TI in Focus: AP® Calculus

2019 AP® Calculus Exam: AB-3/BC-3
Properties of Definite Integrals

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TEXAS INSTRUMENTS

Outline

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Background

(1) Consider a limit of the form

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \, \Delta x = \lim_{n \to \infty} \left[f(x_1^*) \, \Delta x + f(x_2^*) \, \Delta x + \cdots + f(f_n^*) \, \Delta x \right]$$

(2) Other cases: distance traveled by an object, length of a curve, volume of a solid, centers of mass.

Definite Integral

If f is a function defined on a closed interval $[a, b]$, let P be a partition of $[a, b]$ defined by

$$P = \{a = x_0, x_1, x_2, x_3, \ldots, x_n = b\}$$

Choose sample points x_i^* in $[x_{i-1}, x_i]$, the ith subinterval, and let $\Delta x_i = x_i - x_{i-1}$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \, \Delta x_i$$

provided the limit exists and gives the same value for all possible choices of sample points. If the limit does exist, we say that f is **integrable** on $[a, b]$.

A Closer Look

(1) The precise meaning of the limit that defines the definite integral involves an ϵ and N argument.

(2) Another way to define the definite integral involves the norm of a partition P.
$\|P\| = \max\{\Delta x_i\}$.

$$\int_a^b f(x)\,dx = \lim_{\|P\| \to 0} \sum_{i=1}^n f(x_i^*)\,\Delta x_i$$

(3) The function $f(x)$ is the integrand.

The values a and b are the limits of integration.

a is the lower limit and b is the upper limit.

The symbol dx indicates the independent variable is x.

(4) The definite integral is a number; it does not depend upon x.

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(k)\,dk$$

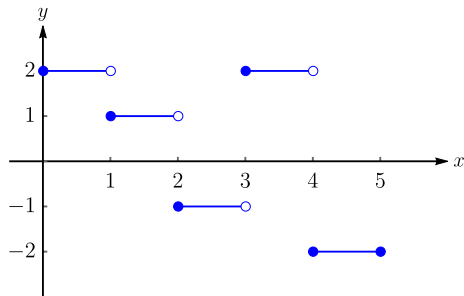(5) $\displaystyle\sum_{i=1}^n f(x_i^*)\,\Delta x_i$ is a **Riemann Sum**.

Theorem

If f is continuous on the interval $[a, b]$, or if f has a finite number of jump discontinuities, then f is integrable on $[a, b]$, that is, the definite integral $\displaystyle\int_a^b f(x)\,dx$ exists.

Example 1 Jump Discontinuities

The figure shows the graph of a function f. How would you find the value of the definite integral $\displaystyle\int_0^5 f(x)\,dx$.

Evaluating a Definite Integral

(1) By definition: limit of a Riemann sum.

(2) Interpreting the definite integral as area.

Notes

(1) In the definition of $\int_a^b f(x)\,dx$, we assume $a < b$.

If $a > b \implies \Delta x = \dfrac{a-b}{n} = -\dfrac{b-a}{n} \implies \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

(2) If $a = b \implies \Delta x = 0 \implies \int_a^a f(x)\,dx = 0$

Properties of the Definite Integral

(1) $\displaystyle\int_a^b c\,dx = c(b-a),$ where c is any constant

(2) $\displaystyle\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(3) $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx,$ where c is any constant

(4) $\displaystyle\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

A Closer Look

(1) Property 1: the definite integral of a constant function $f(x) = c$ is the constant times the length of the interval.
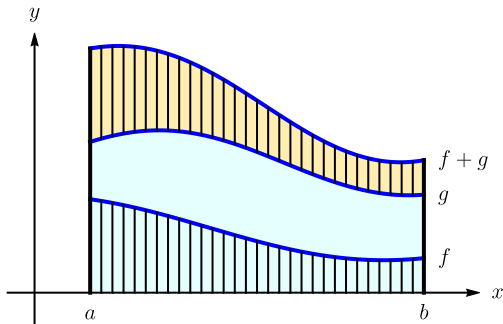
If $c > 0$ and $a < b$:

A Closer Look (Continued)

(2) Property 2: the definite integral of a sum is the sum of the definite integrals.

For positive functions: area under $f + g$ is the area under f plus the area under g.

This property is proved using the definition of the definite integral.
The limit of a sum is the sum of the limits.

A Closer Look (Continued)

(3) Property 3: the definite integral of a constant times a function is the constant times the definite integral of the function.

Constants pass freely through definite integral signs.

Proved using the definition of the definite integral.

(4) Property 4: $f - g = f + (-g)$

Use properties 2 and 3 with $c = -1$.

The definite integral of a difference is the difference of the corresponding definite integrals.

Example 2 Property Value

Given $\displaystyle\int_a^b x\,dx = \frac{1}{2}(b^2 - a^2)$; $\displaystyle\int_a^b x^2\,dx = \frac{1}{3}(b^3 - a^3)$

Use the properties of definite integrals to evaluate each of the following.

