TI in Focus: AP® Calculus

2018 AP® Calculus Exam: BC-6
Scoring Guidelines

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Outline

(1) Free Response Question

(2) Scoring Guidelines

(3) Student performance

(4) Interpretation

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6. The Maclaurin series for $\ln(1+x)$ is given by

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots.$$

On its interval of convergence, this series converges to $\ln(1+x)$. Let f be the function defined by

$f(x) = x\ln\left(1 + \frac{x}{3}\right).$

(a) Write the first four nonzero terms and the general term of the Maclaurin series for f.

(b) Determine the interval of convergence of the Maclaurin series for f. Show the work that leads to your answer.

(c) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Use the alternating series error bound to find an upper bound for $\left|P_4(2) - f(2)\right|$.

(a) The first four nonzero terms are $\frac{x^2}{3} - \frac{x^3}{2 \cdot 3^2} + \frac{x^4}{3 \cdot 3^3} - \frac{x^5}{4 \cdot 3^4}$.

The general term is $(-1)^{n+1} \frac{x^{n+1}}{n \cdot 3^n}$.

$$2 : \begin{cases} 1 : \text{first four terms} \\ 1 : \text{general term} \end{cases}$$

(b) $$\lim_{n\to\infty}\left|\frac{\dfrac{(-1)^{n+2}x^{n+2}}{(n+1)\left(3^{n+1}\right)}}{\dfrac{(-1)^{n+1}x^{n+1}}{n\cdot 3^n}}\right| = \lim_{n\to\infty}\left|\frac{-x}{3}\cdot\frac{n}{(n+1)}\right| = \left|\frac{x}{3}\right|$$

$\left|\frac{x}{3}\right| < 1$ for $|x| < 3$

Therefore, the radius of convergence of the Maclaurin series for f is 3.

— OR —

The radius of convergence of the Maclaurin series for $\ln(1+x)$ is 1, so the series for $f(x) = x\ln\left(1+\frac{x}{3}\right)$ converges absolutely for $\left|\frac{x}{3}\right| < 1$.

$\left|\frac{x}{3}\right| < 1 \Rightarrow |x| < 3$

Therefore, the radius of convergence of the Maclaurin series for f is 3.

5 :
- 1 : sets up ratio
- 1 : computes limit of ratio
- 1 : radius of convergence
- 1 : considers both endpoints
- 1 : analysis and interval of convergence

— OR —

5 :
- 1 : radius for $\ln(1+x)$ series
- 1 : substitutes $\frac{x}{3}$
- 1 : radius of convergence
- 1 : considers both endpoints
- 1 : analysis and interval of convergence

When $x = -3$, the series is $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-3)^{n+1}}{n \cdot 3^n} = \sum_{n=1}^{\infty}\frac{3}{n}$, which diverges by comparison to the harmonic series.

When $x = 3$, the series is $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3^{n+1}}{n \cdot 3^n} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{3}{n}$, which converges by the alternating series test.

The interval of convergence of the Maclaurin series for f is $-3 < x \le 3$.

(c) By the alternating series error bound, an upper bound for $|P_4(2) - f(2)|$ is the magnitude of the next term of the alternating series.

$$|P_4(2) - f(2)| < \left| -\frac{2^5}{4 \cdot 3^4} \right| = \frac{8}{81}$$

2 : { 1 : uses fifth-degree term as error bound
1 : answer }

Student Performance

Part (a)

- Some students tried to compute the coefficients by differentiating the function and evaluating, but had difficulty with the algebra.
- Some students were able to use function composition and multiplication to find the Maclaurin series.
- Some students found the Maclaurin series for $\dfrac{x}{3}\ln(1+x)$

Part(b)

- Most students applied the ratio test appropriately.
- And most students understood it was necessary to check the endpoints.
- Common error: endpoint conclusion without supporting work.

Student Performance

Part (c)

- It appeared many students did not understand the question.
- Some students used the wrong term in the series.
- Notation issues: fifth term versus fifth degree term.

Part (a) 1: first four terms

- Must be the correct answer.
- The answer does not need to be simplified, but must distribute.
- First four terms may be in a list.

Examples

(1) $\dfrac{x^2}{3} - \dfrac{x^3}{18} + \dfrac{x^4}{81} - \dfrac{x^5}{324}$ 1 - ?

