TI in Focus: AP® Calculus

2018 AP® Calculus Exam: BC-6
Technology Solutions and Problem Extensions

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Outline

(1) Free Response Question

(2) Scoring Guidelines

(3) Solutions using technology

(4) Problem Extensions

6. The Maclaurin series for $\ln(1+x)$ is given by

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots.$$

On its interval of convergence, this series converges to $\ln(1+x)$. Let f be the function defined by

$f(x) = x\ln\left(1 + \frac{x}{3}\right)$.

(a) Write the first four nonzero terms and the general term of the Maclaurin series for f.

(b) Determine the interval of convergence of the Maclaurin series for f. Show the work that leads to your answer.

(c) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Use the alternating series error bound to find an upper bound for $\left|P_4(2) - f(2)\right|$.

(a) The first four nonzero terms are $\frac{x^2}{3} - \frac{x^3}{2 \cdot 3^2} + \frac{x^4}{3 \cdot 3^3} - \frac{x^5}{4 \cdot 3^4}$.

The general term is $(-1)^{n+1} \frac{x^{n+1}}{n \cdot 3^n}$.

$2 : \begin{cases} 1 : \text{first four terms} \\ 1 : \text{general term} \end{cases}$

(b) $\lim_{n\to\infty}\left|\dfrac{\dfrac{(-1)^{n+2}x^{n+2}}{(n+1)\left(3^{n+1}\right)}}{\dfrac{(-1)^{n+1}x^{n+1}}{n\cdot 3^{n}}}\right| = \lim_{n\to\infty}\left|\dfrac{-x}{3}\cdot\dfrac{n}{(n+1)}\right| = \left|\dfrac{x}{3}\right|$

$\left|\dfrac{x}{3}\right| < 1$ for $|x| < 3$

Therefore, the radius of convergence of the Maclaurin series for f is 3.

— OR —

The radius of convergence of the Maclaurin series for $\ln(1+x)$ is 1, so the series for $f(x) = x\ln\left(1+\dfrac{x}{3}\right)$ converges absolutely for $\left|\dfrac{x}{3}\right| < 1$.

$\left|\dfrac{x}{3}\right| < 1 \Rightarrow |x| < 3$

Therefore, the radius of convergence of the Maclaurin series for f is 3.

5 :
- 1 : sets up ratio
- 1 : computes limit of ratio
- 1 : radius of convergence
- 1 : considers both endpoints
- 1 : analysis and interval of convergence

— OR —

5 :
- 1 : radius for $\ln(1+x)$ series
- 1 : substitutes $\dfrac{x}{3}$
- 1 : radius of convergence
- 1 : considers both endpoints
- 1 : analysis and interval of convergence

When $x = -3$, the series is $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-3)^{n+1}}{n \cdot 3^n} = \sum_{n=1}^{\infty}\frac{3}{n}$, which diverges by comparison to the harmonic series.

When $x = 3$, the series is $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3^{n+1}}{n \cdot 3^n} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{3}{n}$, which converges by the alternating series test.

The interval of convergence of the Maclaurin series for f is $-3 < x \le 3$.

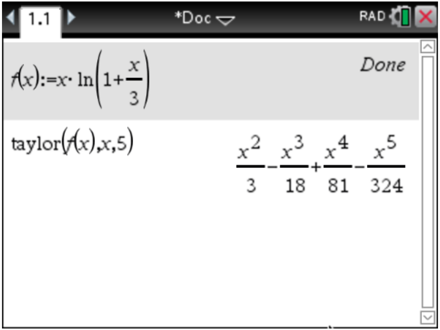
(c) By the alternating series error bound, an upper bound for $|P_4(2) - f(2)|$ is the magnitude of the next term of the alternating series.

