TI in Focus: AP® Calculus

2018 AP® Calculus Exam: AB-5
L'Hospital's Rule

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Outline

(1) Background

(2) Indeterminate Forms: Definitions

(3) L'Hospital's Rule

(4) Special Case: Visualization

(5) Examples

(6) Other Indeterminate Forms

Example 1 Behavior Near 1

Suppose we need to analyze and sketch a graph of the function $F(x) = \dfrac{\ln x}{x-1}$

Near 1:

$\displaystyle\lim_{x\to 1} \frac{\ln x}{x-1}$: $\displaystyle\lim_{x\to 1} \ln x = 0$ and $\displaystyle\lim_{x\to 1} (x-1) = 0$

A limit of the form $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)}$ where both $\displaystyle\lim_{x\to a} f(x) = 0$ and $\displaystyle\lim_{x\to a} g(x) = 0$ is called an **indeterminate form of type $\mathbf{\frac{0}{0}}$** and the limit may or may not exist.

Other solution techniques

(1) $\displaystyle\lim_{x\to 1} \frac{x^2+2x-3}{x^2-1} = \lim_{x\to 1} \frac{(x-1)(x+3)}{(x-1)(x+1)} = \lim_{x\to 1} \frac{x+3}{x+1} = \frac{4}{2} = 2$

(2) $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1$ Geometric argument.

Example 2 Behavior as x Increases Without Bound

Consider another situation in which there is no obvious answer.

$$\lim_{x\to\infty} \frac{\ln x}{x-1}$$

A limit of the form $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)}$ where both $\displaystyle\lim_{x\to a} f(x) = \infty$ (or $-\infty$) and $\displaystyle\lim_{x\to a} g(x) = \infty$ (or $-\infty$) is called an **indeterminate form of type $\frac{\infty}{\infty}$** and the limit may or may not exist.

Other solution techniques

$$\lim_{x\to\infty} \frac{2x^2+1}{5x^2-3} = \lim_{x\to\infty} \frac{2+\frac{1}{x^2}}{5-\frac{3}{x^2}} = \frac{2+0}{5-1} = \frac{2}{5}$$

L'Hospital's Rule

Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval that contains a (except possibly at a). Suppose that

$$\lim_{x\to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x\to a} g(x) = 0$$

or that

$$\lim_{x\to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x\to a} g(x) = \pm\infty$$

(The limit $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ is in an indeterminate form of type $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$.)

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

A Closer Look

(1) L'Hospital's Rule: The limit of a quotient of functions is equal to the limit of the quotient of their derivatives, provided the conditions are satisfied.

The conditions must be satisfied before using L'Hospital's Rule.

(2) L'Hospital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity.

$x \to a$: $x \to a^+$, $x \to a^-$, $x \to \infty$, $x \to -\infty$

(3) Consider the special case: $f(a) = g(a) = 0$, f' and g' continuous, $g'(a) \neq 0$.

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}$$

f' and g' are continuous; $g'(a) \neq 0$.

$$= \frac{\displaystyle\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{\displaystyle\lim_{x\to a} \frac{g(x)-g(a)}{x-a}} = \lim_{x\to a} \frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}}$$

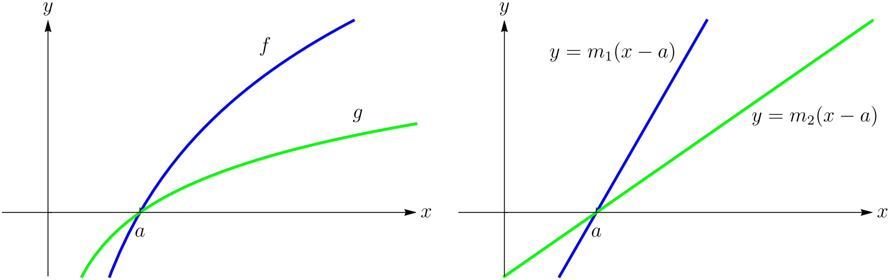
Definition of a derivative; Limit Laws.

$$= \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x\to a} \frac{f(x)}{g(x)}$$

Simplify; $f(a) = g(a) = 0$.

A Closer Look

Illustration of this same situation graphically.



Zoom in near the point $(a, 0)$: Both graphs look linear.

Therefore, their ratio is the quotient of the linear approximations.

$$\frac{f(x)}{g(x)} \approx \frac{m_1(x-a)}{m_2(x-a)} = \frac{m_1}{m_2}$$

In this case, the quotient simplifies to the ratio of their derivatives.

