TI in Focus: AP® Calculus

2019 AP® Calculus Exam: AB-3/BC-3
Technology Solutions and Problem Extensions

Stephen Kokoska
Professor, Bloomsburg University
Former AP® Calculus Chief Reader

Texas Instruments

Outline

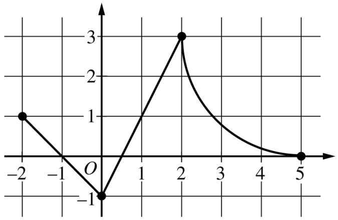
(1) Free Response Question

(2) Scoring Guidelines

(3) Solutions in greater detail

(4) Solutions using technology

(5) Problem Extensions



Graph of f

3. The continuous function f is defined on the closed interval $-6 \leq x \leq 5$. The figure above shows a portion of the graph of f, consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3, 3 - \sqrt{5})$ is on the graph of f.

(a) If $\int_{-6}^{5} f(x)\,dx = 7$, find the value of $\int_{-6}^{-2} f(x)\,dx$. Show the work that leads to your answer.

(b) Evaluate $\int_{3}^{5} (2f'(x) + 4)\,dx$.

(c) The function g is given by $g(x) = \int_{-2}^{x} f(t)\, dt$. Find the absolute maximum value of g on the interval $-2 \le x \le 5$. Justify your answer.

(d) Find $\lim\limits_{x \to 1} \dfrac{10^x - 3f'(x)}{f(x) - \arctan x}$.

(a) $\int_{-6}^{5} f(x)\,dx = \int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx$

$$\Rightarrow 7 = \int_{-6}^{-2} f(x)\,dx + 2 + \left(9 - \frac{9\pi}{4}\right)$$

$$\Rightarrow \int_{-6}^{-2} f(x)\,dx = 7 - \left(11 - \frac{9\pi}{4}\right) = \frac{9\pi}{4} - 4$$

$$3 : \begin{cases} 1 : \int_{-6}^{5} f(x)\,dx = \int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx \\ 1 : \int_{-2}^{5} f(x)\,dx \\ 1 : \text{answer} \end{cases}$$

(b) $\int_{3}^{5} (2f'(x) + 4)\,dx = 2\int_{3}^{5} f'(x)\,dx + \int_{3}^{5} 4\,dx$

$$= 2(f(5) - f(3)) + 4(5-3)$$
$$= 2\left(0 - \left(3 - \sqrt{5}\right)\right) + 8$$
$$= 2\left(-3 + \sqrt{5}\right) + 8 = 2 + 2\sqrt{5}$$

— OR —

$$\int_{3}^{5} (2f'(x) + 4)\,dx = \left[2f(x) + 4x\right]_{x=3}^{x=5}$$
$$= (2f(5) + 20) - (2f(3) + 12)$$
$$= (2 \cdot 0 + 20) - \left(2\left(3 - \sqrt{5}\right) + 12\right)$$
$$= 2 + 2\sqrt{5}$$

$$2 : \begin{cases} 1 : \text{Fundamental Theorem of Calculus} \\ 1 : \text{answer} \end{cases}$$

(c) $g'(x) = f(x) = 0 \Rightarrow x = -1,\ x = \frac{1}{2},\ x = 5$

x	$g(x)$
-2	0
-1	$\frac{1}{2}$
$\frac{1}{2}$	$-\frac{1}{4}$
5	$11 - \frac{9\pi}{4}$

On the interval $-2 \le x \le 5$, the absolute maximum value of g is $g(5) = 11 - \frac{9\pi}{4}$.

$$3 : \begin{cases} 1 : g'(x) = f(x) \\ 1 : \text{identifies } x = -1 \text{ as a candidate} \\ 1 : \text{answer with justification} \end{cases}$$

(d) $$\lim_{x \to 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x} = \frac{10^1 - 3f'(1)}{f(1) - \arctan 1} = \frac{10 - 3 \cdot 2}{1 - \arctan 1} = \frac{4}{1 - \frac{\pi}{4}}$$

1 : answer

Part (a)

Analytical Solution:

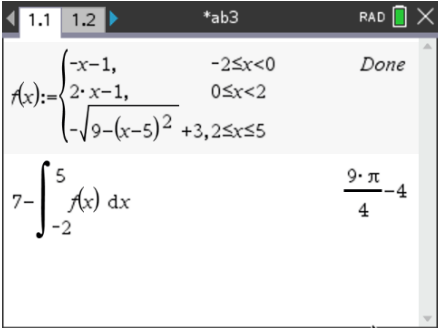
$$\int_{-5}^{6} f(x)\,dx = \int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx$$ Property of integrals

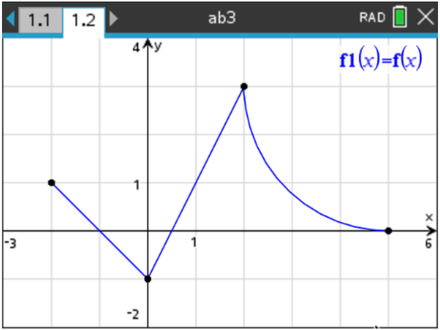
$$7 = \int_{-6}^{-2} f(x)\,dx + 2 + \left(9 - \frac{9\pi}{4}\right)$$ Area associated with integral

$$\int_{-6}^{-2} f(x)\,dx = 7 - \left(11 - \frac{\pi}{4}\right) = \frac{9\pi}{4} - 4$$ Solve for desired integral

Technology Solution

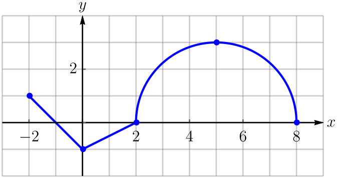
- Define the function f piecewise.
- Check the graph of f.
- Find the value of $\int_{-6}^{-2} f(x)\,dx$.





