

## TI in Focus: AP<sup>®</sup> Calculus

2018 AP<sup>®</sup> Calculus Exam: BC-6

Technology Solutions and Problem Extensions

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## Outline

- (1) Free Response Question
- (2) Scoring Guidelines
- (3) Solutions using technology
- (4) Problem Extensions

6. The Maclaurin series for  $\ln(1+x)$  is given by

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1} \frac{x^n}{n} + \cdots.$$

On its interval of convergence, this series converges to  $\ln(1+x)$ . Let  $f$  be the function defined by

$$f(x) = x \ln\left(1 + \frac{x}{3}\right).$$

- (a) Write the first four nonzero terms and the general term of the Maclaurin series for  $f$ .
- (b) Determine the interval of convergence of the Maclaurin series for  $f$ . Show the work that leads to your answer.
- (c) Let  $P_4(x)$  be the fourth-degree Taylor polynomial for  $f$  about  $x = 0$ . Use the alternating series error bound to find an upper bound for  $|P_4(2) - f(2)|$ .

(a) The first four nonzero terms are  $\frac{x^2}{3} - \frac{x^3}{2 \cdot 3^2} + \frac{x^4}{3 \cdot 3^3} - \frac{x^5}{4 \cdot 3^4}$ .

The general term is  $(-1)^{n+1} \frac{x^{n+1}}{n \cdot 3^n}$ .

2 :  $\begin{cases} 1 : \text{first four terms} \\ 1 : \text{general term} \end{cases}$

$$(b) \lim_{n \rightarrow \infty} \left| \frac{\frac{(-1)^{n+2} x^{n+2}}{(n+1)(3^{n+1})}}{\frac{(-1)^{n+1} x^{n+1}}{n \cdot 3^n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{-x}{3} \cdot \frac{n}{(n+1)} \right| = \left| \frac{x}{3} \right|$$

$$\left| \frac{x}{3} \right| < 1 \text{ for } |x| < 3$$

Therefore, the radius of convergence of the Maclaurin series for  $f$  is 3.

— OR —

The radius of convergence of the Maclaurin series for  $\ln(1+x)$  is 1, so the series for  $f(x) = x \ln\left(1 + \frac{x}{3}\right)$  converges absolutely for  $\left|\frac{x}{3}\right| < 1$ .

$$\left| \frac{x}{3} \right| < 1 \Rightarrow |x| < 3$$

Therefore, the radius of convergence of the Maclaurin series for  $f$  is 3.

5 :  $\left\{ \begin{array}{l} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{radius of convergence} \\ 1 : \text{considers both endpoints} \\ 1 : \text{analysis and interval of convergence} \end{array} \right.$

— OR —

5 :  $\left\{ \begin{array}{l} 1 : \text{radius for } \ln(1+x) \text{ series} \\ 1 : \text{substitutes } \frac{x}{3} \\ 1 : \text{radius of convergence} \\ 1 : \text{considers both endpoints} \\ 1 : \text{analysis and interval of convergence} \end{array} \right.$

When  $x = -3$ , the series is  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-3)^{n+1}}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{3}{n}$ , which diverges by comparison to the harmonic series.

When  $x = 3$ , the series is  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^{n+1}}{n \cdot 3^n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{3}{n}$ , which converges by the alternating series test.

The interval of convergence of the Maclaurin series for  $f$  is  $-3 < x \leq 3$ .

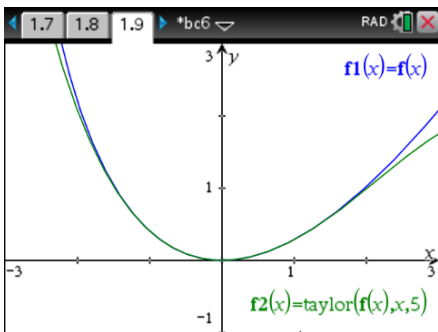
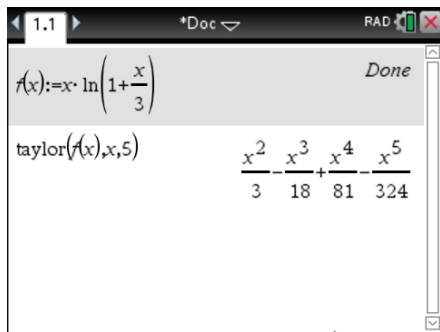
- (c) By the alternating series error bound, an upper bound for  $|P_4(2) - f(2)|$  is the magnitude of the next term of the alternating series.

$$|P_4(2) - f(2)| < \left| -\frac{2^5}{4 \cdot 3^4} \right| = \frac{8}{81}$$

2 :  $\begin{cases} 1 : \text{uses fifth-degree term} \\ \quad \text{as error bound} \\ 1 : \text{answer} \end{cases}$

## Part (a)

First for terms and the general term of the Maclaurin series for  $f$ .