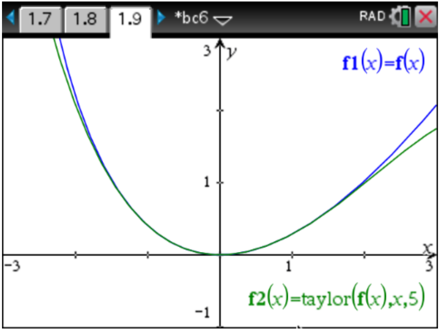
$$|P_4(2) - f(2)| < \left|-\frac{2^5}{4 \cdot 3^4}\right| = \frac{8}{81}$$

$$2 : \begin{cases} 1 : \text{uses fifth-degree term} \\ \quad\ \text{as error bound} \\ 1 : \text{answer} \end{cases}$$

Part (a)

First for terms and the general term of the Maclaurin series for f.

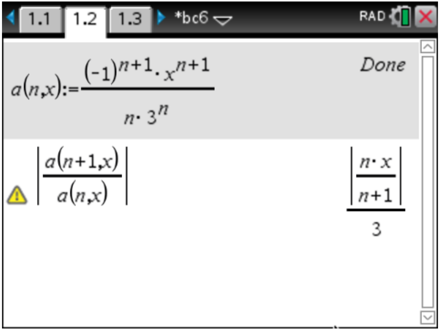


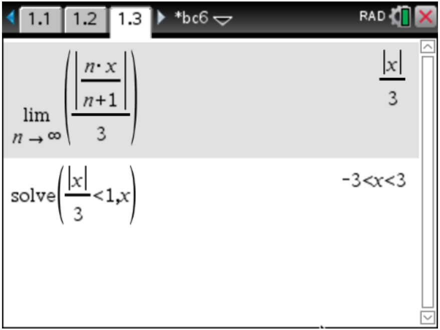


The general term is $(-1)^{n+1} \dfrac{x^{n+1}}{n \cdot 3^n}$

Part (b)

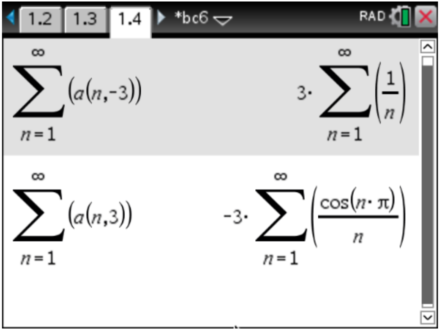
- Use the Ratio Test to determine the radius of convergence R.
- Define the nth term of the series.
- Compute the quotient for the Ratio Test.
- Consider the limit associated with the Ratio Test.

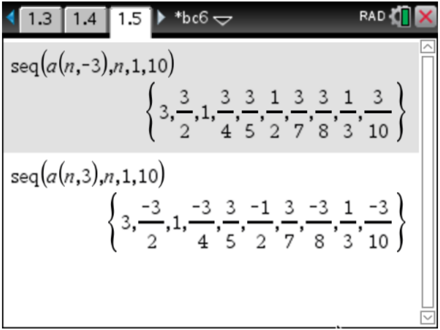




Part (b)

Consider the endpoints.





The interval of convergence of the Maclaurin series for f is $-3 < x \leq 3$.

Example 1 Mainstream Maclaurin

Write the first four nonzero terms and the general term of the Maclaurin series for f. Determine the interval of convergence of the Maclaurin series for f.

(a) $f(x) = \dfrac{x}{3}\ln(1+x)$

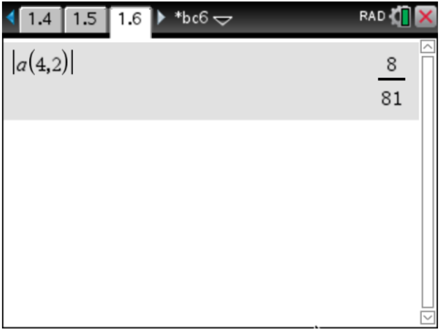
(b) $f(x) = x\ln(1+3x)$

(c) $f(x) = x^2\ln(1+x^2)$

(d) $f(x) = \ln\left(1 - \dfrac{x^2}{3}\right)$

Part (c)

By the alternating series error bound, an upper bound for $|P_4(2) - f(2)|$ is the magnitude of the next term of the alternating series.



Example 2 Error Bounds

Suppose the first four non-zero terms are used to approximate the sum of the series. Find an upper bound on the error of estimation.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 3}{2^n}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n^2)}$

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