TI in Focus: AP® Calculus

2017 AP® Calculus Exam: BC-5
Improper Integrals; Type 1: Infinite Intervals

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Outline

(1) Background: Type 1 Improper Integrals

(2) Geometric interpretation

(3) Definition; limits

(4) Examples

(5) Convergence/Divergence by Comparison

Background

- In the definition of the definite integral $\displaystyle\int_a^b f(x)\,dx$:

  We assumed that the function f was defined on a finite interval $[a, b]$

  (and that f did not have an infinite discontinuity)

- Extend the concept of a definite integral to the case in which the interval is infinite

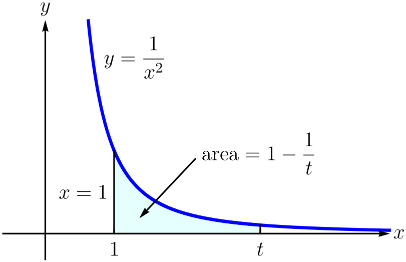
  (and also to the case in which f has an infinite discontinuity in $[a, b]$)

- In either case, the integral is called an *improper* integral.

Understanding the Definition

- Consider the infinite region R that lies under the curve $y = \dfrac{1}{x^2}$, above the x-axis, and to the right of the line $x = 1$.
- It seems reasonable that since R is infinite in extent, its area must be infinite.
- The area of the part of R that lies to the left of the line $x = t$:

$$A(t) = \int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\bigg]_1^t = 1 - \frac{1}{t}$$



Understanding the Definition

- $A(t) = 1 - \dfrac{1}{t} < 1$ no matter how large a value of t is selected.

- Consider the limit as t increases without bound.

  $$\lim_{t\to\infty} A(t) = \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1$$

- This limit indicates that the area of the shaded region approaches 1 as $t \to \infty$.

  It also suggests the *area* of the infinite region R is 1.

- We express this result mathematically using the following notation.

  $$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx = 1$$

Improper Integral of Type 1

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

Improper Integral of Type 1

(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

In part (c) any real number a can be used.

A Closer Look

(1) Remember: ∞ is a symbol, not a number.

You cannot simply find an antiderivative of f and evaluate at ∞ or $-\infty$.

(2) Evaluating a Type 1 improper integral is a two-step process.

First, evaluate a definite integral (and simplify), and then evaluate the limit.

(3) Any of the improper integrals in this definition can be interpreted as an area provided that f is a positive function.

For example, in case (a) if $f(x) \geq 0$ and the integral $\displaystyle\int_a^\infty f(x)\,dx$ is convergent, then we define the area of the region

$R = \{(x,y) \,|\, x \geq a, 0 \leq y \leq f(x)\}$ in $A(R) = \displaystyle\int_a^\infty f(x)\,dx$

Example 1 Infinite Region Finite Area

Determine whether the integral is convergent or divergent.

$$\int_1^\infty \frac{e^{-1/x}}{x^2}\,dx$$

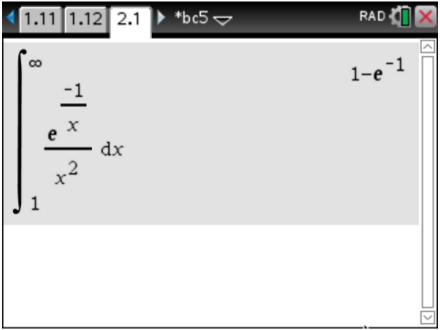
Solution

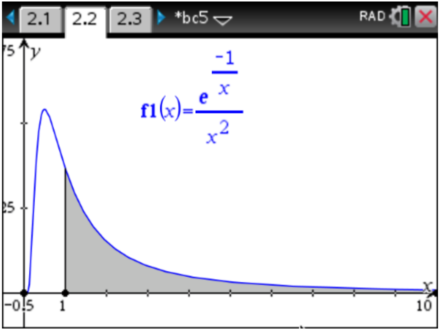
$$\int_1^\infty \frac{e^{-1/x}}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{e^{-1/x}}{x^2}\,dx \qquad \text{Type 1 Improper Integral}$$

$$= \lim_{t\to\infty}\left[e^{-1/x}\right]_1^t \qquad \text{Substitution: } u = -1/x$$

$$= \lim_{t\to\infty} e^{-1/t} - e^{-1} = 1 - \frac{1}{e} \qquad \text{FTC; evaluate limit}$$

Solution


1.11
1.12
2.1
*bc5
RAD
$\int_1^\infty \frac{e^{\frac{-1}{x}}}{x^2}\,dx$
$1-e^{-1}$


2.1
2.2
2.3
*bc5
RAD
y
$f1(x)=\frac{e^{\frac{-1}{x}}}{x^2}$
x
-0.5
1
10

Example 2 Infinite Region to the Left

Determine whether the integral is convergent or divergent.

$$\int_{-\infty}^{0} \frac{1}{5-7x}\,dx$$

Solution

$$\int_{-\infty}^{0} \frac{1}{5-7x}\,dx = \lim_{t\to-\infty}\int_{t}^{0} \frac{1}{5-7x}\,dx$$ Type 1 Improper Integral

