TI in Focus: AP® Calculus

2017 AP® Calculus Exam: BC-6

Technology Solutions and Problem Extensions

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Outline

(1) Free Response Question

(2) Scoring Guidelines

(3) Solutions using technology

(4) Problem extensions

$$f(0) = 0$$
$$f'(0) = 1$$
$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \text{ for all } n \geq 1$$

6. A function f has derivatives of all orders for $-1 < x < 1$. The derivatives of f satisfy the conditions above. The Maclaurin series for f converges to $f(x)$ for $|x| < 1$.

(a) Show that the first four nonzero terms of the Maclaurin series for f are $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$, and write the general term of the Maclaurin series for f.

(b) Determine whether the Maclaurin series described in part (a) converges absolutely, converges conditionally, or diverges at $x = 1$. Explain your reasoning.

(c) Write the first four nonzero terms and the general term of the Maclaurin series for $g(x) = \int_0^x f(t)\,dt$.

(d) Let $P_n\left(\frac{1}{2}\right)$ represent the nth-degree Taylor polynomial for g about $x = 0$ evaluated at $x = \frac{1}{2}$, where g is the function defined in part (c). Use the alternating series error bound to show that

$$\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right| < \frac{1}{500}.$$

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(a) $f(0) = 0$

$f'(0) = 1$

$f''(0) = -1(1) = -1$

$f'''(0) = -2(-1) = 2$

$f^{(4)}(0) = -3(2) = -6$

The first four nonzero terms are

$$0 + 1x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-6}{4!}x^4 = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}.$$

The general term is $\dfrac{(-1)^{n+1}x^n}{n}$.

$3 : \begin{cases} 1 : f''(0),\ f'''(0), \text{ and } f^{(4)}(0) \\ 1 : \text{verify terms} \\ 1 : \text{general term} \end{cases}$

(b) For $x = 1$, the Maclaurin series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

The series does not converge absolutely because the harmonic series diverges.

The series alternates with terms that decrease in magnitude to 0 and therefore the series converges conditionally.

2 : converges conditionally with reason

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(c) $\int_0^x f(t)\,dt = \int_0^x \left(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots + \frac{(-1)^{n+1} t^n}{n} + \cdots \right) dt$

$$= \left[\frac{t^2}{2} - \frac{t^3}{3 \cdot 2} + \frac{t^4}{4 \cdot 3} - \frac{t^5}{5 \cdot 4} + \cdots + \frac{(-1)^{n+1} t^{n+1}}{(n+1)n} + \cdots \right]_{t=0}^{t=x}$$

$$= \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20} + \cdots + \frac{(-1)^{n+1} x^{n+1}}{(n+1)n} + \cdots$$

$3 : \begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \\ 1 : \text{general term} \end{cases}$

(d) The terms alternate in sign and decrease in magnitude to 0. By the alternating series error bound, the error $\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right|$ is bounded by the magnitude of the first unused term, $\left| -\frac{(1/2)^5}{20} \right|$.

Thus, $\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right| \le \left| -\frac{(1/2)^5}{20} \right| = \frac{1}{32 \cdot 20} < \frac{1}{500}$.

1 : error bound

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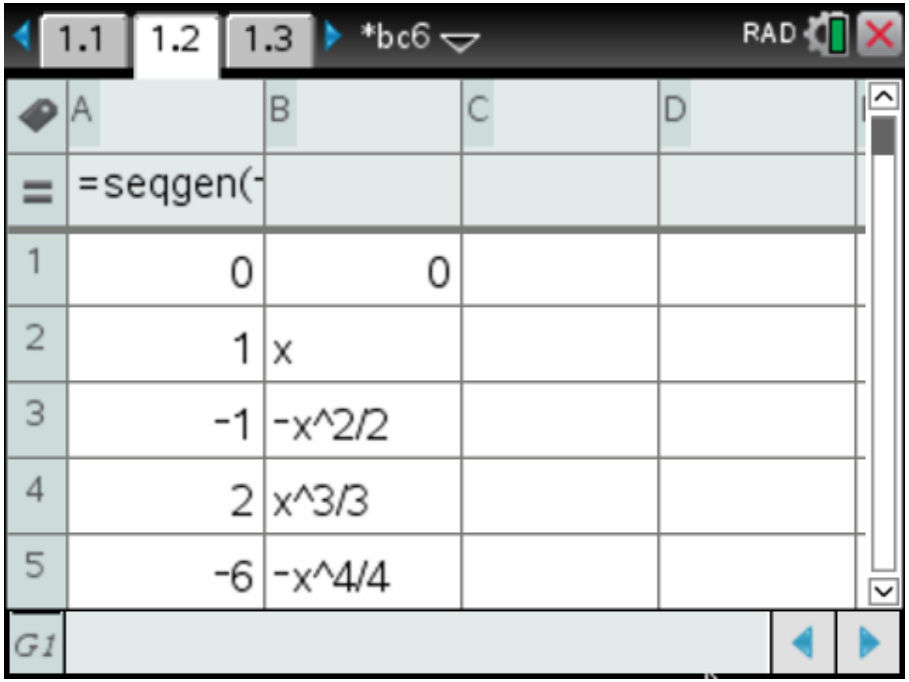
Part (a)

- Generate a sequence to find $f''(0)$, $f'''(0)$, and $f^{(4)}(0)$.
- Use these values to find the first four nonzero terms of the Maclaurin series.
- The general term is $\dfrac{(-1)^{n+1}x^n}{n}$.

