TI in Focus: AP® Calculus

2019 AP® Calculus Exam: BC-6
Taylor Series and e^x

Stephen Kokoska
Professor, Bloomsburg University
Former AP® Calculus Chief Reader

*AP is a registered trademark of the College Board, which was not involved in the production of, and does not endorse, this product.

Outline

(1) Power Series Representation

(2) Taylor Series

(3) Maclaurin Series

(4) Maclaurin Series for e^x

(5) Taylor Polynomials and Visualization

(6) Some Typical Questions

Questions

(1) Which functions have power series representations?

(2) How can we find these representations?

Suppose f is any function that can be represented by a power series.

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

A power series in $(x-a)$

A power series centered at a

A power series about a.

Try to determine the value of each coefficient c_n in terms of f.

$$f(a) = c_0$$

In search of the coefficients.

- A power series with radius of convergence R is differentiable.

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$$

Let $x = a$: $\quad f'(a) = c_1$

- Differentiate again.

$$f''(x) = 2c_2 + 2 \cdot 3c_3(x-a) + 3 \cdot 4(x-a)^2 + \cdots$$

Let $x = a$: $\quad f''(a) = 2c_2$

- Differentiate again.

$$f'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4(x-a) + 3 \cdot 4 \cdot 5c_5(x-a)^2 + \cdots$$

Let $x = a$: $\quad f'''(a) = 2 \cdot 3c_3 = 3!\, c_3$

In search of the coefficients.

- Recognize the pattern. Continue to differentiate and let $x = a$.

$$f^{(n)}(a) = 2 \cdot 3 \cdot 4 \cdot \cdots \cdot n c_n = n!\, c_n$$

- Solve this equation for the nth coefficient.

$$c_n = \frac{f^{(n)}(a)}{n!}$$

This formula is valid even for $n = 0$: $0! = 1$ and $f^{(0)} = f$

Theorem

If f has a power series representation at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Power Series, Taylor Series Connection

- Use the formula for c_n.
  If f has a power series expansion at a, then it must look like this.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

- This is called the Taylor series of the function f at a, or about a, or centered at a.

- Special case: $a = 0$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

  This is called a Maclaurin series.

A Closer Look

(1) If f can be represented as a power series about a, then f is equal to the sum of its Taylor series.

(2) There exist functions that are not equal to the sum of their Taylor series.

Example 1 Maclaurin Series

Find the Maclaurin series of the function $f(x) = e^x$ and its radius of convergence.

Solution

If $f(x) = e^x \quad \Rightarrow \quad f^{(n)}(x) = e^x \quad \Rightarrow \quad f^{(n)}(0) = 1$ for all n

Therefore, the Taylor series for f about 0, that is, the Maclaurin series, is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

To find the radius of convergence, let $a_n = \dfrac{x^n}{n!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \frac{|x|}{n+1}\ ; \qquad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x|}{n+1} = 0 < 1$$

By the Ratio Test, the series converges for all x and the radius of convergence is $R = \infty$.

A Closer Look

(1) If e^x has a power series expansion at 0, then we found it!

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

(2) Question: Does e^x really have a power series representation?

(3) More generally: Under what circumstances (conditions) is a function equal to the sum of its Taylor series?

If f has derivatives of all orders, when is it true that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Taylor Polynomials

- If the Taylor series converges to $f(x)$, then $f(x)$ is the limit of the sequence of partial sums.

- The partial sums:

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

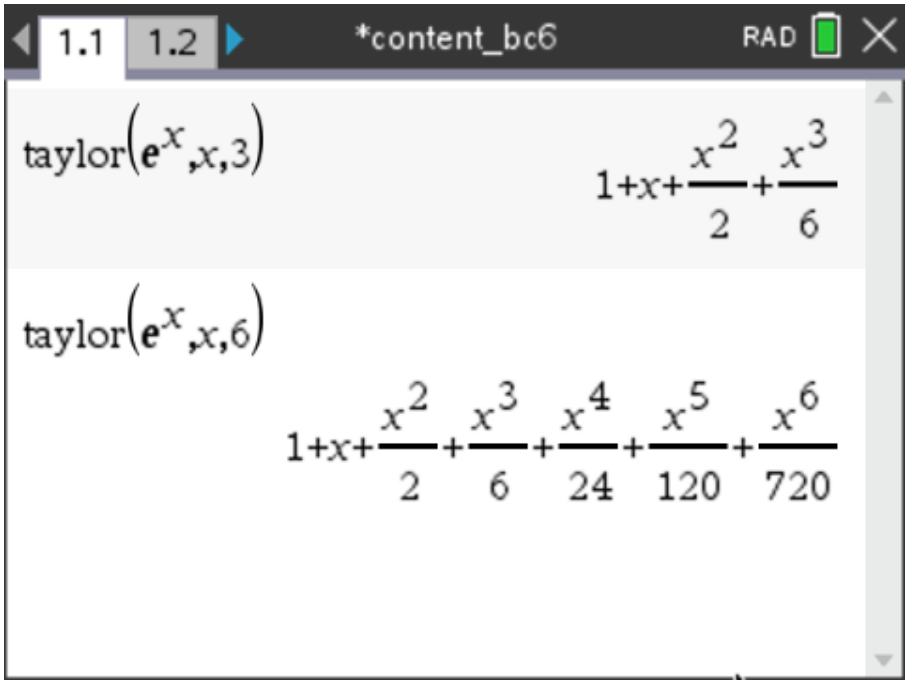
$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