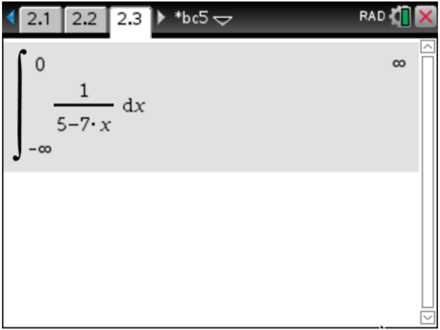
$$= \lim_{t\to-\infty}\left[-\frac{1}{7}\ln|5-7x|\right]_{t}^{0}$$ Substitution: $u = 5-7x$

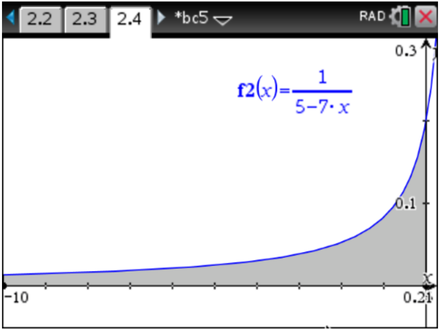
$$= -\frac{1}{7}\lim_{t\to-\infty}\left[\ln|5| - \ln|5-7t|\right]$$ FTC

$$= \infty$$ Evaluate limit

The integral is divergent.

Solution


2.1 2.2 2.3 *bc5
RAD
$\int_{-\infty}^{0} \frac{1}{5-7\cdot x}\,dx$
∞


2.2 2.3 2.4 *bc5
RAD
$f2(x)=\frac{1}{5-7\cdot x}$
0.3
0.1
x
-10
0.21

Example 3 To Infinity and Beyond

Determine whether the integral is convergent or divergent.

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$$

Solution

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

$$\int_{0}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \int_{0}^{t} \frac{1}{1+x^2}\,dx$$ Type 1 Improper Integral

$$= \lim_{t\to\infty} \left[\tan^{-1} x\right]_0^t$$ Antidifferentiation rule

$$= \lim_{t\to\infty} \left(\tan^{-1} t - \tan^{-1} 0\right)$$ FTC

$$= \frac{\pi}{2}$$ Evaluate limit

Solution

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty}\int_{t}^{0} \frac{1}{1+x^2}\,dx \qquad \text{Type 1 Improper Integral}$$

$$= \lim_{t\to-\infty}\left[\tan^{-1} x\right]_{t}^{0} \qquad \text{Antidifferentiation rule}$$

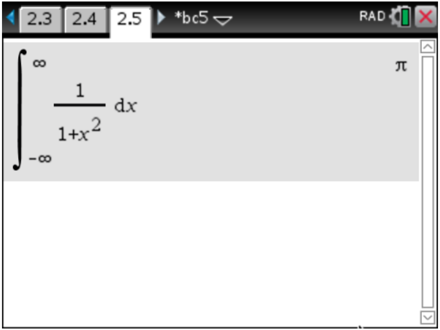
$$= \lim_{t\to-\infty}\left(\tan^{-1} 0 - \tan^{-1} t\right) \qquad \text{FTC}$$

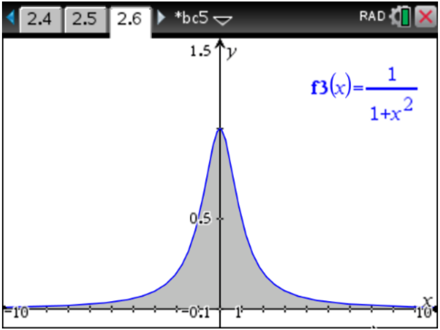
$$= -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2} \qquad \text{Evaluate limit}$$

The (original) integral converges.

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Solution


2.3
2.4
2.5
*bc5
RAD
$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$
π


2.4
2.5
2.6
*bc5
RAD
1.5
y
$f3(x)=\frac{1}{1+x^2}$
0.5
-10
-0.1
1
10
x

Comparison Test

- Sometimes it is impossible to find the exact value of an improper integral but it is still important to know whether it is convergent or divergent.
- It seems reasonable to compare known (old) integrals to a given (new) integral in order to draw a conclusion.

Comparison Theorem

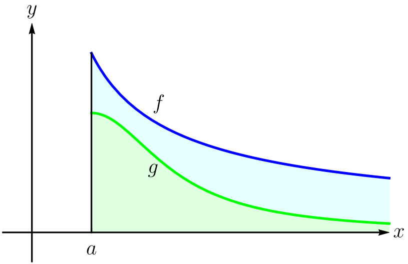
Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.

(b) if $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

A Closer Look

(1) An illustration of the Comparison Theorem.



(2) The reverse of this theorem is not necessarily true:

If $\displaystyle\int_a^\infty g(x)\,dx$ is convergent, then $\displaystyle\int_a^\infty f(x)\,dx$ may or may not be convergent.

And if $\displaystyle\int_a^\infty f(x)\,dx$ is divergent, then $\displaystyle\int_a^\infty g(x)\,dx$ may or may not be divergent.

Example 4 Exercises

(a) Determine whether the integral is convergent or divergent.

(a) $\displaystyle\int_1^\infty \frac{2x}{e^{x^2}}\,dx$

(b) $\displaystyle\int_3^\infty \frac{1}{x(\ln x)^2}\,dx$

(c) $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$

(d) $\displaystyle\int_2^\infty \frac{1}{\sqrt{x}-1}\,dx$

(b) Find the value of C for which the integral

$$\int_0^\infty \left(\frac{1}{\sqrt{x^2+4}} - \frac{C}{x+2}\right)dx$$

converges. Evaluate the integral for this value of C.

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