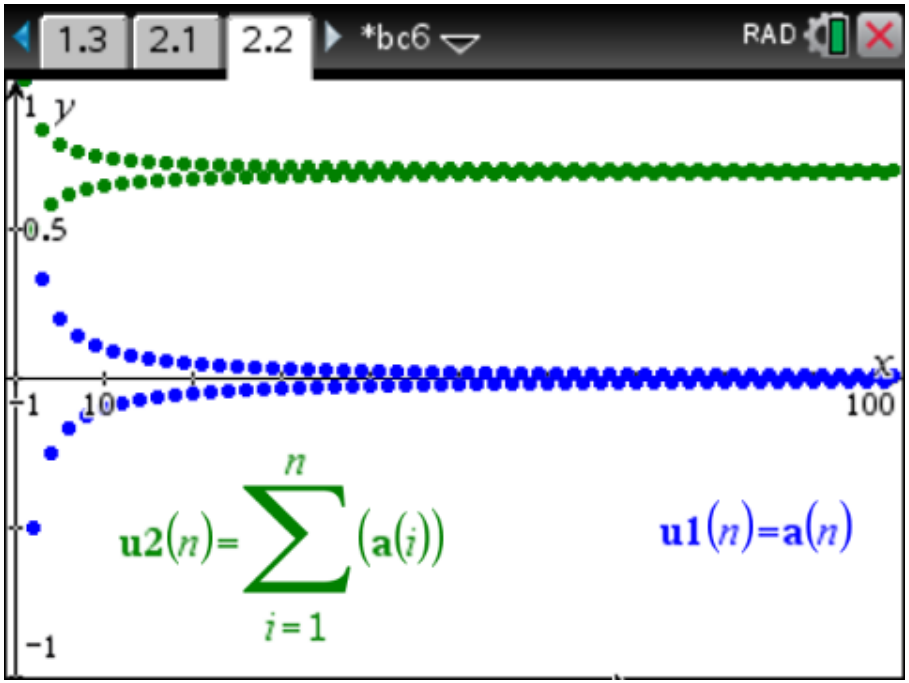


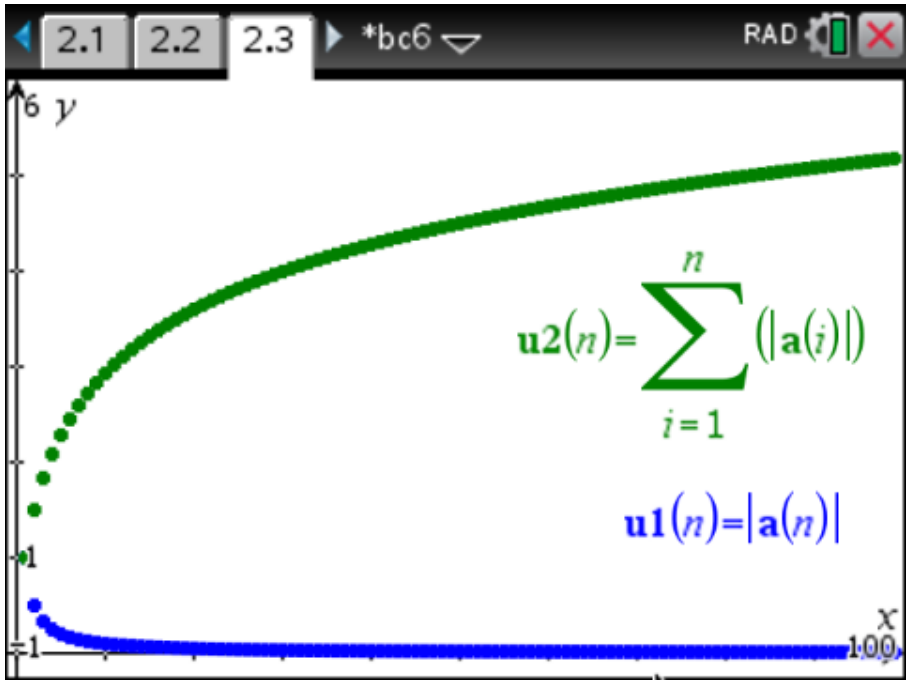
Part (b)

- For $x = 1$, the Maclaurin series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

  Converges, alternating harmonic series, or AST.