The general term is  $(-1)^{n+1} \frac{x^{n+1}}{n \cdot 3^n}$

## Part (b)

- Use the Ratio Test to determine the radius of convergence  $R$ .
- Define the  $n$ th term of the series.
- Compute the quotient for the Ratio Test.
- Consider the limit associated with the Ratio Test.

TI-84 Plus calculator screen showing the definition of the  $n$ th term  $a(n,x)$  and the ratio test quotient.

Top: 1.1 1.2 1.3 \*bc6 ▾ RAD

Equation Editor:  $a(n,x) := \frac{(-1)^{n+1} \cdot x^{n+1}}{n \cdot 3^n}$  Done

Bottom:  $\left| \frac{a(n+1,x)}{a(n,x)} \right|$   $\frac{\left| \frac{n \cdot x}{n+1} \right|}{3}$

TI-84 Plus calculator screen showing the limit of the ratio test quotient and the resulting inequality for  $x$ .

Top: 1.1 1.2 1.3 \*bc6 ▾ RAD

Equation Editor:  $\lim_{n \rightarrow \infty} \left( \frac{\left| \frac{n \cdot x}{n+1} \right|}{3} \right)$   $\frac{|x|}{3}$

Bottom:  $\text{solve}\left(\frac{|x|}{3} < 1, x\right)$   $-3 < x < 3$



## Part (b)

Consider the endpoints.

TI-84 Plus calculator screen showing two summations for  $x = -3$ . The mode is set to RAD and the variable mode is set to \*bc6.

Top row:  $\sum_{n=1}^{\infty} (a(n, -3))$  and  $3 \cdot \sum_{n=1}^{\infty} \left( \frac{1}{n} \right)$

Bottom row:  $\sum_{n=1}^{\infty} (a(n, 3))$  and  $-3 \cdot \sum_{n=1}^{\infty} \left( \frac{\cos(n \cdot \pi)}{n} \right)$

TI-84 Plus calculator screen showing two sequences for  $x = -3$  and  $x = 3$ . The mode is set to RAD and the variable mode is set to \*bc6.

Top row:  $\text{seq}(a(n, -3), n, 1, 10)$  and  $\left\{ 3, \frac{3}{2}, 1, \frac{3}{4}, \frac{3}{5}, \frac{1}{2}, \frac{3}{7}, \frac{3}{8}, \frac{1}{3}, \frac{3}{10} \right\}$

Bottom row:  $\text{seq}(a(n, 3), n, 1, 10)$  and  $\left\{ 3, \frac{-3}{2}, 1, \frac{-3}{4}, \frac{3}{5}, \frac{-1}{2}, \frac{3}{7}, \frac{-3}{8}, \frac{1}{3}, \frac{-3}{10} \right\}$

The interval of convergence of the Maclaurin series for  $f$  is  $-3 < x \leq 3$ .

## Example 1 Mainstream Maclaurin

Write the first four nonzero terms and the general term of the Maclaurin series for  $f$ . Determine the interval of convergence of the Maclaurin series for  $f$ .

(a)  $f(x) = \frac{x}{3} \ln(1+x)$

(b)  $f(x) = x \ln(1+3x)$

(c)  $f(x) = x^2 \ln(1+x^2)$

(d)  $f(x) = \ln\left(1 - \frac{x^2}{3}\right)$

## Part (c)

By the alternating series error bound, an upper bound for  $|P_4(2) - f(2)|$  is the magnitude of the next term of the alternating series.



## Example 2 Error Bounds

Suppose the first four non-zero terms are used to approximate the sum of the series. Find an upper bound on the error of estimation.

(a)  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$

(b)  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$

(c)  $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 3}{2^n}$

(d)  $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n^2)}$

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