TI in Focus: AP® Calculus

2019 AP® Calculus Exam: BC-6

Technology Solutions and Problem Extensions

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Outline

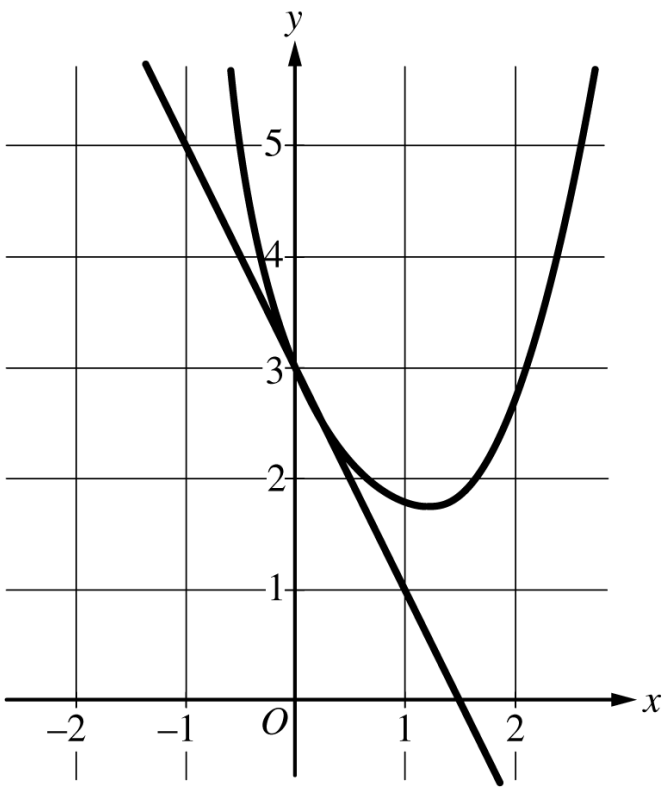
(1) Free Response Question

(2) Scoring Guidelines

(3) Solutions in greater detail

(4) Solutions using technology

(5) Problem Extensions



n	$f^{(n)}(0)$
2	3
3	$-\frac{23}{2}$
4	54

6. A function f has derivatives of all orders for all real numbers x. A portion of the graph of f is shown above, along with the line tangent to the graph of f at $x = 0$. Selected derivatives of f at $x = 0$ are given in the table above.

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(a) Write the third-degree Taylor polynomial for f about $x = 0$.

(b) Write the first three nonzero terms of the Maclaurin series for e^x. Write the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$.

(c) Let h be the function defined by $h(x) = \int_0^x f(t)\,dt$. Use the Taylor polynomial found in part (a) to find an approximation for $h(1)$.

(d) It is known that the Maclaurin series for h converges to $h(x)$ for all real numbers x. It is also known that the individual terms of the series for $h(1)$ alternate in sign and decrease in absolute value to 0. Use the alternating series error bound to show that the approximation found in part (c) differs from $h(1)$ by at most 0.45.

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(a) $f(0) = 3$ and $f'(0) = -2$

The third-degree Taylor polynomial for f about $x = 0$ is

$$3 - 2x + \frac{3}{2!}x^2 + \frac{-\frac{23}{2}}{3!}x^3 = 3 - 2x + \frac{3}{2}x^2 - \frac{23}{12}x^3.$$

$2 : \begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \end{cases}$

(b) The first three nonzero terms of the Maclaurin series for e^x are $1 + x + \frac{1}{2!}x^2$.

The second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$ is

$$\begin{aligned} &3\left(1 + x + \frac{1}{2!}x^2\right) - 2x(1 + x) + \frac{3}{2}x^2(1) \\ &\quad = 3 + (3 - 2)x + \left(\frac{3}{2} - 2 + \frac{3}{2}\right)x^2 \\ &\quad = 3 + x + x^2. \end{aligned}$$

$2 : \begin{cases} 1 : \text{three terms for } e^x \\ 1 : \text{three terms for } e^x f(x) \end{cases}$

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(c) $h(1) = \int_0^1 f(t)\,dt$

$$\approx \int_0^1 \left(3 - 2t + \frac{3}{2}t^2 - \frac{23}{12}t^3\right) dt$$

$$= \left[3t - t^2 + \frac{1}{2}t^3 - \frac{23}{48}t^4\right]_{t=0}^{t=1}$$

$$= 3 - 1 + \frac{1}{2} - \frac{23}{48} = \frac{97}{48}$$

$2 : \begin{cases} 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(d) The alternating series error bound is the absolute value of the first omitted term of the series for $h(1)$.

