

TI in Focus: AP[®] Calculus

2018 AP[®] Calculus Exam: AB-4/BC-4

Technology Solutions and Problem Extensions

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Outline

- (1) Free Response Question
- (2) Scoring Guidelines
- (3) Solutions using technology
- (4) Problem Extensions

t (years)	2	3	5	7	10
$H(t)$ (meters)	1.5	2	6	11	15

4. The height of a tree at time t is given by a twice-differentiable function H , where $H(t)$ is measured in meters and t is measured in years. Selected values of $H(t)$ are given in the table above.
- (a) Use the data in the table to estimate $H'(6)$. Using correct units, interpret the meaning of $H'(6)$ in the context of the problem.
- (b) Explain why there must be at least one time t , for $2 < t < 10$, such that $H'(t) = 2$.
- (c) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the average height of the tree over the time interval $2 \leq t \leq 10$.
- (d) The height of the tree, in meters, can also be modeled by the function G , given by $G(x) = \frac{100x}{1+x}$, where x is the diameter of the base of the tree, in meters. When the tree is 50 meters tall, the diameter of the base of the tree is increasing at a rate of 0.03 meter per year. According to this model, what is the rate of change of the height of the tree with respect to time, in meters per year, at the time when the tree is 50 meters tall?

$$(a) \quad H'(6) \approx \frac{H(7) - H(5)}{7 - 5} = \frac{11 - 6}{2} = \frac{5}{2}$$

$H'(6)$ is the rate at which the height of the tree is changing, in meters per year, at time $t = 6$ years.

$$(b) \quad \frac{H(5) - H(3)}{5 - 3} = \frac{6 - 2}{2} = 2$$

Because H is differentiable on $3 \leq t \leq 5$, H is continuous on $3 \leq t \leq 5$.

By the Mean Value Theorem, there exists a value c , $3 < c < 5$, such that $H'(c) = 2$.

2 : $\left\{ \begin{array}{l} 1 : \text{estimate} \\ 1 : \text{interpretation with units} \end{array} \right.$

2 : $\left\{ \begin{array}{l} 1 : \frac{H(5) - H(3)}{5 - 3} \\ 1 : \text{conclusion using} \\ \quad \text{Mean Value Theorem} \end{array} \right.$

- (c) The average height of the tree over the time interval $2 \leq t \leq 10$ is given by $\frac{1}{10-2} \int_2^{10} H(t) dt$.

$$\begin{aligned} \frac{1}{8} \int_2^{10} H(t) dt &\approx \frac{1}{8} \left(\frac{1.5+2}{2} \cdot 1 + \frac{2+6}{2} \cdot 2 + \frac{6+11}{2} \cdot 2 + \frac{11+15}{2} \cdot 3 \right) \\ &= \frac{1}{8} (65.75) = \frac{263}{32} \end{aligned}$$

The average height of the tree over the time interval $2 \leq t \leq 10$ is $\frac{263}{32}$ meters.

- (d) $G(x) = 50 \Rightarrow x = 1$

$$\frac{d}{dt}(G(x)) = \frac{d}{dx}(G(x)) \cdot \frac{dx}{dt} = \frac{(1+x)100 - 100x \cdot 1}{(1+x)^2} \cdot \frac{dx}{dt} = \frac{100}{(1+x)^2} \cdot \frac{dx}{dt}$$

$$\left. \frac{d}{dt}(G(x)) \right|_{x=1} = \frac{100}{(1+1)^2} \cdot 0.03 = \frac{3}{4}$$

According to the model, the rate of change of the height of the tree with respect to time when the tree is 50 meters tall is $\frac{3}{4}$ meter per year.

$$2: \begin{cases} 1: \text{trapezoidal sum} \\ 1: \text{approximation} \end{cases}$$

$$3: \begin{cases} 2: \frac{d}{dt}(G(x)) \\ 1: \text{answer} \end{cases}$$

Note: max 1/3 [1-0] if
no chain rule

Part (a)

$$H'(6) \approx \frac{H(7) - H(5)}{7 - 5} = \frac{11 - 6}{2} = \frac{5}{2}$$

$H'(6)$ is the **rate** at which the height of the tree is changing, **in meters per year**, at time $t = 6$ years.

Part (b)

$$\frac{H(5) - H(3)}{5 - 3} = \frac{6 - 2}{2} = 2$$

Because H is differentiable on $3 \leq t \leq 5$, H is continuous on $3 \leq t \leq 5$.

By the MVT, there exists a value c , $3 < c < 5$, such that $H'(c) = 2$.

Example 1 MVT and IVT

- (a) Explain why there must be at least one time t , for $2 < t < 10$, such that $H'(t) = 1.9$.
- (b) Use the MVT (twice) and the IVT (once) to explain why there must be at least one time t , for $2 < t < 10$, such that $H'(t) = 2$.

Solution

$$(a) \frac{H(7) - H(2)}{7 - 2} = \frac{11 - 1.5}{5} = 1.9$$

Because H is differentiable on $2 \leq t \leq 7$, H is continuous on $2 \leq t \leq 7$.

