

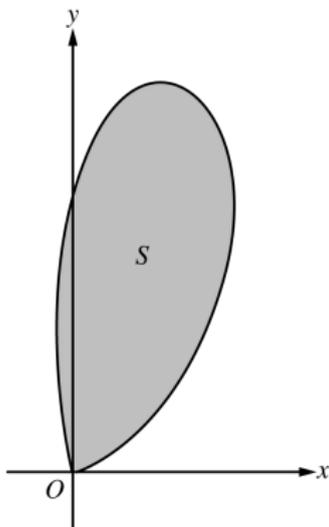
# TI in Focus: AP<sup>®</sup> Calculus

2019 AP<sup>®</sup> Calculus Exam: BC-2  
Scoring Guidelines

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## Outline

- (1) Free Response Question
- (2) Scoring Guidelines
- (3) Student performance
- (4) Interpretation
- (5) Common errors
- (6) Specific scoring examples



2. Let  $S$  be the region bounded by the graph of the polar curve  $r(\theta) = 3\sqrt{\theta} \sin(\theta^2)$  for  $0 \leq \theta \leq \sqrt{\pi}$ , as shown in the figure above.

- (a) Find the area of  $S$ .
- (b) What is the average distance from the origin to a point on the polar curve  $r(\theta) = 3\sqrt{\theta} \sin(\theta^2)$  for  $0 \leq \theta \leq \sqrt{\pi}$ ?
- (c) There is a line through the origin with positive slope  $m$  that divides the region  $S$  into two regions with equal areas. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of  $m$ .
- (d) For  $k > 0$ , let  $A(k)$  be the area of the portion of region  $S$  that is also inside the circle  $r = k \cos \theta$ . Find  $\lim_{k \rightarrow \infty} A(k)$ .

$$(a) \frac{1}{2} \int_0^{\sqrt{\pi}} (r(\theta))^2 d\theta = 3.534292$$

The area of  $S$  is 3.534.

$$(b) \frac{1}{\sqrt{\pi} - 0} \int_0^{\sqrt{\pi}} r(\theta) d\theta = 1.579933$$

The average distance from the origin to a point on the curve  $r = r(\theta)$  for  $0 \leq \theta \leq \sqrt{\pi}$  is 1.580 (or 1.579).

2 :  $\begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

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$$(c) \tan \theta = \frac{y}{x} = m \Rightarrow \theta = \tan^{-1} m$$

$$\frac{1}{2} \int_0^{\tan^{-1} m} (r(\theta))^2 d\theta = \frac{1}{2} \left( \frac{1}{2} \int_0^{\sqrt{\pi}} (r(\theta))^2 d\theta \right)$$

(d) As  $k \rightarrow \infty$ , the circle  $r = k \cos \theta$  grows to enclose all points to the right of the  $y$ -axis.

$$\begin{aligned} \lim_{k \rightarrow \infty} A(k) &= \frac{1}{2} \int_0^{\pi/2} (r(\theta))^2 d\theta \\ &= \frac{1}{2} \int_0^{\pi/2} (3\sqrt{\theta} \sin(\theta^2))^2 d\theta = 3.324 \end{aligned}$$

3 :  $\left\{ \begin{array}{l} 1 : \text{equates polar areas} \\ 1 : \text{inverse trigonometric function} \\ \text{applied to } m \text{ as limit of} \\ \text{integration} \\ 1 : \text{equation} \end{array} \right.$

2 :  $\left\{ \begin{array}{l} 1 : \text{limits of integration} \\ 1 : \text{answer with integral} \end{array} \right.$

## Student Performance

### Part (a)

- Most students were able to set up the definite integral for area, using  $[r(\theta)]^2$  and including the constant  $\frac{1}{2}$ .
- Some presentation errors, parentheses errors.

### Part (b)

- Some students used the formula for arc length.
- Some students used  $x = r \cos \theta$  and  $y = r \sin \theta$  to construct a formula for distance.
- Common error:  $\frac{1}{\sqrt{\pi}} \int_0^{\sqrt{\pi}} [r(\theta)]^2 d\theta$ ; sort of an average area.
- Presentation errors:  $\sin(\theta)^2$  versus  $\sin(\theta^2)$

## Student Performance

### Part (c)

- Some students had difficulty connecting the slope of the line to a polar equation.
- Common error: use of  $r(\theta)$  instead of  $[r(\theta)]^2$ .
- Some students used  $m$  or  $\tan \theta$  as a bound on a definite integral.

### Part (d)

- Written justification was not required.
- Some students had difficulty connecting the concept of a limit to area.
- Some students did not present a definite integral or simply tried to use their result for area from part (a).