$$\int_0^1 \left(\frac{54}{4!}t^4\right) dt = \left[\frac{9}{20}t^5\right]_{t=0}^{t=1} = \frac{9}{20}$$

$$\text{Error} \le \left|\frac{9}{20}\right| = 0.45$$

$3 : \begin{cases} 1 : \text{uses fourth-degree term of Maclaurin series for } f \\ 1 : \text{uses first omitted term of series for } h(1) \\ 1 : \text{error bound} \end{cases}$

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Part (a)

Solution

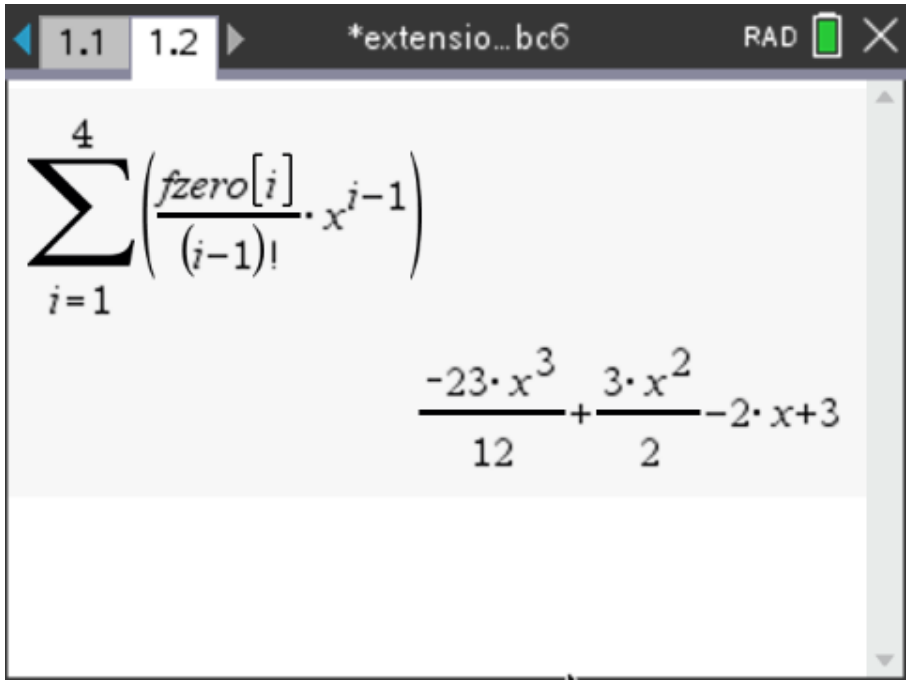
Using the graph: $f(0) = 3$; $f'(0) = -2$

$$T_3(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$= 3 + (-2)x + \frac{3}{2}x^2 + \frac{-23/2}{6}x^3$$

$$= 3 - 2x + \frac{3}{2}x^2 - \frac{23}{12}x^3$$

Technology Solution



Example 1 Taylor Series from Scratch

Suppose the function f has derivatives of all orders for all real numbers. If $f^{(n)}(0) = (n+2)!$, for $n = 0, 1, 2, \ldots$, find the fourth degree Taylor polynomial for f about $x = 0$.

Part (b)

Solution

$$f(x) = e^x \quad \Rightarrow \quad f^{(n)}(x) = e^x \quad \Rightarrow \quad f^{(n)}(0) = e^0 = 1$$

$$T_2(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2$$

$$= 1 + \frac{1}{1}x + \frac{1}{2!}x^2$$

$$= 1 + x + \frac{x^2}{2!}$$

Solution (Continued)

Multiply the appropriate expressions for e^x and $f(x)$.

Collect terms to find the second-degree Taylor polynomial.

$$T_2(x) = 3\left(1 + x + \frac{x^2}{2}\right) - 2x(1+x) + \frac{3}{2}x^2(1)$$

$$= 3 + 3x + \frac{3}{2}x^2 - 2x - 2x^2 + \frac{3}{2}x^2$$

$$= 3 + x + x^2$$

Technology Solution

1.1 | 1.2 | 2.1 | *extensio...bc6 | RAD

$$\left(1+x+\frac{x^2}{2}\right)\cdot\left(3-2\cdot x+\frac{3}{2}\cdot x^2-\frac{23}{12}\cdot x^3\right)$$

$$\frac{-\left(x^2+2\cdot x+2\right)\cdot\left(23\cdot x^3-18\cdot x^2+24\cdot x-36\right)}{24}$$

1.1 | 1.2 | 2.1 | *extensio...bc6 | RAD

$$\text{expand}\left(\frac{-\left(x^2+2\cdot x+2\right)\cdot\left(23\cdot x^3-18\cdot x^2+24\cdot x\right.}{24}\right.$$

$$\frac{-23\cdot x^5}{24}-\frac{7\cdot x^4}{6}-\frac{17\cdot x^3}{12}+x^2+x+3$$

Example 2 Out With the Old, In With the New

Write the third-degree Taylor polynomial for each function about $x = 0$.