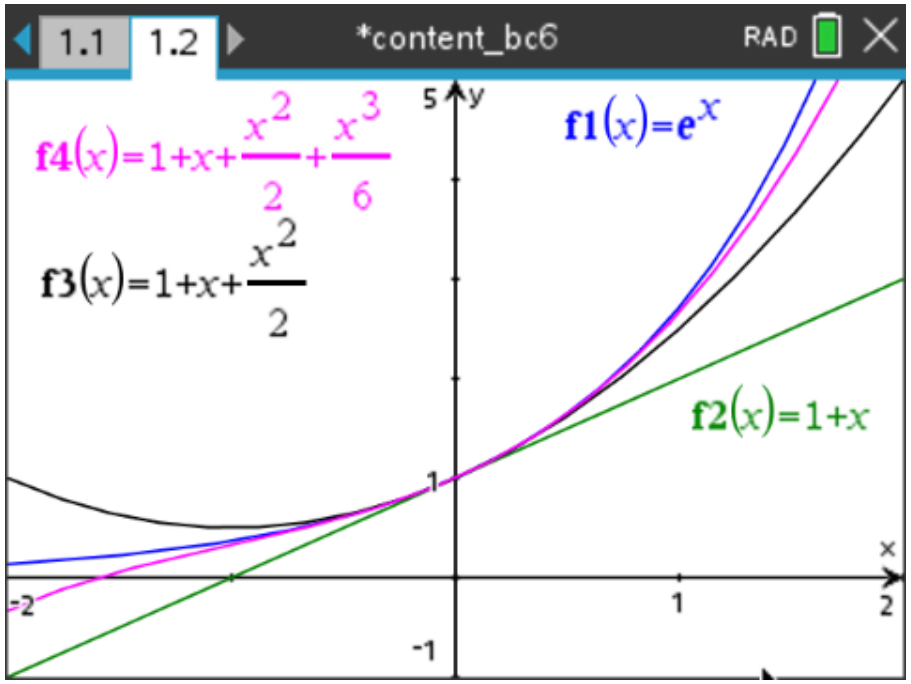
$T_n(x)$: the nth-degree Taylor polynomial of f at a.

Example 2 Taylor Polynomials for e^x

If $f(x) = e^x$, the Taylor polynomials at 0 (or Maclaurin polynomials) with $n = 1$, 2, and 3 are:

$$T_1(x) = 1 + x \; ; \qquad T_2(x) = 1 + x + \frac{x^2}{2!} \; ; \qquad T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$





The Remainder

- $f(x)$ is the sum of its Taylor series if $f(x) = \lim_{n\to\infty} T_n(x)$
- To evaluate this limit, and to determine when a function is equal to the sum of its Taylor series, consider the remainder.

$$R_n(x) = f(x) - T_n(x); \qquad f(x) = T_n(x) + R_n(x)$$

- If $\lim_{n\to\infty} R_n(x) = 0$ then

$$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} [f(x) - R_n(x)] = f(x) - \lim_{n\to\infty} R_n(x) = f(x)$$

Theorem

If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a and

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x - a| < R$, then f is equal to the sum of its Taylor series on the interval $|x - a| < R$.

Notes

(1) To show $\lim_{n\to\infty} R_n(x) = 0$ for a specific function f, we often use Taylor's Inequality. This provides a bound for $|R_n(x)|$.

(2) We can prove that e^x is equal to the sum of its Maclaurin series.

Examples

(1) Find the Taylor series for $f(x) = e^x$ at $a = 3$.

(2) Find the Maclaurin series for $\cos x$. Graph several Taylor polynomials and $\cos x$ on the same coordinate axes.

(3) Use the series in (2) to find the Maclaurin series for $\sin x$.

(4) Find the Maclaurin series for $f(x) = x^2 \sin x$.

(5) Evaluate $\int e^{-x^2}\, dx$ as an infinite series.

(6) Find the first three nonzero terms in the Maclaurin sereis for $e^x \cos x$.

education.ti.com