

TI in Focus: AP[®] Calculus

2018 AP[®] Calculus Exam: AB-2

Displacement and Total Distance Traveled

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Outline

- (1) The Fundamental Theorem of Calculus; Net Change Theorem
- (2) Applications
- (3) Displacement
- (4) Total distance traveled
- (5) Examples

Background

- The Fundamental Theorem of Calculus, Part 2:

If f is continuous on $[a, b]$ and F is an antiderivative of f , then

$$\int_a^b f(x) dx = F(b) - F(a)$$

- F is an antiderivative of f : $F' = f$.

$$\int_a^b F'(x) dx = F(b) - F(a)$$

- $F'(x)$: represents the rate of change of $y = F(x)$ with respect to x .

$F(b) - F(a)$: net change in y as x changes from a to b .

Net Change Theorem

The definite integral of a rate of change (F') may be interpreted as the net change in the original function F .

$$\int_a^b F'(x) dx = F(b) - F(a)$$

A Closer Look

(1) Another interpretation:

$\int_a^b F'(x) dx$: an accumulation of the change in F over the interval $[a, b]$.

(2) A practical approach for solving many problems:

$$\underbrace{F(b)}_{\text{End amount}} = \underbrace{F(a)}_{\text{Start amount}} + \underbrace{\int_a^b F'(x) dx}_{\text{Net change}}$$

Net Change Theorem Applications

(1) $V'(t)$: rate at which water flows into a reservoir at time t .

$V(t)$: volume of water in the reservoir at time t .

$$\int_{t_1}^{t_2} V'(t) dt = V(t_2) - V(t_1)$$

Net change in the amount of water in the reservoir between times t_1 and t_2 .

(2) $C(t)$: concentration of the product of a chemical reaction at time t .

$C'(t)$: rate of the reaction.

$$\int_{t_1}^{t_2} C'(t) dt = C(t_2) - C(t_1)$$

Change in concentration of the chemical from time t_1 to time t_2 .

Net Change Theorem Applications

- (3) $m(x)$: mass of a rod measured from the left end to a point x .

$m'(x) = \rho(x)$: linear density

$$\int_a^b \rho(x) dx = m(b) - m(a)$$

Mass of the segment of the rod that lies between $x = a$ and $x = b$.

- (4) $P'(t)$: rate of growth of a population.

$$\int_{t_1}^{t_2} P'(t) dt = P(t_2) - P(t_1)$$

Net change in population during the time period from t_1 to t_2 .

- (5) $C(x)$: cost of producing x units of a commodity. $C'(x)$: marginal cost.

$$\int_{x_1}^{x_2} C'(x) dx = C(x_2) - C(x_1)$$

Increase in cost when production is increased from x_1 units to x_2 units.

Net change in cost.

Displacement

If a particle moves along a horizontal line so that its position at time t is given by $s(t)$, then its velocity is $v(t) = s'(t)$, and

$$\int_{t_1}^{t_2} v(t) dt = s(t_2) - s(t_1)$$

is the net change in position, or **displacement**, of the particle over the time period from t_1 to t_2

A Closer Look

- (1) Examples in which this was true for a particle moving only in a positive direction.
- (2) Now, more general: net change in position of the particle over the interval $[t_1, t_2]$.

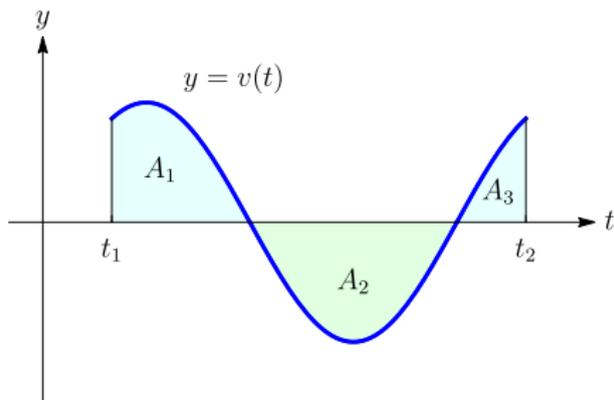
Total Distance Traveled

- Find the total distance traveled by a particle over a time interval.
- Could consider separately intervals where $v(t) \geq 0$ and where $v(t) \leq 0$.
- Simplify this process by considering $|v(t)|$, the speed of the particle.

- $\int_{t_1}^{t_2} |v(t)| dt =$ total distance traveled from time t_1 to time t_2 .