## Notes

- Two points in this question specifically for the *integral*.  
This means the limits as well as the integrand.  
The constants in front, or the coefficients on the integral, are not part of the *integral*.
- This was a calculator active question.  
So, some students used degrees rather than radians.
- The given function contains a square.  
$$3\sqrt{\theta} \sin(\theta^2) \neq [3\sqrt{\theta} \sin(\theta^2)]^2$$
- Presentation errors; inoculation.
- Philosophy:  
Communicate an understanding of how polar area is determined.  
Aware of the parts of a polar integral (limits as angles).  
Apply and connect concepts.

**Part (a) 1: integral**

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(1) Must see  $\int_0^{\sqrt{\pi}} [r(\theta)]^2 d\theta$  or  $\int_0^{\sqrt{\pi}} [3\sqrt{\theta} \sin(\theta^2)]^2 d\theta$

(2) The coefficient  $1/2$  is not part of this point.

(3) A missing  $d\theta$  is not penalized, as long as it does not alter the integrand.

(4)  $\int_0^{\sqrt{\pi}} 3\sqrt{\theta} \sin(\theta^2) d\theta$  does not earn the point.

**Part (a) 1: answer**

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- (1) No bald answers.
- (2) Must be our answer: 3.534
- (3) May be the exact symbolic answer:  $\frac{9\pi}{8}$
- (4) Missing  $1/2$  does not earn this point.

**Part (a) Examples**

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$$(1) \frac{1}{2} \int_0^{\sqrt{\pi}} [3\sqrt{\theta} \sin(\theta^2)]^2 d\theta = 3.534 \qquad 1 - 1$$

$$(2) \int_0^{\sqrt{\pi}} \frac{1}{2} [3\sqrt{\theta} \sin(\theta^2)] d\theta = 1.400 \qquad 0 - 0$$

$$(3) \frac{1}{2} \int_0^{\sqrt{\pi}} [3\sqrt{\theta} \sin(\theta^2)] d\theta = 3.534 \qquad 0 - 1$$

**Part (b) 1: integral**

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(1) Must see  $\int_0^{\sqrt{\pi}} r(\theta) d\theta$  or  $\int_0^{\sqrt{\pi}} 3\sqrt{\theta} \sin(\theta^2) d\theta$

(2) The coefficient,  $\frac{1}{\sqrt{\pi} - 0}$ , is not part of this point.

(3) Attempts at arc length calculation do not earn this point

$$\int_0^{\sqrt{\pi}} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta \quad \int_0^{\sqrt{\pi}} \sqrt{(r)^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

(4) Distance formula approach:  $x = r \cos \theta$ ,  $y = r \sin \theta$ .

Can earn 1 - 1

$$\frac{1}{\sqrt{\pi}} \int_0^{\sqrt{\pi}} \sqrt{[3\sqrt{\theta} \sin(\theta^2) \cos \theta]^2 + [3\sqrt{\theta} \sin(\theta^2) \sin \theta]^2} d\theta = 1.580$$

**Part (b) 1: answer**

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- (1) No bald answers
- (2) Must be our answer: 1.580, 1.58, or 1.579
- (3) Missing  $\frac{1}{\sqrt{\pi}}$  does not earn this point.

**Part (b) Examples**

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$$(1) \frac{1}{\sqrt{\pi} - 0} \int_0^{\sqrt{\pi}} r(\theta) d\theta = 1.580 \quad 1 - 1$$

$$(2) \frac{1}{\sqrt{\pi} - 0} \int_0^{\sqrt{\pi}} [3\sqrt{\theta} \sin(\theta^2)]^2 d\theta = 1.580 \quad 0 - 1$$

$$(3) \frac{\frac{1}{2} \int [3\sqrt{\theta} \sin(\theta^2)]^2}{\sqrt{\pi} - 0} = \frac{3.354}{\sqrt{\pi}} = 1.99 \quad 0 - 0$$

$$(4) \frac{\int_0^{\sqrt{\pi}} \sqrt{[r(\theta) \cos \theta]^2 + [r(\theta) \sin \theta]^2}}{\sqrt{\pi} - 0} = 1.580 \quad 1 - 1$$

$$(5) \frac{1}{\sqrt{\pi} - 0} \int_0^{\sqrt{\pi}} \sqrt{x^2 + y^2} d\theta = 1.58$$
$$x = r \cos \theta \quad y = r \sin \theta \quad 1 - 1$$

**Part (c) 1: equates polar areas**

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- (1) Must be a structurally correct equation involving at least one polar area integral.
- (2) A presentation error in one integral: does not earn the third point. (Parenthesis error, lack of square, misrepresentation of  $r\theta$ .)

