# TI in Focus: AP® Calculus

2019 AP® Calculus Exam: BC-5
Scoring Guidelines

Stephen Kokoska
Professor, Bloomsburg University
Former AP® Calculus Chief Reader

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## Outline

(1) Free Response Question
(2) Scoring Guidelines
(3) Student performance
(4) Interpretation
(5) Common errors
(6) Specific scoring examples

5. Consider the family of functions $f(x) = \dfrac{1}{x^2 - 2x + k}$, where $k$ is a constant.

(a) Find the value of $k$, for $k > 0$, such that the slope of the line tangent to the graph of $f$ at $x = 0$ equals 6.

(b) For $k = -8$, find the value of $\int_0^1 f(x)\,dx$.

(c) For $k = 1$, find the value of $\int_0^2 f(x)\,dx$ or show that it diverges.

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(a) $f'(x) = \dfrac{-(2x-2)}{\left(x^2 - 2x + k\right)^2}$

$f'(0) = \dfrac{2}{k^2} = 6 \Rightarrow k^2 = \dfrac{1}{3} \Rightarrow k = \dfrac{1}{\sqrt{3}}$

$3 : \begin{cases} 1 : \text{denominator of } f'(x) \\ 1 : f'(x) \\ 1 : \text{answer} \end{cases}$

(b) $\dfrac{1}{x^2 - 2x - 8} = \dfrac{1}{(x-4)(x+2)} = \dfrac{A}{x-4} + \dfrac{B}{x+2}$

$\Rightarrow 1 = A(x+2) + B(x-4)$

$\Rightarrow A = \dfrac{1}{6},\ B = -\dfrac{1}{6}$

$$\int_0^1 f(x)\,dx = \int_0^1 \left( \frac{\frac{1}{6}}{x-4} - \frac{\frac{1}{6}}{x+2} \right) dx$$

$$= \left[ \frac{1}{6}\ln|x-4| - \frac{1}{6}\ln|x+2| \right]_{x=0}^{x=1}$$

$$= \left( \frac{1}{6}\ln 3 - \frac{1}{6}\ln 3 \right) - \left( \frac{1}{6}\ln 4 - \frac{1}{6}\ln 2 \right) = -\frac{1}{6}\ln 2$$

$3 : \begin{cases} 1 : \text{partial fraction decomposition} \\ 1 : \text{antiderivatives} \\ 1 : \text{answer} \end{cases}$

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(c) $$\int_0^2 \frac{1}{x^2 - 2x + 1}\,dx = \int_0^2 \frac{1}{(x-1)^2}\,dx = \int_0^1 \frac{1}{(x-1)^2}\,dx + \int_1^2 \frac{1}{(x-1)^2}\,dx$$

$$= \lim_{b \to 1^-} \int_0^b \frac{1}{(x-1)^2}\,dx + \lim_{b \to 1^+} \int_b^2 \frac{1}{(x-1)^2}\,dx$$

$$= \lim_{b \to 1^-} \left( -\frac{1}{x-1} \Big|_{x=0}^{x=b} \right) + \lim_{b \to 1^+} \left( -\frac{1}{x-1} \Big|_{x=b}^{x=2} \right)$$

$$= \lim_{b \to 1^-} \left( -\frac{1}{b-1} - 1 \right) + \lim_{b \to 1^+} \left( -1 + \frac{1}{b-1} \right)$$

Because $\lim_{b \to 1^-} \left( -\frac{1}{b-1} \right)$ does not exist, the integral diverges.

$$3 : \begin{cases} 1 : \text{improper integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer with reason} \end{cases}$$

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## Student Performance

### Part (a)

- Most students were able to connect the idea of the slope of the tangent line and the derivative $f'(x)$.
- Common errors: computation of the derivative, algebra errors (missing parentheses, negative sign issues), missing $-1$ as part of the Chain Rule, errors in the power in the denominator, solving for $k$.

### Part (b)

- Most students knew to find a partial fraction decomposition.
- Most students were able to find the partial fraction decomposition.
- Good integration: some missing absolute value symbols on the $x - 4$ term.
- Simplification errors, properties of logarithms.

## Student Performance

### Part (c)

- Many students did not recognize that the integral was improper.
- Many correct antiderivatives, and incorrect (finite) answers.
- For those students who recognized the integral is improper:
  Many were able to set up a limit.
  However, very few used a one-sided limit.
  Evaluation issues.
  Many recognized the integral diverges, but presented insufficient justification.

## Part (a) 1: denominator of $f'(x)$

(1) Two ways to see the denominator:

$$\frac{**}{(x^2-2x+k)^2} \qquad\qquad * \, (x^2-2x+k)^2(**)$$

## Part (a) 1: $f'(x)$

(1) If the first point is earned, check the remaining parts of $f'(x)$.

(2) If the first point is not earned, cannot earn the second point.

(3) If $f'(x)$ is presented correctly, but simplified incorrectly: earns the second point but not eligible for the third point.

(4) Example:

$$f'(x) = \frac{-(2x-2)}{(x^2-2x+k)^2} = \frac{-2x-2}{(x^2-2x+k)^2}$$

1 - 1 - 0

## Part (a) 1: answer

(1) If both first and second points earned: look for correct and consistent answer.

(2) Watch for issues related to minus signs.

(3) Linkage errors: cannot earn the third point.