(2) $\dfrac{x\left(\frac{x}{3}\right)}{1} - \dfrac{x\left(\frac{x}{3}\right)^2}{2} + \dfrac{x\left(\frac{x}{3}\right)^3}{3} - \dfrac{x\left(\frac{x}{3}\right)^4}{4}$ 1 - ?

(3) $x\left(\dfrac{x}{3} - \dfrac{x^2}{18} + \dfrac{x^3}{81} - \dfrac{x^4}{324}\right)$? - ?

(4) $\dfrac{x^2}{3} - \dfrac{x^3}{18} + \dfrac{x^4}{81} - \dfrac{x^5}{108}$ 0 - ?

Part (a) 1: general term

- Must be our general term.
- Does not need to be simplified.
- If the summation is included, start at $n = 0$ or $n = 1$.
- General term must be declared in part (a) to earn the point.

Examples

(1) $(-1)^{n+1}\dfrac{x\left(\frac{x}{3}\right)^n}{n}$? - 1

(2) $(-1)^{n+1}\dfrac{\left(x\cdot\frac{x}{3}\right)^n}{n}$? - 0

(3) $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{n+1}}{n\cdot 3^n}$? - 1

Part (b) 1: sets up ratio

- This point is earned for the correct ratio. Setup point.
- Limit and/or absolute value not necessary.
- If no general term given in part (a), then it must be our ratio.

Examples

(1) $\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+2}}{3^{n+1}(n+1)}\cdot\frac{3^n(n)}{(-1)^{n+1}x^{n+1}}\right|$ 1 - ? - ? - ? - ?

(2) $\frac{x^{n+2}}{(n+1)3^{n+1}}\cdot\frac{n3^n}{x^{n+1}}$ 1 - ? - ? - ? - ?

Part (b) 1: computes limit of ratio

- Point is earned for the correct limit.
- Must see an explicit indication of a limit: $\lim$ or $\to$
- Absolute value symbols may suddenly appear.
- If the response indicates the limit is $= \frac{0}{0}$, this point is not earned.

Example

(1) $$\lim_{n\to\infty} \frac{x^{n+2}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{x^{n+1}} = \left|\frac{x}{3}\right|$$

Part (b) 1: radius of convergence

- Eligibility: limit $\neq 0$, $|x|$, $|x^2|$, ∞ from consecutive terms.
- Philosophy: Need a ratio and a non-trivial radius of convergence.
- Read with the presented limit.
- Earned for identifying the radius of convergence.

  $-3 < x < 3 \qquad |x| < 3.$
- If the response is $\left|\frac{x}{3}\right|$, need additional work to convey the radius of convergence.

Part (b) 1: considers both endpoints

- Eligibility: radius of converence $\neq 0,\ \infty$ and centered at 0.
- Philosophy: Considers the endpoints, not simply identifies the endpoints.
- There may be errors but a response can still earn the point.

$$(-1)^{n+1}\,\frac{3^{n+1}}{n\cdot 3^n} \quad \text{and} \quad (-1)^{n+1}\,\frac{-3^{n+1}}{n\cdot 3^n}$$

- May be earned verbally.

Part (b) 1: analysis and interval of convergence

- Correct justification for both endpoints and the correct interval.
- Correct interval: $(-3, 3]$. Endpoints must be correct.
- Acceptable justification for $x = 3$:
  converges by AST
  converges, alternating harmonic
  converges, it is 3 times alternating harmonic.
- Acceptable justification for $x = -3$:
  diverges, it is harmonic
  diverges, it is 3 times harmonic
  diverges by the P-test
  diverges by the comparison test, comparing with the harmonic series

Part (c) 1: uses fifth-degree term as error bound

- Must use fifth-degree term.
- $\left|-\dfrac{x^5}{4\cdot 3^4}\right|$ or $\dfrac{x^5}{4\cdot 3^4}$ 1 - ?
- $\left|\dfrac{x^6}{5\cdot 3^5}\right|$ (Fifth term) 0 - ?

Part (c) 1: answer

- First and second point can be earned in one step.
- $\dfrac{2^5}{4\cdot 3^4}$ 1 - 1
- $\dfrac{2^6}{5\cdot 3^5}$? - 0
- $\dfrac{-2^5}{4\cdot 3^4}$ 1 - ?

Imports

Eligible for points if and only if the general term is a Maclaurin series.

(1) $3n$

The $3n$ response is any Maclaurin series with a general term that does not include a 3^n.

(2) Incorrect Maclaurin series in part (a) that contains 3^n.

(3) Factorial case.

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