt{5}$$

Use value for C

$$h(t) = \left(-\frac{1}{20} t + \sqrt{5} \right)^2$$

Solve for h in terms of t

Part (c)

Technology Solution

deSolve($h' = \frac{-1}{10} \cdot \sqrt{h}$ and $h(0) = 5, t, h$)

$$\sqrt{h} = \sqrt{5} - \frac{t}{20}$$

solve($\sqrt{h} = \sqrt{5} - \frac{t}{20}, h$)

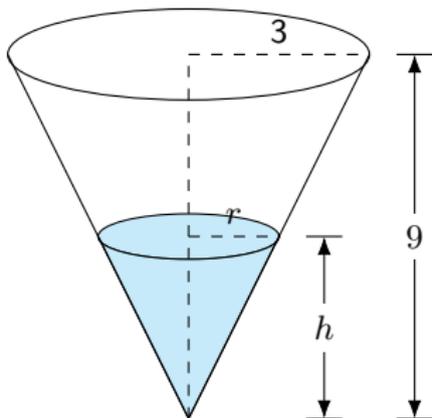
$$h = \frac{(t - 20 \cdot \sqrt{5})^2}{400} \text{ and } t - 20 \cdot \sqrt{5} \leq 0$$

Example 2 Slope Field and Solution Curve

- (a) Sketch the slope field for the differential equation $\frac{dh}{dt} = -\frac{1}{10}\sqrt{h}$.
- (b) Sketch the solution curve on the slope field that satisfies the initial condition $h(0) = 5$.

Example 3 Inverted Cone

A water tank has the shape of an inverted circular cone with base radius 3 m and height 9 m. Water is being pumped into the tank at a rate of $5/2$ m³/min. Find the rate at which the water level is rising when the water is 4 m deep.



education.ti.com