Interpretations

Displacement and total distance traveled can be interpreted in terms of areas under a velocity curve.



$$\text{displacement} = \int_{t_1}^{t_2} v(t) dt = A_1 - A_2 + A_3$$

$$\text{total distance traveled} = \int_{t_1}^{t_2} |v(t)| dt = A_1 + A_2 + A_3$$

Example 1 Displacement versus Total Distance Traveled

A particle moves along a horizontal axis so that its velocity at time t , $t \geq 0$, is given by $v(t) = t^3 - 6t^2 - t + 30$, where v is measured in meters per second

- Find the displacement of the particle during the time interval $[2, 8]$.
- Find the total distance traveled by the particle during the time interval $[2, 8]$.
- Suppose the particle is at position $x = 0$ at time $t = 2$. Sketch a graph of the position function for the time interval $[0, 8]$.

Solution

$$\begin{aligned} \text{(a)} \quad \int_2^8 v(t) dt &= \int_2^8 (t^3 - 6t^2 - t + 30) dt \\ &= \left[\frac{t^4}{4} - 2t^3 - \frac{t^2}{2} + 30t \right]_2^8 \\ &= \left[\frac{8^4}{4} - 2 \cdot 8^3 - \frac{8^2}{2} + 30 \cdot 8 \right] - \left[\frac{2^4}{4} - 2 \cdot 2^3 - \frac{2^2}{2} + 30 \cdot 2 \right] = 162 \end{aligned}$$

The particle's net change in position over the time interval $[2, 8]$ is 162 meters to the right.

Solution

(b) Find the intervals on which $v(t) \geq 0$ and those on which $v(t) \leq 0$.

$$v(t) = t^3 - 6t^2 - t + 30 = (t + 2)(t - 3)(t - 5)$$

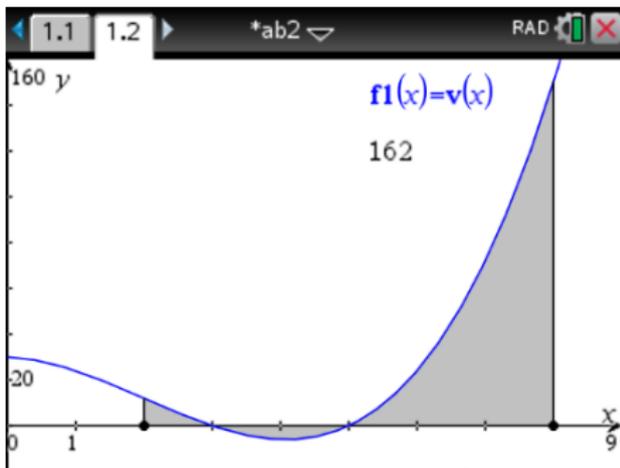
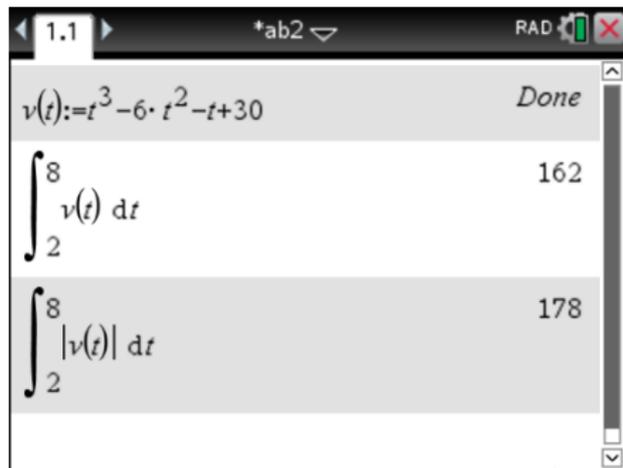
$$v(t) \geq 0: [0, 3], [5, \infty) \quad v(t) \leq 0: [3, 5]$$

$$\begin{aligned} \int_2^8 |v(t)| dt &= \int_2^3 v(t) dt + \int_3^5 -v(t) dt + \int_5^8 v(t) dt \\ &= \dots = 178 \end{aligned}$$

The particle travels a total distance of 178 meters over the time interval $[2, 8]$.

Solution

A technology solution and a visualization of the displacement.



Example 2 Velocity Tell-All

A particle moves along a horizontal axis so that its velocity at time t , $0 \leq t \leq 12$, is given by $v(t) = \sin(t) e^{-t^2/40}$, where v is measured in meters per second.

- (a) Find the displacement of the particle during the time interval $[1, 10]$.
- (b) Find the total distance traveled by the particle during the time interval $[1, 10]$.
- (c) The position of the particle is $x = 2$ at $t = 0$. Find the position of the particle at $t = 6$.
- (d) Find the time at which the particle is farthest to the right. Find its position at that time.
- (e) Find the maximum velocity of the particle.
- (f) Find the average speed of the particle over the time interval $[3, 8]$.
- (g) Sketch a graph of the position function.

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