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Example 3 $\frac{0}{0}$ and L'Hospital's Rule

Find $\lim\limits_{x \to 1} \dfrac{\ln x}{x - 1}$

Solution

Verify the conditions.

$$\lim_{x \to 1} \ln x = 0 \quad \text{and} \quad \lim_{x \to 1} (x - 1) = 0$$

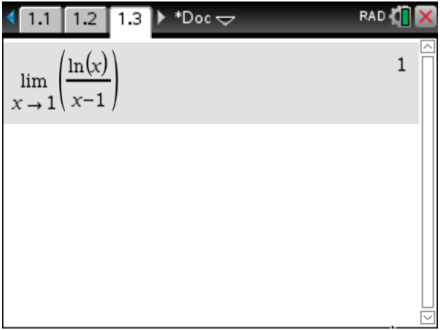
$$\lim_{x \to 1} \frac{\ln x}{x - 1} = \lim_{x \to 1} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x - 1)} = \lim_{x \to 1} \frac{\frac{1}{x}}{1}$$

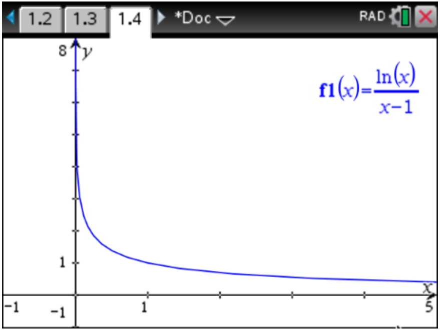
Differentiate the numerator and the denominator

$$= \lim_{x \to 1} \frac{1}{x} = 1$$

Simplify; Limit Laws.

Technology Solution





Example 4 Double L'Hospital

Find $\lim\limits_{x\to\infty} \dfrac{x^2}{e^x}$

Solution

Verify the conditions.

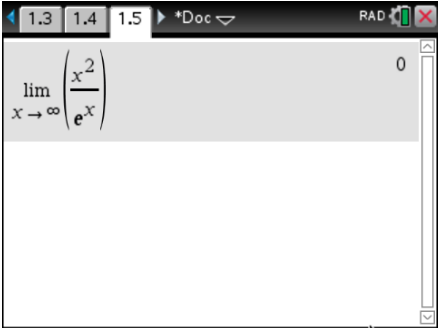
$$\lim_{x\to\infty} x^2 = \infty \quad \text{and} \quad \lim_{x\to\infty} e^x = \infty$$

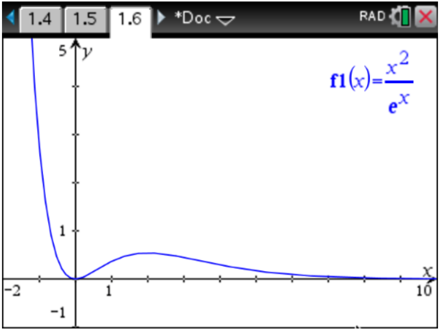
$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{\dfrac{d}{dx}(x^2)}{\dfrac{d}{dx}(e^x)} = \lim_{x\to\infty} \frac{2x}{e^x}$$

$$\lim_{x\to\infty} 2x = \infty \quad \text{and} \quad \lim_{x\to\infty} e^x = \infty$$

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0$$

Technology Solution





Example 5 More L'Hospital

Find each limit. Use L'Hospital's Rule where appropriate. If L'Hospital's Rule does not apply, explain why.

(a) $\lim\limits_{x \to 0} \dfrac{e^{2x} - 1}{\sin x}$

(b) $\lim\limits_{x \to \infty} \dfrac{\ln \sqrt{x}}{x^2}$

(c) $\lim\limits_{x \to 0^+} \dfrac{\ln x}{x}$

(d) $\lim\limits_{x \to 0} \dfrac{\cos mx - \cos nx}{x^2}$

(e) $\lim\limits_{x \to 0} \dfrac{e^x - e^{-x} - 2x}{x - \sin x}$

Indeterminate Products

Suppose $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$ (or $-\infty$).

$\lim_{x \to a} [f(x)g(x)]$ **Indeterminate form of type $0 \cdot \infty$**

Indeterminate Differences

Suppose $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$

$\lim_{x \to a} [f(x) - g(x)]$ **Indeterminate form of type $\infty - \infty$**

Indeterminate Powers

Consider the limit $\lim_{x \to a} [f(x)]^{g(x)}$

(1) $\lim_{x \to a} f(x) = 0$	and	$\lim_{x \to a} g(x) = 0$	type 0^0
(2) $\lim_{x \to a} f(x) = \infty$	and	$\lim_{x \to a} g(x) = 0$	type ∞^0
(3) $\lim_{x \to a} f(x) = 1$	and	$\lim_{x \to a} g(x) = \pm\infty$	type 1^∞

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