

TI in Focus: AP[®] Calculus

2018 AP[®] Calculus Exam: BC-5

Tangents to Polar Curves

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Outline

- (1) Slope of the tangent line
- (2) Horizontal tangent lines
- (3) Vertical tangent lines
- (4) Slope at the pole
- (5) Example

Tangents to Polar Curves

- Find the tangent line to a polar curve described by $r = f(\theta)$.
- Consider θ as a parameter and write the parametric equations that describe the curve.
- $x = r \cos \theta = f(\theta) \cos \theta$ $y = r \sin \theta = f(\theta) \sin \theta$
- Use the method for finding the slope of a curve defined by parametric equations and the Product Rule.

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta}$$

- Note: $\frac{dr}{d\theta}$ is the derivative of r with respect to θ , not the slope of the tangent line to the graph of $r = f(\theta)$.

Special Tangents

- Horizontal tangent line: $\frac{dy}{d\theta} = 0$, provided $\frac{dx}{d\theta} \neq 0$.
- Vertical tangent line: $\frac{dx}{d\theta} = 0$, provided $\frac{dy}{d\theta} \neq 0$.
- Tangent line at the pole: let $r = 0$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta} \sin \theta + 0 \cdot \cos \theta}{\frac{dr}{d\theta} \cos \theta - 0 \cdot \sin \theta} = \frac{\frac{dr}{d\theta} \sin \theta}{\frac{dr}{d\theta} \cos \theta} = \tan \theta$$

$$\text{if } \frac{dr}{d\theta} \neq 0 \text{ and } \cos \theta \neq 0$$

A Closer Look

- It isn't necessary to memorize the expression for $\frac{dy}{dx}$.
- Apply the method used to derive this equation.
- Suppose $r = 1 + \cos \theta$.

$$x = r \cos \theta = (1 + \cos \theta) \cos \theta = \cos \theta + \cos^2 \theta$$

$$y = r \sin \theta = (1 + \cos \theta) \sin \theta = \sin \theta + \frac{1}{2} \sin 2\theta$$

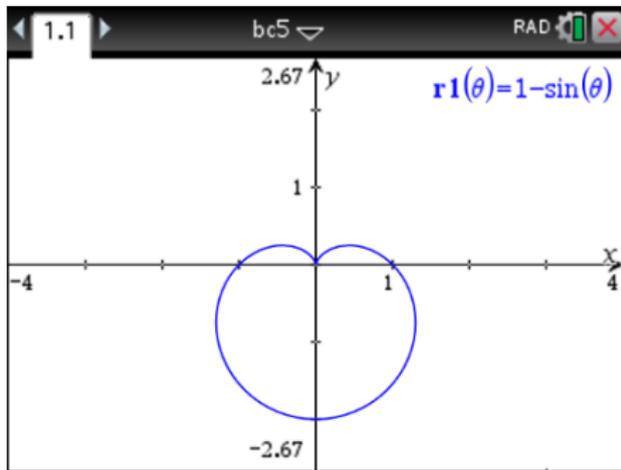
$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cos \theta + \cos 2\theta}{-\sin \theta - 2 \cos \theta \sin \theta} = \frac{\cos \theta + \cos 2\theta}{-\sin \theta - \sin 2\theta}$$

Example 1 Heart Exam

Consider the cardioid described by the polar equation $r = 1 - \sin \theta$.

- (a) Find the slope of the tangent line to the cardioid when $\theta = \frac{\pi}{3}$.
- (b) Find the points on the cardioid where the tangent line is horizontal or vertical.

Solution



Solution

$$(a) \quad x = r \cos \theta = (1 - \sin \theta) \cos \theta = \cos \theta - \frac{1}{2} \sin 2\theta$$

$$y = r \sin \theta = (1 - \sin \theta) \sin \theta = \sin \theta - \sin^2 \theta$$

$$\frac{dy}{dx} = \frac{\cos \theta - 2 \sin \theta \cos \theta}{-\sin \theta - \cos 2\theta} = \frac{\cos \theta - \sin 2\theta}{-\sin \theta - \cos 2\theta}$$

Slope of the tangent line at the point where $\theta = \frac{\pi}{3}$

$$\begin{aligned} \left. \frac{dy}{dx} \right|_{\theta=\pi/3} &= \frac{\cos \frac{\pi}{3} - \sin \frac{2\pi}{3}}{-\sin \frac{\pi}{3} - \cos \frac{2\pi}{3}} \\ &= \frac{\frac{1}{2} - \frac{\sqrt{3}}{2}}{-\frac{\sqrt{3}}{2} - \left(-\frac{1}{2}\right)} = \frac{\frac{1-\sqrt{3}}{2}}{\frac{-\sqrt{3}+1}{2}} \\ &= \frac{1 - \sqrt{3}}{-\sqrt{3} + 1} = 1 \end{aligned}$$