**Acceptable Equation Forms**

$$(1) \frac{1}{2} \int_0^? [r(\theta)]^2 d\theta = \frac{1}{2} \left[ \frac{1}{2} \int_0^{\sqrt{\pi}} [r(\theta)]^2 d\theta \right]$$

$$(2) \frac{1}{2} \int_0^? [r(\theta)]^2 d\theta = \frac{3.534}{2} \quad \text{or equivalent } [?, \sqrt{\pi}]$$

$$(3) \frac{1}{2} \int_0^? [r(\theta)]^2 d\theta = \frac{\text{part (a) answer}}{2} \quad \text{or equivalent } [?, \sqrt{\pi}]$$

$$(4) \int_0^? [r(\theta)]^2 d\theta = \frac{1}{2} \int_0^{\sqrt{\pi}} [r(\theta)]^2 d\theta$$

**Part (c) Examples: first point**

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$$(1) \frac{1}{2} \int_0^m [r(\theta)]^2 d\theta = \frac{3.534}{2} \quad \text{Yes}$$

$$(2) \frac{1}{2} \int_0^\theta [r(\theta)]^2 d\theta = \frac{1}{4} \int_0^{\sqrt{\pi}} [r(\theta)]^2 d\theta \quad \text{Yes}$$

$$(3) \int_0^k [r(\theta)]^2 d\theta = \int_k^{\sqrt{\pi}} [r(\theta)]^2 d\theta \quad \text{Yes}$$

$$(4) \frac{1}{2} \int_0^\theta [r(\theta)]^2 d\theta = \int_0^{\sqrt{\pi}} [r(\theta)]^2 d\theta \quad \text{No}$$

$$(5) \int_0^m [r(\theta)]^2 d\theta + \int_m^{\sqrt{\pi}} [r(\theta)]^2 d\theta = \int_0^{\sqrt{\pi}} [r(\theta)]^2 d\theta \quad \text{No}$$

**Part (c) inverse trig function applied to  $m$  as limit of integration**

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- (1) There must be an integral with limits that are two angles.  
One limit must involve an inverse trig function that incorporates  $m$ .
- (2) Point may be earned even if the integral does not represent area.

$$\int_0^{\arctan m} r(\theta) d\theta \quad \int_{\arctan m}^{\sqrt{\pi}} r(\theta) d\theta$$

- (3) Acceptable limits:  $\arctan m$ , Inverse trig( $m$ ),  $k$ , where  $\tan k = m$   
Unacceptable limits:  $m$ ,  $mx$

## Part (c) 1: equation

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- (1) Eligibility: first two points must be earned.
- (2) Must be one of our acceptable forms from the first point with correct limits.

**Part (c) Examples**

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$$(1) \int_0^{\theta} \frac{1}{2} r^2 d\theta = \frac{1}{2} \int_0^{\sqrt{\pi}} \frac{1}{2} r^2 d\theta; \quad m = \tan \theta \quad 1 - 1 - 1$$

$$(2) \frac{1}{2} \int_0^{\cos^{-1}(1/m)} [3\sqrt{\theta} \sin(\theta^2)]^2 d\theta = \frac{1}{2} \int_{\cos^{-1}(1/m)}^{\sqrt{\pi}} [3\sqrt{\theta} \sin(\theta^2)]^2 d\theta \quad 1 - 1 - 0$$

- (3) Let  $q$  be the value of  $\theta$  such that the line with slope  $m$  passes through  $q$  to divide  $S$  into two equal-area regions.

$$\int_0^q [r(\theta)]^2 d\theta = \int_q^{\sqrt{\pi}} [r(\theta)]^2 d\theta \quad 1 - 0 - 0$$

$$(4) 3.534 = \frac{1}{2} \int_0^m [r(\theta)]^2 d\theta + \frac{1}{2} \int_m^{\sqrt{\pi}} [r(\theta)]^2 d\theta \quad 0 - 0 - 0$$

**Part (d) 1: limits of integration**

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- (1) Must see a definite integral with limits 0 and  $\frac{\pi}{2}$  (or 1.571)
- (2) Note: using 1.571 does not yield the correct answer.
- (3) This demonstrates understanding of the limit process.

**Part (d) 1: answer with integral**

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- (1) Eligibility: first point must have been earned.
- (2) 0 - 1: not possible.
- (3) Must be our answer, 3.324, associated with a definite integral that earns the first point.
- (4) If 1.571 is used as a bound on the definite integral and the correct answer is presented: 1 - 1

## Part (d) Examples

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$$(1) \int_0^{\pi/2} 3\sqrt{\theta} \sin(\theta^2) d\theta = 2.544$$

1 - 0

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