

TI in Focus: AP[®] Calculus

2018 AP[®] Calculus Exam: AB-6

Euler's Method

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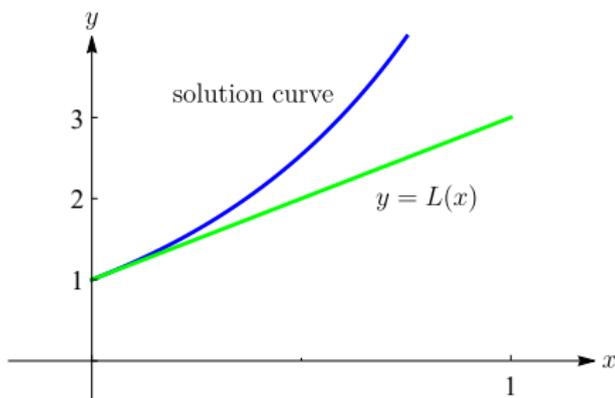
Outline

- (1) Background
- (2) Generalization, Illustration
- (3) Euler's Method
- (4) Example
- (5) Technology

Linear Approximation

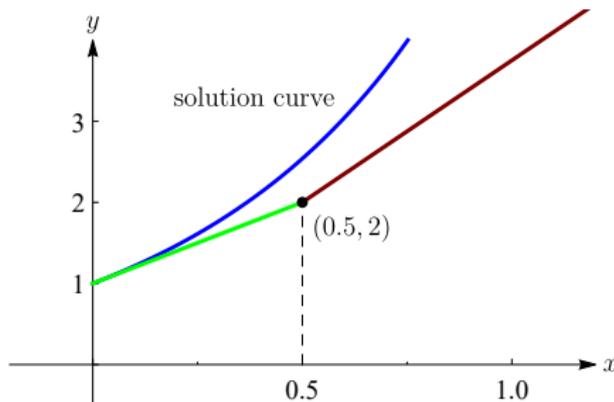
- Consider the initial-value problem $y' = -x + 2y$; $y(0) = 1$
- Using the differential equation: $y'(0) = -0 + 2(1) = 2$
- Therefore, the solution curve has slope 2 at the point $(0, 1)$.
- A first approximation to the solution curve: the linear approximation.

$$L(x) = 1 + 2(x - 0) = 1 + 2x$$



Approximation Improvement

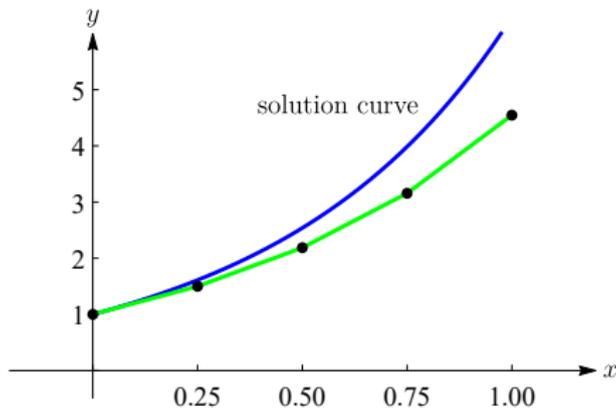
- Travel a short distance along this tangent line and make a midcourse correction. Change direction as indicated by the slope field.
- The figure shows what happens if we stop at $x = 0.5$. The horizontal distance traveled is the **step size**.



- Since $L(0.5) = 1 + 2(0.5) = 2$, then $y(0.5) \approx 2$.
- Take $(0.5, 2)$ as the new starting point.
Using the DE: $y'(0.5) = -0.5 + 2(2) = 3.5$
New linear function: $y = 2 + 3.5(x - 0.5) = 0.25 + 3.5x$

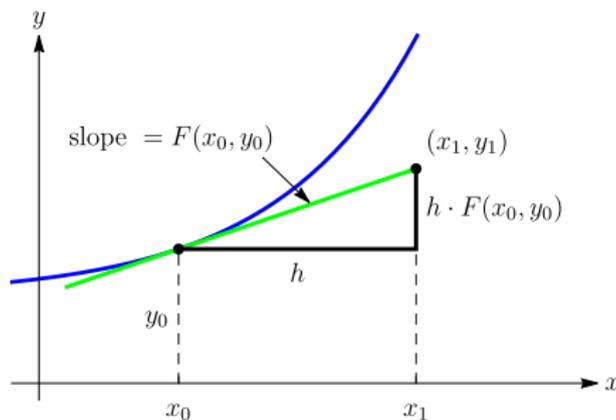
More Improvement: Euler's Method

- Start at the initial value and proceed in the direction indicated by the slope field.
- Stop after a short distance; consider the slope at the new location; proceed in the new direction.
- Continue stopping and changing direction according to the slope field.