Solution

(a) Technology solution

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$r(t) := 1 - \sin(t)$	Done
$x(t) := r(t) \cdot \cos(t)$	Done
$y(t) := r(t) \cdot \sin(t)$	Done
$\frac{d}{dt}(y(t))$	1
$\frac{d}{dt}(x(t)) \Big _{t=\frac{\pi}{3}}$	$\frac{\pi}{3}$

Solution

$$(b) \frac{dy}{d\theta} = \cos \theta(1 - 2 \sin \theta) = 0$$

$$\cos \theta = 0 \quad \text{or} \quad 1 - 2 \sin \theta = 0 \implies \sin \theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2} \quad \text{or} \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\frac{dx}{d\theta} = -\sin \theta - \cos 2\theta = (2 \sin \theta + 1)(\sin \theta - 1) = 0$$

$$2 \sin \theta + 1 = 0 \implies \sin \theta = -\frac{1}{2} \implies \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\sin \theta - 1 = 0 \implies \sin \theta = 1 \implies \theta = \frac{\pi}{2}$$

Solution

(b) Horizontal tangent lines at the points: $\left(2, \frac{3\pi}{2}\right)$, $\left(\frac{1}{2}, \frac{\pi}{6}\right)$, $\left(\frac{1}{2}, \frac{5\pi}{6}\right)$

Vertical tangent lines at the points: $\left(\frac{3}{2}, \frac{7\pi}{6}\right)$, $\left(\frac{3}{2}, \frac{11\pi}{6}\right)$

At $\theta = \frac{\pi}{2}$: both $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ are 0.

$$\begin{aligned}\lim_{\theta \rightarrow (\pi/2)^+} \frac{dy}{dx} &= \left[\lim_{\theta \rightarrow (\pi/2)^+} \frac{1 - 2 \sin \theta}{2 \sin \theta + 1} \right] \left[\lim_{\theta \rightarrow (\pi/2)^+} \frac{\cos \theta}{\sin \theta - 1} \right] \\ &= -\frac{1}{3} \cdot \lim_{\theta \rightarrow (\pi/2)^+} \frac{\cos \theta}{\sin \theta - 1} = -\frac{1}{3} \cdot \lim_{\theta \rightarrow (\pi/2)^+} \frac{-\sin \theta}{\cos \theta} \\ &= -\frac{1}{3} \cdot \frac{-1}{(-)} = -\infty\end{aligned}$$

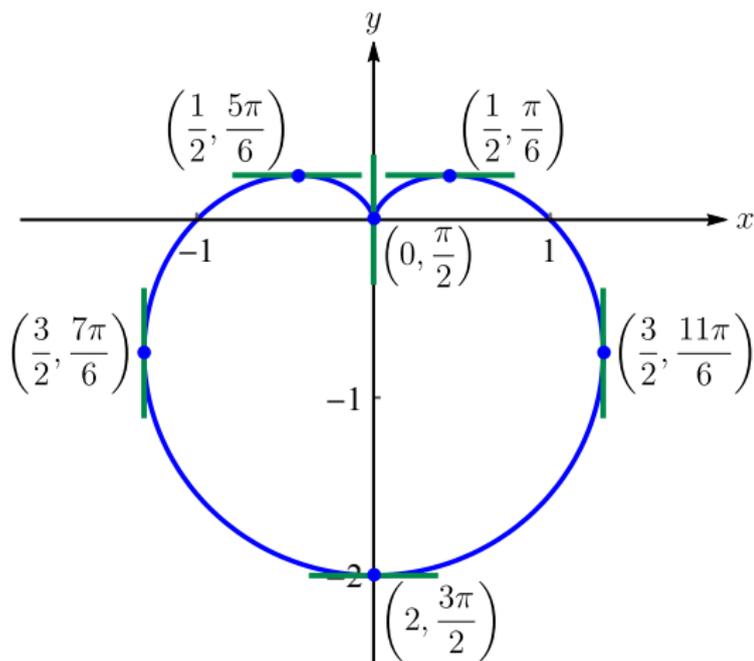


Solution

(b) Similarly, by symmetry, $\lim_{\theta \rightarrow (\pi/2)^-} \frac{dy}{dx} = \infty$

Therefore, the graph has a vertical tangent line at the pole.

Here's a graph of the polar equation in the Cartesian plane.



Example 2 Slopes

Find the slope of the tangent line to the graph of polar equation at the point specified by the value of θ .

(a) $r = 2 + \sin\left(\frac{3\theta}{2}\right), \quad \theta = \frac{\pi}{4}$

(b) $r = \sin 2\theta, \quad \theta = \frac{\pi}{3}$

Example 3 Horizontal or Vertical

Find the points on the graph of the polar equation where the tangent line is horizontal or vertical.

(a) $r = e^{-\theta/2}$

(b) $r = \cos\left(\frac{\theta}{2}\right)$

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