TI in Focus: AP® Calculus

2018 AP® Calculus Exam: AB-3/BC-3
Concavity and Points of Inflection

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Outline

(1) Concavity: Motivation, Definition

(2) Concavity Test

(3) Points of Inflection: Definition

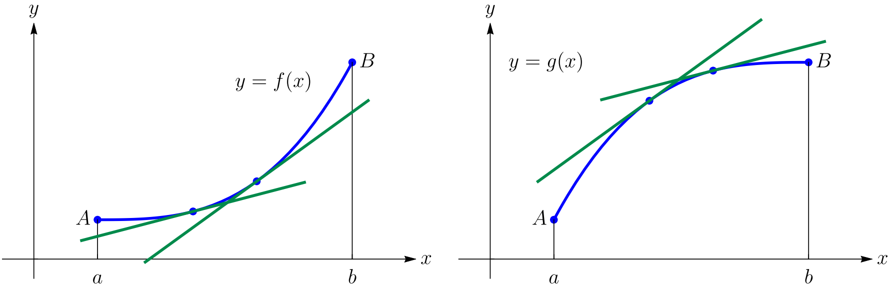
(4) Determining Inflection Points

(5) The Second Derivative Test

(6) Examples

What does f'' say about f?

Consider the graphs of two increasing functions on the interval $[a, b]$.



Notice:

The graph of f lies above each tangent line.

The graph of g lies below each tangent line.

Definition

Let f be a differentiable function.

f is **concave up** at a if the graph of f is above the tangent line to f at a for all x in a neighborhood containing a (but not equal to a).

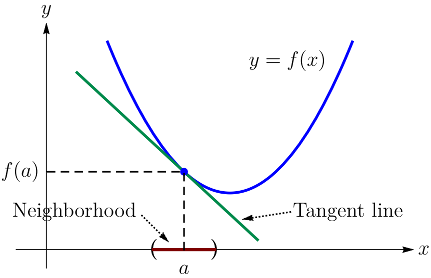
f is **concave down** at a if the graph of f is below the tangent line to f at a for all x in a neighborhood containing a (but not equal to a).

A Closer Look

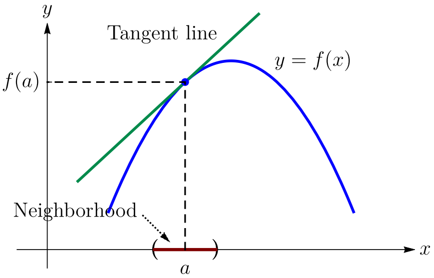
(1) Neighborhood containing a: an open interval containing a.

(2) Concavity is defined in terms of a single number, not an interval.

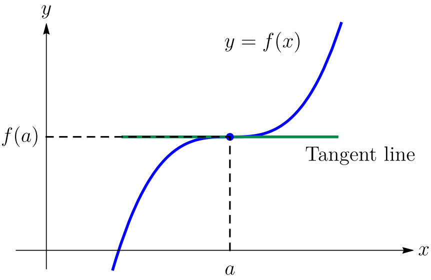
The graph of a function can be concave up, concave down, or have no concavity at a number.



The graph of f lies above the tangent line in a neighborhood of a.

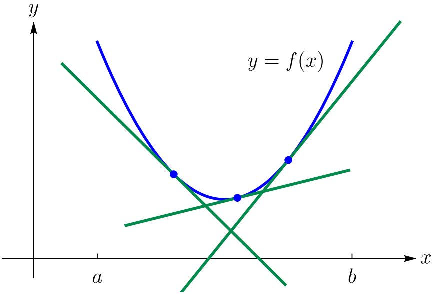


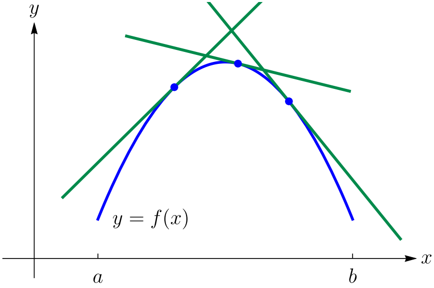
The graph of f lies below the tangent line in a neighborhood of a.



There is no neighborhood of a such that the graph of f is always above or always below the tangent line.

The second derivative can be used to help determine intervals of concavity.





What happens to the slope of the tangent line as x moves from left to right?

Concavity Test

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave up on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave down on I.

A Closer Look

(1) The Concavity Test allows us to draw a conclusion about the behavior of the graph of f if $f''(x) > 0$ or $f''(x) < 0$.

It doesn't say anything about the numbers a where $f''(a) = 0$ or $f''(a)$ DNE.

(2) The graph of a function is either concave up, concave down, or has no concavity at a number a.

(3) The point on the graph of f at which the graph changes concavity: inflection point.

Definition

A point P on the graph of f is called an **inflection point** (IP) if f is continuous there and the graph changes from concave up to concave down or from concave down to concave up at P.