- $\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ Diverges, harmonic series.





Part (b)

Some numerical confirmation.

2.2 2.3 2.4 *bc6 RAD

	A n	B	C	D
1	10.	0.645635	2.92897	
2	100.	0.688172	5.18738	
3	1000.	0.692647	7.48547	
4	10000.	0.693097	9.78761	
5	100000.	0.693142	12.0901	

C6

2.3 2.4 2.5 *bc6 RAD

$$\sum_{i=1}^{50}(a(i)) \qquad \frac{211741335802448128775 3}{30990445042459967064 00}$$

$$\sum_{i=1}^{50}(|a(i)|) \qquad \frac{13943237577224054960759}{30990445042459967064 00}$$

Example 1 Radius and Interval of Convergence

Find the radius of convergence and the interval of convergence of the Maclaurin series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}$$

Solution

Let $a_n = \dfrac{(-1)^{n+1}x^n}{n}$ and consider the quotient for the Ratio Test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}x^{n+1}}{n+1} \cdot \frac{n}{(-1)^{n+1}x^n}\right|$$ Definition of nth term; division of fractions

$$= \frac{n}{n+1}|x|$$ Simplify; property of absolute value

Solution

Consider the limit associated with the Ratio Test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n}{n+1}|x| = |x|$$

Divide numerator and denominator by n; evaluate limit

By the Ratio Test, the series converges if $|x| < 1$ and diverges if $|x| > 1$.

The radius of convergence is $R = 1$.

$a = 0$

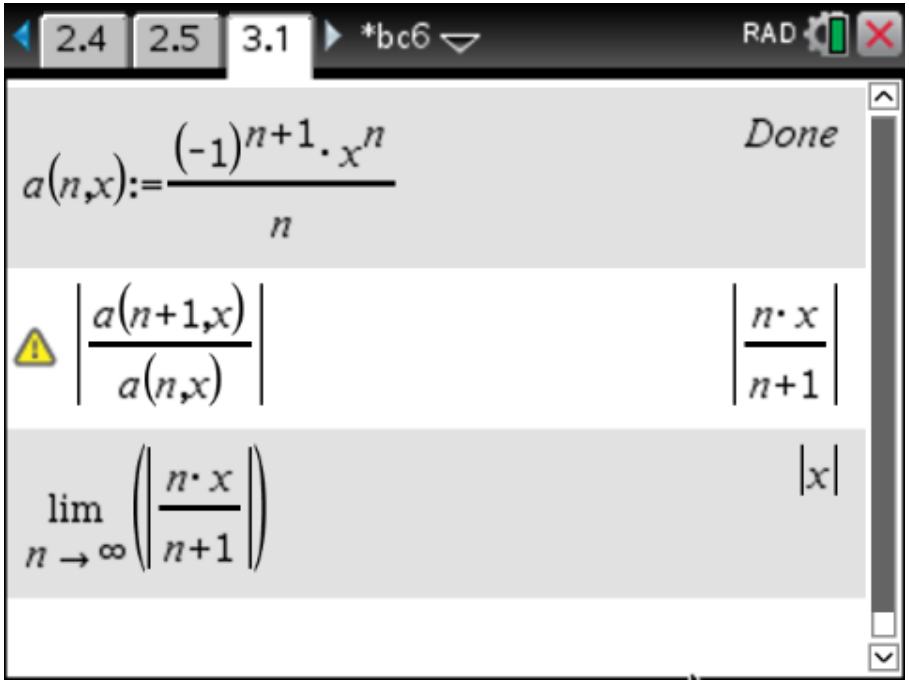
$$x = 1 : \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} \quad \text{Converges, AST}$$

$$x = -1 : \sum_{n=1}^{\infty}\frac{(-1)n + 1(-1)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{2n+1}}{n} = (-1)\sum_{n=1}^{\infty}\frac{1}{n} \quad \text{Diverges}$$

The interval of convergence is $(-1, 1]$.