Example 1 Integral and Area

The continuous function f is defined on the closed interval $-2 \leq x \leq 8$. The figure shows the graph of f, consisting of two line segments and a half circle centered at the point $(5, 0)$. The function g is given by $g(x) = \int_{-2}^{x} f(t)\, dt$



(a) Find $g(x)$ for $x = -2, -1, 0, 1, 2, 5, 8$.

(b) Use these values of g to sketch a rough graph of g.

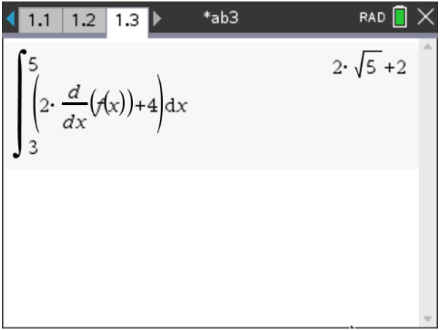
(c) Explain the relationship between the graph of g and the graph of f.

(d) Find an analytical definition for the function g. Check your solution using technology.

Part (b)

$$\int_3^5 f'(x)\,dx = 2f(x) + 4x\Big]_3^5$$
$$= (2f(5) + 4\cdot 5) - (2f(3) + 4\cdot 3)$$
$$= (2\cdot 0 + 20) - (2(3-\sqrt{5}) + 12))$$
$$= 2 + 2\sqrt{5}$$

Technology Solution



Part (c)

$$g(x) = \int_{-2}^{x} f(t)\,dt$$

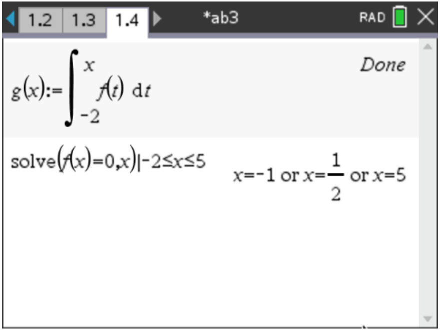
$g'(x) = f(x) = 0 \implies x = -1,\ \frac{1}{2},\ 5$ (endpoint)

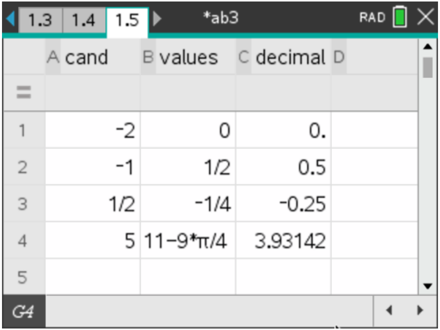
Construct a table of values.

x	$g(x)$
-2	0
-1	$\frac{1}{2}$
$\frac{1}{2}$	$-\frac{1}{4}$
5	$11 - \frac{9\pi}{4}$

On the interval $-2 \le x \le 5$, the absolute maximum value of g is $g(5) = 11 - \dfrac{9\pi}{4}$

Technology Solution





Part (d)

Solve this limit problem by direct substitution.

$$\lim_{x\to 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x} = \frac{10^1 - 3f'(1)}{f(1) - \arctan 1} \quad \text{Direct substitution}$$

$$= \frac{10 - 3\cdot 2}{1 - \arctan 1} \quad \text{Use the graph of } f$$

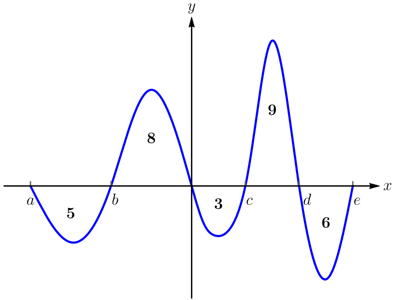
$$= \frac{4}{1 - \frac{\pi}{4}} = \frac{16}{4 - \pi}$$

Example 2 Limit Extension

Find $\displaystyle\lim_{x\to 1} \frac{x^2 f(x) - 1}{\arctan x - \frac{\pi}{4}}$

Example 3 Integral as Area

The figure shows the graph of a function f. The areas of the regions between the graph of f and the x-axis are labeled.



Determine the value of each definite integral.

(a) $\int_a^e f(x)\,dx$ (b) $\int_a^c |f(x)|\,dx$ (c) $\int_b^d f(x)\,dx$

(d) $\left|\int_a^c f(x)\,dx\right|$ (e) $\int_0^e (f(x)-2)\,dx$ (f) $\int_a^0 2f(x)\,dx$

Example 4 The Least I Can Do

Consider the portion of the graph of f as given in the statement of the Free Response Question. The function g is given by $g(x) = x^2 - \int_{-2}^{x} 3f(t)\,dt$.

(a) Find the intervals on which the graph of g is increasing. Decreasing.

(b) Find the intervals on which the graph of g is concave up. Concave down. Find the inflection point(s) on the graph of g.

(c) Find the absolute minimum value of g on the interval $-2 \leq x \leq 5$. Justify your answer.

(d) Carefully sketch a graph of the function g over the interval $-2 \leq x \leq 5$.

education.ti.com