A Closer Look

(1) If $f''(a)$ exists and $f''(a) \neq 0$: concavity is known, graph cannot change concavity at $(a, f(a))$.

$f''(x)$ can change sign only when $f''(x) = 0$ or $f''(x)$ DNE.

(2) Concavity Test: IP only where second derivative changes sign.

Use a sign chart for the second derivative.

Procedure for Determining Inflection Points

(1) Find the IP candidates:

Those x in the domain of f such that $f''(x) = 0$ or $f''(x)$ DNE.

(2) Screen the IP candidates:

Check for a change in sign of f'' at each candidate.

If a change in sign occurs, then $(x, f(x))$ is a point of inflection.

If no change in sign, then $(x, f(x))$ is not a point of inflection.

Example 1 Concavity and IPs

Let $f(x) = -x^4 + 3x^2 + 2$.

Find the interval(s) on which f is concave up or f is concave down, and find all inflection points.

Solution

Find f'' and factor completely.

$$f'(x) = -4x^3 + 6x$$

$$f''(x) = -12x^2 + 6 = -6(2x^2 - 1) = -6(\sqrt{2}x + 1)(\sqrt{2}x - 1)$$

Consider the IP candidates.

$$f''(x) = 0 \implies x = -\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}$$

$f''(x)$ DNE: none

Solution

Construct a sign chart for $f''(x)$.

$-$	0	$+$	0	$-$	$f''(x)$
	$-\frac{1}{\sqrt{2}}$		$\frac{1}{\sqrt{2}}$		x
CD	IP	CU	IP	CD	$f(x)$

$f''(x) > 0$ on $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$f''(x) < 0$ on $\left(-\infty, -\frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, \infty\right)$

f is concave up on $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

f is concave down on $\left(-\infty, -\frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, \infty\right)$

Solution

$f''(x)$ changes sign from negative to positive at $x = -\frac{1}{\sqrt{2}}$.

Therefore, f has an inflection point at $x = -\frac{1}{\sqrt{2}}$.

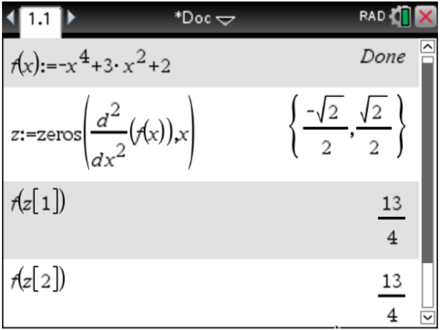
The inflection point is $\left(-\frac{1}{\sqrt{2}}, f\left(-\frac{1}{\sqrt{2}}\right)\right) = \left(-\frac{1}{\sqrt{2}}, \frac{13}{4}\right)$

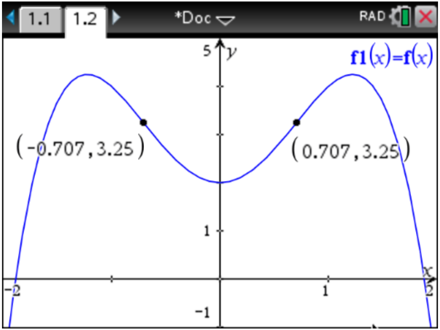
$f''(x)$ changes sign from positive to negative at $x = \frac{1}{\sqrt{2}}$.

Therefore, f has an inflection point at $x = \frac{1}{\sqrt{2}}$.

The inflection point is $\left(\frac{1}{\sqrt{2}}, f\left(\frac{1}{\sqrt{2}}\right)\right) = \left(\frac{1}{\sqrt{2}}, \frac{13}{4}\right)$

Solution


1.1
*Doc
RAD
f(x):=-x^4+3·x^2+2
Done
z:=zeros(d^2/dx^2(f(x)),x)
{-√2/2, √2/2}
f(z[1])
13/4
f(z[2])
13/4


1.1
1.2
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RAD
f1(x)=f(x)
y
5
1
-1
x
-2
1
2
(-0.707,3.25)
(0.707,3.25)

The Second Derivative Test

Suppose f'' is continuous in a neighborhood of c.

(a) If $f'(c) = 0$ and $f''(c) > 0$ then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$ then f has a local maximum at c.

A Closer Look

(1) Suppose $f''(x) > 0$ near c. Then f is concave up near c.
Therefore, the graph of f lies above the horizontal tangent line at c.
And, f has a local minimum at c.

(2) If $f'(c) = 0$ and $f''(c) = 0$: Second Derivative Test gives no information.

(3) Second Derivative Test cannot be used if $f''(x)$ DNE.

Example 2 Take A Chance

Let $f(x) = e^{-x^2/2}$

(a) Find the asymptote on the graph of f.

(b) Find the interval(s) of concavity and the inflection point(s).

(c) Find the local maximum and minimum values of f.

(d) Consider the function $g(x) = e^{-x^2/2\sigma^2}$ where σ is a positive constant. Answer parts (a), (b), and (c) for the function g.

(e) Graph the function g for several values of σ. Explain how the value of σ affects the shape of the graph.

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