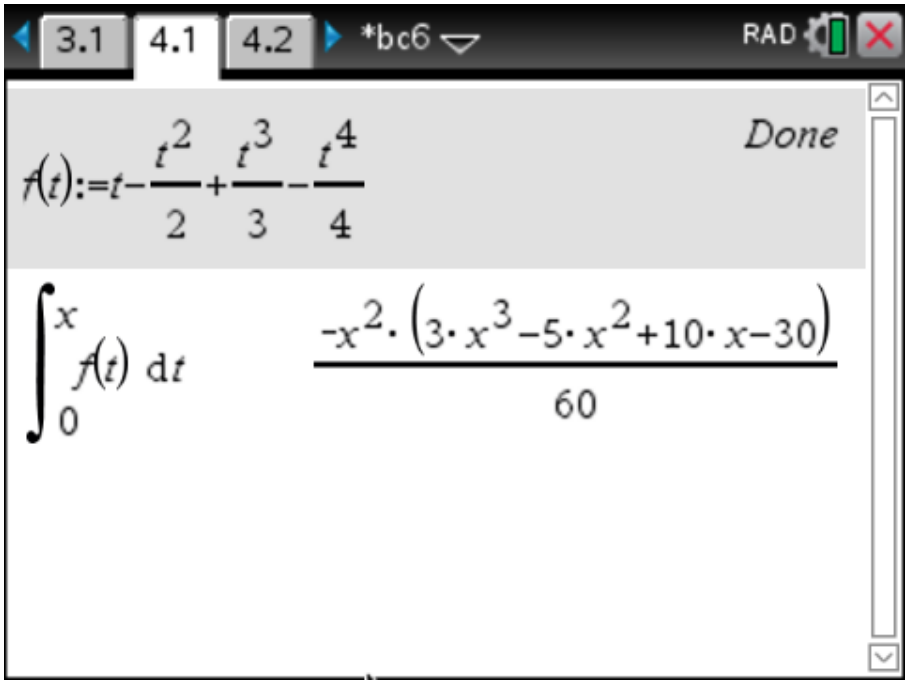
Solution

- Define the nth term of the series.
- Consider the quotient for the Ratio Test.
- Consider the limit associated with the Ratio Test.



Part (c)

- Define a function $f(t)$ (with four terms).
- Evaluate the definite integral.



Example 2 Radius and Interval of Convergence, Part 2

Consider the function f defined by its Maclaurin series

$$f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$$

(a) Find the interval of convergence for $\int f(x)\,dx$

(b) Find $f'(x)$ and find the interval of convergence for $f'(x)$.

Note:

If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has a radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

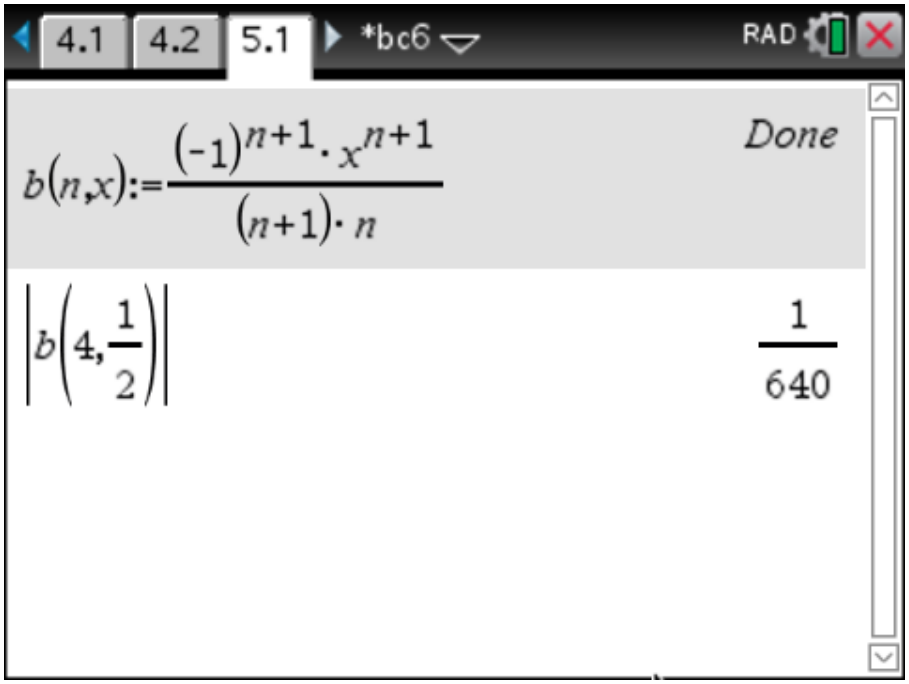
is differentiable on the interval $(a-R, a+R)$.

What is the radius of convergence for $f'(x)$ and $\int f(x)\,dx$?

What is the interval of convergence for $f'(x)$ and $\int f(x)\,dx$?

Part (d)

- Define the nth term of the series.
- Find the magnitude (absolute value) of the first unused term.



Example 3 An Examination of Errors

Consider the function f defined by its Maclaurin series

$$f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$$

(a) Use the Maclaurin series to approximate $f(0.5)$ to six decimal places.

(b) Find a closed form expression for $f(x)$.

(c) Let $T_4(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$. Use your expression found in part (b) to sketch a graph of the remainder, $|R_4(x)| = |f(x) - T_4(x)|$ on the interval $[-0.5, 0.5]$ and justify the general shape of this graph.

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