Generalization

- Consider the first-order initial value problem: $y' = F(x, y)$, $y(x_0) = y_0$
- Goal: find approximate values for the solution at equally spaced numbers: $x_0, x_1 = x_0 + h, x_2 = x_1 + h, \dots$ where h is the step size
- Use the DE to find the slope at (x_0, y_0) ; $y' = F(x_0, y_0)$.



$$y_1 = y_0 + h \cdot F(x_0, y_0)$$

$$y_2 = y_1 + h \cdot F(x_1, y_1)$$

$$y_n = y_{n-1} + h \cdot F(x_{n-1}, y_{n-1})$$

Euler's Method

Approximate values for the solution of the initial-value problem $y' = F(x, y)$, $y(x_0) = y_0$, with step size h , at $x_n = x_{n-1} + h$, are

$$y_n = y_{n-1} + h \cdot F(x_{n-1}, y_{n-1}) \quad n = 1, 2, 3, \dots$$

A Closer Look

- (1) The step size in Euler's Method could be negative.
For example, h could be -0.1 .
- (2) For more accurate approximations: decrease the step size.
- (3) Alternative notation: Let $h = \Delta x$ and use Leibniz notation for the derivative.

$$y_n = y_{n-1} + \Delta x \cdot \frac{dy}{dx}$$

Example 1 One Step at a Time

Use Euler's method with step size 0.1 to construct a table of approximate values for the solution to the initial-value problem

$$y' = -x + 2y \quad y(0) = 1$$

Solution

Given: $h = 0.1$, $x_0 = 0$, $y_0 = 1$, $F(x, y) = -x + 2y$

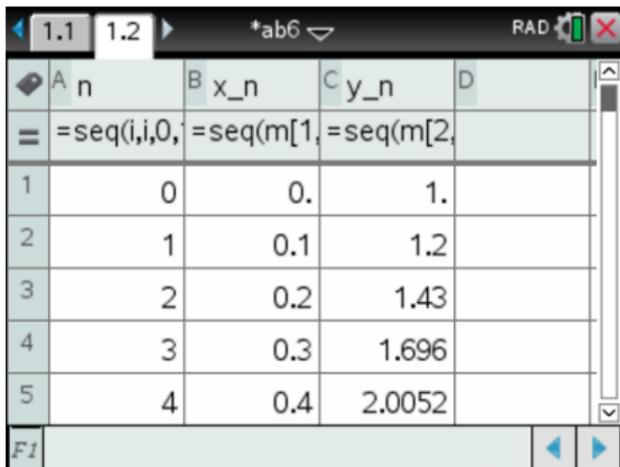
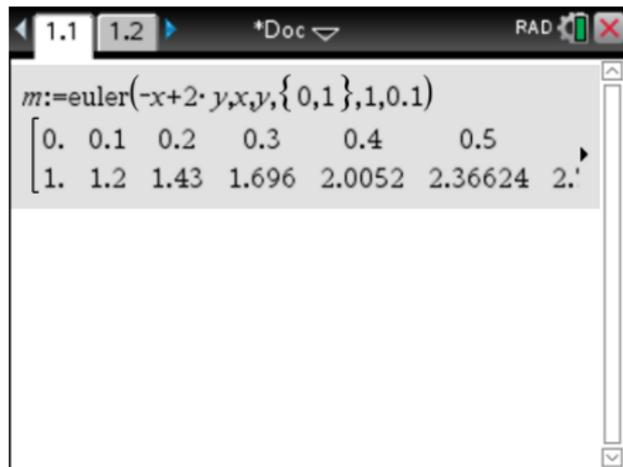
$$y_1 = y_0 + h \cdot F(x_0, y_0) = 1 + 0.1 \cdot (-0 + 2(1)) = 1.2$$

$$y_2 = y_1 + h \cdot F(x_1, y_1) = 1.2 + 0.1 \cdot (-0.1 + 2(1.2)) = 1.43$$

$$y_3 = y_2 + h \cdot F(x_2, y_2) = 1.43 + 0.1 \cdot (-0.2 + 2(1.43)) = 1.696$$

Interpretation: If $y(x)$ is the exact solution to the initial-value problem, then $y(0.3) \approx 1.696$

Technology Solution



Example 2 Decreasing Step Size

Use Euler's method with decreasing step size to approximate $y(0.5)$ and $y(1)$.

What do you notice about these approximations as the step size decreases?

Notes

- (1) Reminder: the step size can be negative.
- (2) A tabular method to find Euler approximations.

x	y	Δx	$\frac{dy}{dx}$	$\Delta y = \Delta x \cdot \frac{dy}{dx}$
0.0	1	0.1	2	0.2
0.1	1.2	0.1	2.3	0.23
0.2	1.43	0.1	2.66	0.266
0.3	1.696	0.1	3.092	0.3092

Example 3 Euler's Method Practice

- (a) Use Euler's method with step size 0.1 to estimate $y(1)$ where $y(x)$ is the solution of the initial-value problem $y' = \sin(x + y)$, $y(0) = 0$.
- (b) Use Euler's method with step size 0.2 to estimate $y(0.6)$ where $y(x)$ is the solution of the initial-value problem $y' = \frac{0.75x}{y}$, $y(0) = 6$.
- (c) Use Euler's method with step size 0.1 to estimate $y(1)$ where $y(x)$ is the solution of the initial-value problem $y' = x - xy^2$, $y(0) = 1$.
- (d) Let $y = f(x)$ be the particular solution to the differential equation $y' = x - y$ having initial condition $f(0) = 1$. Use Euler's method with two equal steps to estimate $f(0.4)$.

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