(a) $\dfrac{e^x}{x}$

(b) $e^x \cos x$

(c) $e^{-x} \sin x$

(d) $e^{2x} - e^{3x}$

Part (c)

Solution

Use the Taylor polynomial, find an antiderivative, and evaluate.

$$h(1) = \int_0^1 f(t)\,dt$$

$$\approx \int_0^1 \left[3 - 2t + \frac{3}{2}t^2 - \frac{23}{12}t^3\right]\,dt$$

$$= \left[3t - t^2 + \frac{1}{2}t^3 - \frac{23}{48}t^4\right]_0^1$$

$$= 3 - 1 + \frac{1}{2} - \frac{23}{48} = \frac{97}{48}$$

Technology Solution

1.2 | 2.1 | 3.1 | *extensio...bc6 | RAD

$$h(x):=\int_0^x \left(3-2\cdot t+\frac{3}{2}\cdot t^2-\frac{23}{12}\cdot t^3\right)dt$$

Done

$h(1)$

$$\frac{97}{48}$$

Part (d)

Solution

The Maclaurin series for h converges to $h(x)$ for all real numbers x.

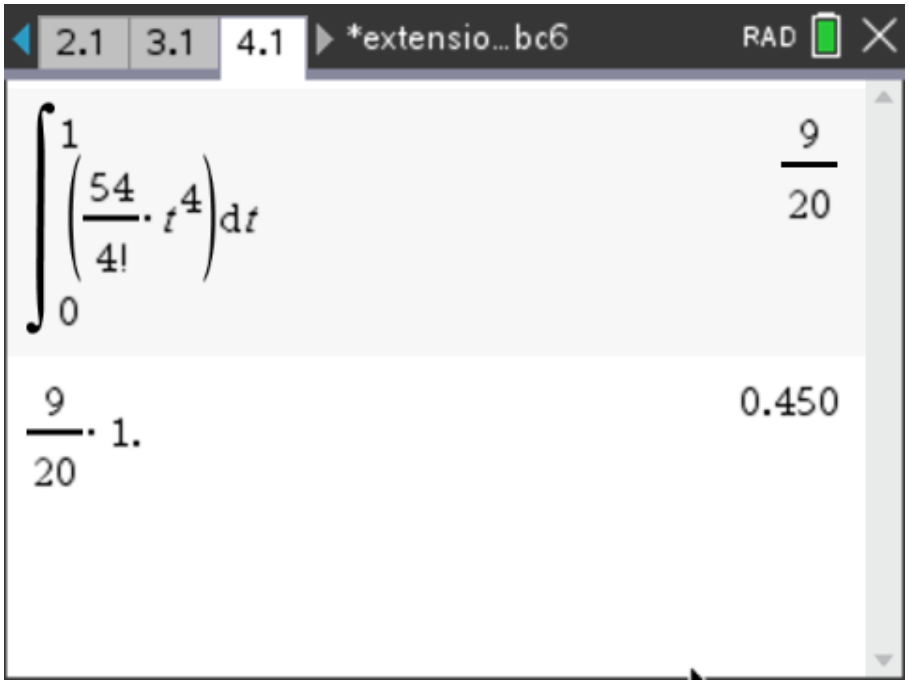
The terms of the series for $h(1)$ alternate in sign and decrease in absolute value to 0.

The alternating series error bound is the absolute value of the first omitted term of the series for $h(1)$.

$$\int_0^1 \left(\frac{54}{4!}t^4\right)\,dt = \left[\frac{54}{120}t^5\right]_0^1 = \left[\frac{9}{20}t^5\right]_0^1 = \frac{9}{20}$$

Therefore, error $\leq \left|\frac{9}{20}\right| \leq 0.45$

Technology Solution



Example 3 Alternating Series

Show that the series is convergent. How many terms of the series are necessary in order to estimate the sum to the indicated accuracy?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1/3)^n}{n}$ $\qquad |\text{error}| < 0.0005$

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n \cdot e^n}{n!}$ $\qquad |\text{error}| < 0.00001$

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