By the MVT, there exists a value c , $2 < c < 7$, such that $H'(c) = 1.9$.

Part (c)

$$\begin{aligned}\int_2^{10} H(t) dt &\approx \frac{1.5 + 2}{2} \cdot 1 + \frac{2 + 6}{2} \cdot 2 + \frac{6 + 11}{2} \cdot 2 + \frac{11 + 15}{2} \cdot 3 \\ &= \frac{263}{4} = 65.75\end{aligned}$$

Average height of the tree over the time interval $2 \leq t \leq 10$ is approximately

$$\frac{1}{8} \int_2^{10} H(t) dt \approx \frac{1}{8} \cdot \frac{263}{4} = \frac{263}{32} = 8.21875$$

Part (c)

Technology Solutions

	A t	B h	C	D
1	2	3/2		
2	3	2		
3	5	6		
4	7	11		
5	10	15		

$$\sum_{i=1}^4 \left(\frac{h[i]+h[i+1]}{2} \cdot (t[i+1]-t[i]) \right) = \frac{263}{4}$$

$$\frac{263}{4} \cdot \frac{1}{8} = \frac{263}{32}$$

$$\frac{263}{4} \cdot \frac{1}{8} = 8.21875$$

Example 2 Sums are Good, More is Better

- (a) Carefully sketch a graph to illustrate the trapezoidal sum found in part (c).
- (b) Use a left Riemann sum with the four subintervals indicated by the data in the table to approximate $\int_2^{10} H(t) dt$.
- (c) Use a right Riemann sum with the four subintervals indicated by the data in the table to approximate $\int_2^{10} H(t) dt$.
- (d) Use your answers in parts (b) and (c) to find the trapezoidal sum.
- (e) Suppose the function H is increasing, which seems pretty reasonable since H represents the height of the tree at time t .

Is the left Riemann sum an overestimate or underestimate for $\int_2^{10} H(t) dt$?
Explain your reasoning.

Is the right Riemann sum an overestimate or underestimate for $\int_2^{10} H(t) dt$?
Explain your reasoning.

Solution

$$(b) L_4 = 1.5 \cdot 1 + 2 \cdot 2 + 6 \cdot 2 + 11 \cdot 3 = 50.5$$

$$(c) R_4 = 2 \cdot 1 + 6 \cdot 2 + 11 \cdot 2 + 15 \cdot 3 = 81$$

$$(d) T_4 = \frac{L_4 + R_4}{2} = \frac{50.5 + 81}{2} = 65.75$$

TI-84 Plus calculator screen showing the calculation of the left Riemann sum L_4 . The expression $l := \sum_{i=1}^4 (h[i] \cdot (t[i+1] - t[i]))$ is entered, and the result $\frac{101}{2}$ is displayed. Below it, the expression $r := \sum_{i=1}^4 (h[i+1] \cdot (t[i+1] - t[i]))$ is entered, and the result 81 is displayed.

TI-84 Plus calculator screen showing the calculation of the trapezoidal sum T_4 . The expression $\frac{l+r}{2}$ is entered, and the result 65.75 is displayed.

Part (d)

$$G(x) = \frac{100x}{1+x} = 50 \implies 100x = 50 + 50x$$

$$50x = 50 \implies x = 1$$

$$\frac{d}{dt}[G(x)] = \frac{d}{dx}[G(x)] \cdot \frac{dx}{dt} = \frac{(1+x)100 - 100x \cdot 1}{(1+x)^2} \cdot \frac{dx}{dt} = \frac{100}{(1+x)^2} \cdot \frac{dx}{dt}$$

$$\left. \frac{d}{dt}[G(x)] \right|_{x=1} = \frac{100}{(1+1)^2} \cdot 0.03 = \frac{3}{4}$$

The rate of change of the height of the tree with respect to time when the tree is 50 meters tall is $\frac{3}{4}$ meters per year.

Part (d)

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1.4 1.5 2.1 *extension_ab4 RAD

$$g(x) := \frac{100 \cdot x}{1+x} \quad \text{Done}$$
$$dg(x) := \frac{d}{dx}(g(x)) \quad \text{Done}$$

⚠ $dg(1) \cdot 0.03$ 0.75

Example 3 Rate of Growth

Suppose the rate of growth of the tree is modeled by a differentiable function h , $2 \leq t \leq 15$, where h is measured in meters per year. Selected values of $h(t)$ are shown in the table. At time $t = 2$ the height of the tree is 1.5 meters.

t (years)	2	3	5	7	10	15
$h(t)$ (m/yr)	0.74	1.24	2.42	2.25	0.61	0.03

- (a) Use a left Riemann sum with five subintervals indicated by the table to estimate the total amount the tree grows over the interval $2 \leq t \leq 15$.
- (b) Use your answer to part (a) to find an estimate of the height of the tree after 15 years.
- (c) Suppose the rate of growth of the tree is modeled by the function g , where $2 \leq t \leq 15$, and

$$g(t) = \frac{457.6e^{-0.65t}}{(1 + 44e^{-0.65t})^2}$$

Use this function to estimate the height of the tree after 15 years.

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