## Reading with the Student

(1) In certain cases, we read with the student with an incorrect $f'(x)$, for the third point.

$$f'(x) = \frac{-(2x-2)}{(x^2-2x+k)^n} \quad n \text{ nonzero integer}$$

$$\frac{2}{k^n} = 6 \quad \Rightarrow \quad k = \left(\frac{1}{3}\right)^{1/n}$$ 0 - 0 - 1

(2) $$f'(x) = \frac{2x-2}{(x^2-2x+k)^2} \quad \text{(missing minus sign)}$$

$$\frac{-2}{k^2} \quad \Rightarrow \quad \text{no real solutions}$$ 1 - 0 - 0

(3) Presentation error with parentheses resolved.
Correct $f'(x)$ used, earns the second point, eligible for the third.
Must have the correct answer with consistent work to earn the third point.

## Part (a) Examples

(1) $f'(x) = -(x^2 - 2x + k)^2(2x - 2)(2)$ 1 - 0 - 0

(2) $f'(x) = -\dfrac{2x - 2}{(x^2 - 2x + k)^2}$ 1 - 1 - ?

(3) $f'(x) = \dfrac{(x^2 - 2x + k)(0) - (2x - 2)}{(x^2 - 2x + k)^2}$ 1 - 1 - ?

(4) $f'(x) = \dfrac{x^2 - 2x + k(0) - (2x - 2)}{(x^2 - 2x + k)^2}$ 1 - 0 - ?

## Part (b) 1: partial fraction decomposition

(1) Point is earned for a correct partial fraction decomposition.

(2) $$\frac{1}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2} \quad \Rightarrow \quad A = \frac{1}{6} \quad B = -\frac{1}{6}$$

(3) $$\int_0^1 f(x)\,dx = \int_0^1 \left[\frac{1/6}{x-4} - \frac{1/6}{x+2}\right] dx$$ earns the point

(4) Certain PFD errors: read with the student.
Example: factorization is $(x-4)(x+2)$.

## Part (b) Examples: PFD

(1) $\dfrac{1}{x^2-2x-8} = \dfrac{A}{x-4} + \dfrac{B}{x+2}$ ? - ? - ?

(2) $\dfrac{1}{x^2-2x-8} = \dfrac{1/6}{x+4} - \dfrac{1/2}{x-2}$ 0 - ? - 0

(3) $\displaystyle\int_0^1 f(x)\,dx = \int_0^1 \left[\frac{1}{6x-24} - \frac{1}{6x+12}\right] dx$ 1 - ? - ?

(4) $\dfrac{1}{x^2-2x-8} = \dfrac{1}{x-4} + \dfrac{1}{x+2}$ 0 - ? - ?

(5) $\dfrac{1}{x^2-2x-8} = \dfrac{1}{x-2} + \dfrac{1}{x+6}$ 0 - 0 - 0

## Part (b) 1: antiderivatives

(1) Must see correct antiderivatives for eligible PFDs, including absolute values for the term corresponding to the factor $x - 4$ (or $x - 2$ in special case).

(2) Different variations based on simplifying and properties of logarithms.

(3) Correct antiderivatives presented, but simplified incorrectly:
Earns the second point but not eligible for the third point.

(4) If no absolute values present in antiderivatives: Second point is not earned.
Possible to earn the third point if (late) absolute values appear.

## Part (b) 1: answer

(1) Remember: numerical answers do not need to be simplified.

(2) There are many variations of the correct answer based on simplifying and properties of logarithms.

## Part (c) 1: improper integral

(1) Need to see evidence of an infinite discontinuity at $x = 1$:

- Break the definite integral at $x = 1$.
- Just $\int_0^1 + \int_1^2$
- $f(x)$ has a vertical asymptote at $x = 1$
- $\lim_{x \to 1} f(x) = \infty$
- $f(x)$ has an infinite discontinuity at $x = 1$

(2) Improper integral non-evidence:

- $\lim_{x \to 1} f(x)$ does not exist
- $f(x)$ has a discontinuity at $x = 1$
- $f(x)$ is undefined at $x = 1$
- A graph of $f(x)$ with no accompanying text.

(3) Must see evidence of an improper integral to earn the third point.

## Part (c) 1: antiderivative

(1) Correct antiderivative in terms of $x$:

$$\int_{*}^{**} \frac{1}{(x-1)^2}\,dx = -\left.\frac{1}{x-1}\right|_{*}^{**}$$

(2) Correct antiderivative in terms of $u$:

$$\int_{*}^{**} \frac{1}{u^2}\,du = -\left.\frac{1}{u}\right|_{*}^{**}$$

Need to see $u = x - 1$.

Will not penalize for incorrect limits.

## Part (c) 1: answer with reason

Must see the following

(1) A relevant one-sided limit, correctly evaluated.

(2) Conclusion of divergence.

Examples

(1) $\lim_{b\to 1^-}\left[-\frac{1}{b-1}-1\right]=\infty$

(2) $\lim_{b\to 1^+}\left[-1+\frac{1}{b-1}\right]=\infty$

(3) $\lim_{b\to 1^-}\left[-\frac{1}{b-1}\right]=\infty$

(4) $\lim_{b\to 1^+}\left[\frac{1}{b-1}\right]=\infty$

## Part (c) 1: answer with reason

Issues

(1) If there is no one-sided limit, cannot earn the third point.

(2) Not sufficient to simply state that the integral equals $\infty$.

(3) If there is no mention of divergence, cannot earn the third point.

(4) Only need one of the two integrals. One or both limits may be evaluated. Presentation must be correct.

(5) Sufficient to write the limit(s) Does Not Exist (DNE).

(6) A limit equal to $\frac{1}{0}$ does not earn the third point.

(7) A limit equal to $\infty - \infty$ does not earn the third point.

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