TI in Focus: AP® Calculus

2018 AP® Calculus Exam: BC-6
Radius of Convergence and Interval of Convergence

Stephen Kokoska
Professor, Bloomsburg University
Former AP® Calculus Chief Reader

Texas Instruments

Outline

(1) Power series and convergence

(2) Radius of convergence

(3) Interval of convergence

(4) Endpoints

(5) Examples

(6) Use of the Ratio Test

Power Series Convergence

- The set of values x for which a power series converges is an interval.
- Finite interval, infinite interval, or a collapsed interval, for example $\{0\}$.

Theorem

For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities.

(1) The series converges only when $x = a$.

(2) The series converges for all x.

(3) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

A Closer Look

(1) The number R is called the **radius of convergence** of the power series.

$R = 0$ in case (1) and $R = \infty$ in case (2).

(2) The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges.

(3) Case (1): the interval is a single number a.

Case (2): the interval is $(-\infty, \infty)$.

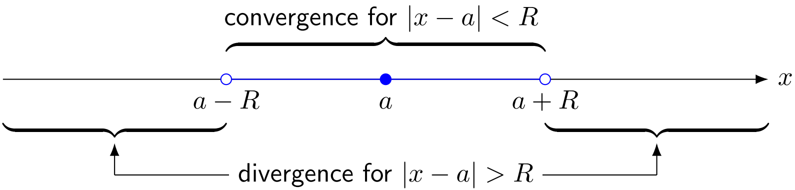
Case (3): the inequality $|x - a| < R$ can be rewritten as $a - R < x < a + R$.

When $x = a \pm R$, an endpoint of the interval, anything can happen.

There are four possibilities for the interval of convergence.

$(a - R, a + R) \quad (a - R, a + R] \quad [a - R, a + R) \quad [a - R, a + R]$

Graphical illustration of the interval of convergence.



Note: We always have to check each endpoint of the interval separately for convergence.

Here are some examples of radius and interval of convergence.

Series	Radius of convergence	Interval of convergence
$\sum_{n=0}^{\infty} x^n$	$R = 1$	$(-1, 1)$
$\sum_{n=0}^{\infty} n!(3x+1)^n$	$R = 0$	$\{-\frac{1}{3}\}$
$\sum_{n=0}^{\infty} \frac{(x-4)^n}{n}$	$R = 1$	$[3, 5)$
$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$R = \infty$	$(-\infty, \infty)$

Determining the Radius of Convergence

- The Ratio Test, or sometimes the Root Test, should be used to determine the radius of convergence R.
- These tests are inconclusive when x is an endpoint of the interval of convergence.
- The endpoints must be checked using some other method.

Example 1 Radius and Interval of Convergence

Find the radius of convergence and interval of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{\sqrt{n}}\, x^n.$$

Solution

Let $a_n = \dfrac{(-1)^n 4^n}{\sqrt{n}}\, x^n$ and consider the quotient for the Ratio Test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}4^{n+1}x^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(-1)^n 4^n x^n}\right|$$

Definition of nth term; division of fractions

$$= \left|-4x\sqrt{\frac{n}{n+1}}\right| = 4\sqrt{\frac{n}{n+1}}\;|x|$$

Simplify; property of absolute value

Solution

Evaluate the limit associated with the Ratio Test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} 4\sqrt{\frac{n}{n+1}}\;|x|$$

$$= \lim_{n\to\infty} 4\sqrt{\frac{1}{1+(1/n)}}\;|x| = 4|x|$$

Series converges if $4|x| < 1$ and diverges if $4|x| > 1$.

Therefore, converges if $|x| < \frac{1}{4}$ and diverges if $|x| > \frac{1}{4}$

Radius of convergence: $R = \frac{1}{4}$

The series converges for x in the interval $\left(-\frac{1}{4}, \frac{1}{4}\right)$

Solution

Test the endpoints of the interval.

$x = -\frac{1}{4}$

$$\sum_{n=1}^{\infty} \frac{(-1)^n 4^n \left(-\frac{1}{4}\right)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots$$

p-series with $p = \frac{1}{2} < 1$. The series diverges.

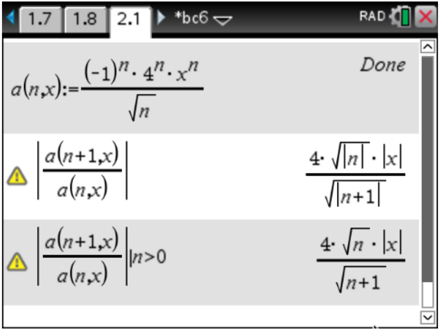
$x = \frac{1}{4}$:

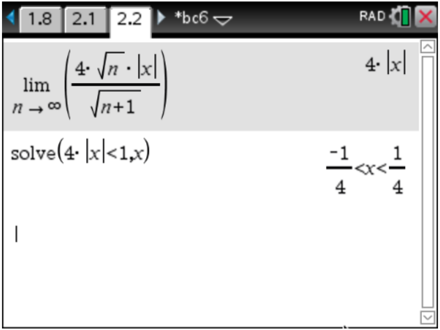
$$\sum_{n=1}^{\infty} \frac{(-1)^n 4^n \left(\frac{1}{4}\right)^n}{\sqrt{n}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

The series converges by the Alternating Series Test.

Interval of convergence: $\left(-\frac{1}{4}, \frac{1}{4}\right]$

Technology Solution





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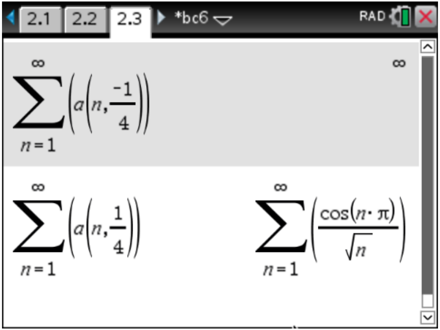

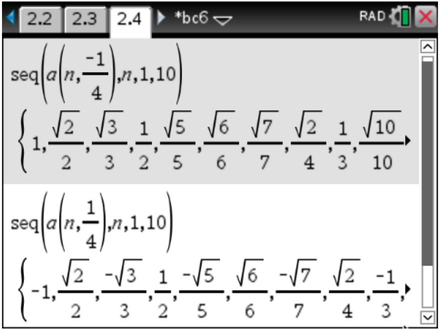

Example 2 Interval Training

Find the radius of convergence and the interval of convergence of the series.

(a) $$\sum_{n=0}^{\infty} \frac{(2x+1)^n}{2^n}$$

(b) $$\sum_{n=1}^{\infty} \frac{x^n}{n^5 5^n}$$

(c) $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n4^n} \, x^n$$

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