(a) $\displaystyle\int_{-2}^3 (2x + 4)\,dx$

(b) $\displaystyle\int_0^3 (3x^2 - 5)\,dx$

(c) $\displaystyle\int_{-3}^3 (x^3 - 2x^2 + 7)\,dx$

Solution

(a)
$$\int_{-2}^{3} (2x+4)\,dx = \int_{-2}^{3} 2x\,dx + \int_{-2}^{3} 4\,dx \qquad \text{Property 2}$$

$$= 2\int_{-2}^{3} x\,dx + \int_{-2}^{3} 4\,dx \qquad \text{Property 3}$$

$$= 2 \cdot \frac{1}{2}\left[3^2 - (-2)^2\right] + 4[3 - (-2)] \qquad \text{Given formula; Property 1}$$
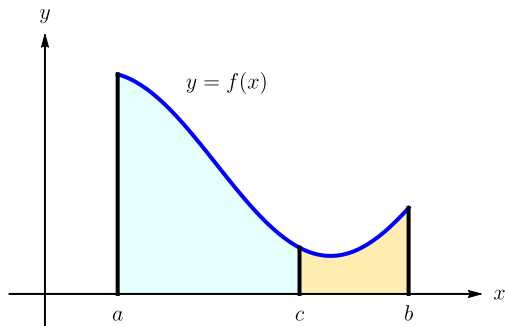
$$= 5 + 20 = 25 \qquad \text{Simplify}$$

Properties of the Definite Integral

(5) Suppose that all of the following definite integrals exist.

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Geometric interpretation if $f(x) \geq 0$ and $a < c < b$:

Example 3 The Sum of All Integrals

Suppose $\int_0^{10} f(x)\,dx = 15$ and $\int_0^7 f(x)\,dx = 5$, find $\int_7^{10} f(x)\,dx$.

Solution

$$\int_0^{10} f(x)\,dx = \int_0^7 f(x)\,dx + \int_7^{10} f(x)\,dx$$

Solve for the unknown integral.

$$\int_7^{10} f(x)\,dx = \int_0^{10} f(x)\,dx - \int_0^7 f(x)\,dx = 15 - 5 = 10$$

Comparison Properties of the Definite Integral

Suppose the following definite integrals exist and $a \le b$.

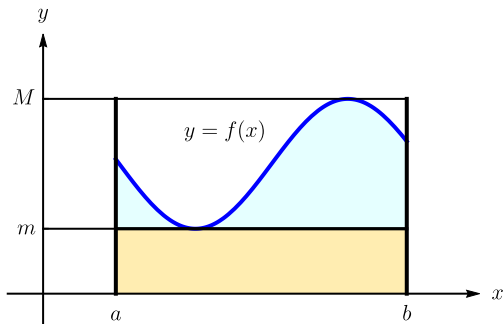(6) If $f(x) \ge 0$ for $a \le x \le b$, then $\displaystyle\int_a^b f(x)\,dx \ge 0$

(7) If $f(x) \ge g(x)$ for $a \le x \le b$, then $\displaystyle\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

(8) If $m \le f(x) \le M$ for $a \le x \le b$, then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

TEXAS INSTRUMENTS

A Closer Look

(1) If $f(x) \geq 0$, then the definite integral represents the area of the region bounded above by the graph of $y = f(x)$, below by the x-axis, and lying between the lines $x = a$ and $x = b$. This area is nonnegative.

(2) A larger function has a larger definite integral.
Follows from properties 6 and 4, and $f - g \geq 0$.

(3) Property 8 illustration:

Example 4 Definite Integral Estimate

Use Property 8 to estimate $\displaystyle\int_0^1 2^{-x^2}\, dx$

Solution

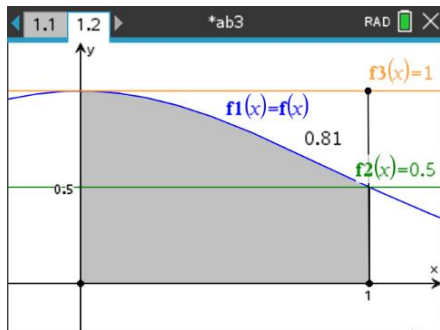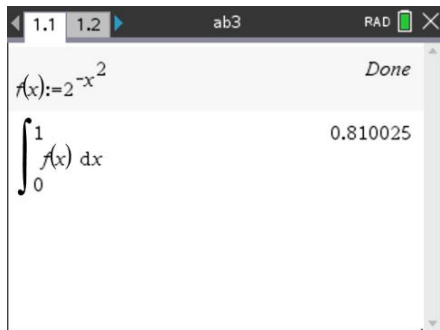$f(x) = 2^{-x^2}$ is decreasing on $[0, 1]$.

$M = f(0) = 1$ and $m = f(1) = 2^{-1} = 0.5$

$$0.5(1 - 0) \leq \int_0^1 2^{-x^2}\, dx \leq 1(1 - 0)$$

$$0.5 \leq \int_0^1 2^{-x^2}\, dx \leq 1$$

Solution

$f(x) := 2^{-x^2}$ — Done

$$\int_0^1 f(x)\,dx = 0.810025$$

f3(x)=1

f1(x)=f(x)

0.81

f2(x)=0.5

0:5

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