TI in Focus: AP® Calculus

2017 AP® Calculus Exam: BC-2

Technology Solutions and Problem Extensions

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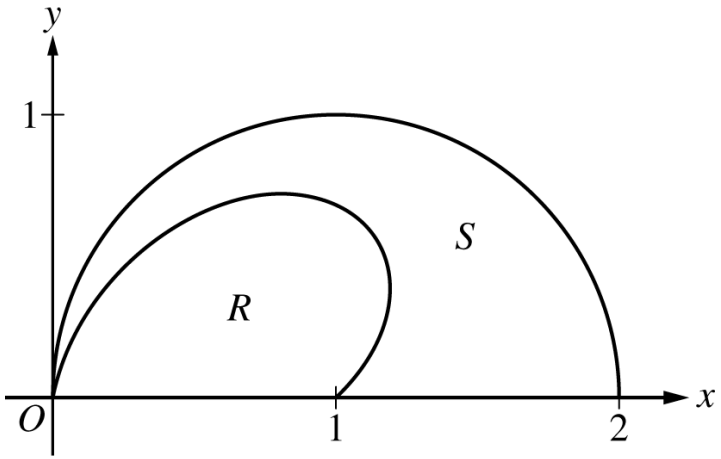
Outline

(1) Free Response Question

(2) Scoring Guidelines

(3) Solutions using technology

(4) Problem extensions



2. The figure above shows the polar curves $r = f(\theta) = 1 + \sin\theta\cos(2\theta)$ and $r = g(\theta) = 2\cos\theta$ for $0 \le \theta \le \frac{\pi}{2}$. Let R be the region in the first quadrant bounded by the curve $r = f(\theta)$ and the x-axis. Let S be the region in the first quadrant bounded by the curve $r = f(\theta)$, the curve $r = g(\theta)$, and the x-axis.

(a) Find the area of R.

(b) The ray $\theta = k$, where $0 < k < \frac{\pi}{2}$, divides S into two regions of equal area. Write, but do not solve, an equation involving one or more integrals whose solution gives the value of k.

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(c) For each θ, $0 \le \theta \le \frac{\pi}{2}$, let $w(\theta)$ be the distance between the points with polar coordinates $(f(\theta), \theta)$ and $(g(\theta), \theta)$. Write an expression for $w(\theta)$. Find w_A, the average value of $w(\theta)$ over the interval $0 \le \theta \le \frac{\pi}{2}$.

(d) Using the information from part (c), find the value of θ for which $w(\theta) = w_A$. Is the function $w(\theta)$ increasing or decreasing at that value of θ ? Give a reason for your answer.

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(a) $\frac{1}{2}\int_0^{\pi/2}(f(\theta))^2\,d\theta = 0.648414$

The area of R is 0.648.

$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$

(b) $\int_0^k\left((g(\theta))^2-(f(\theta))^2\right)d\theta = \frac{1}{2}\int_0^{\pi/2}\left((g(\theta))^2-(f(\theta))^2\right)d\theta$

— OR —

$\int_0^k\left((g(\theta))^2-(f(\theta))^2\right)d\theta = \int_k^{\pi/2}\left((g(\theta))^2-(f(\theta))^2\right)d\theta$

$2: \begin{cases} 1: \text{integral expression} \\ \quad\ \text{for one region} \\ 1: \text{equation} \end{cases}$

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(c) $w(\theta) = g(\theta) - f(\theta)$

$$w_A = \frac{\int_0^{\pi/2} w(\theta)\, d\theta}{\frac{\pi}{2} - 0} = 0.485446$$

The average value of $w(\theta)$ on the interval $\left[0, \frac{\pi}{2}\right]$ is 0.485.

$$3 : \begin{cases} 1 : w(\theta) \\ 1 : \text{integral} \\ 1 : \text{average value} \end{cases}$$

(d) $w(\theta) = w_A$ for $0 \le \theta \le \frac{\pi}{2} \Rightarrow \theta = 0.517688$

$w(\theta) = w_A$ at $\theta = 0.518$ (or 0.517).

$w'(0.518) < 0 \Rightarrow w(\theta)$ is decreasing at $\theta = 0.518$.

$$2 : \begin{cases} 1 : \text{solves } w(\theta) = w_A \\ 1 : \text{answer with reason} \end{cases}$$

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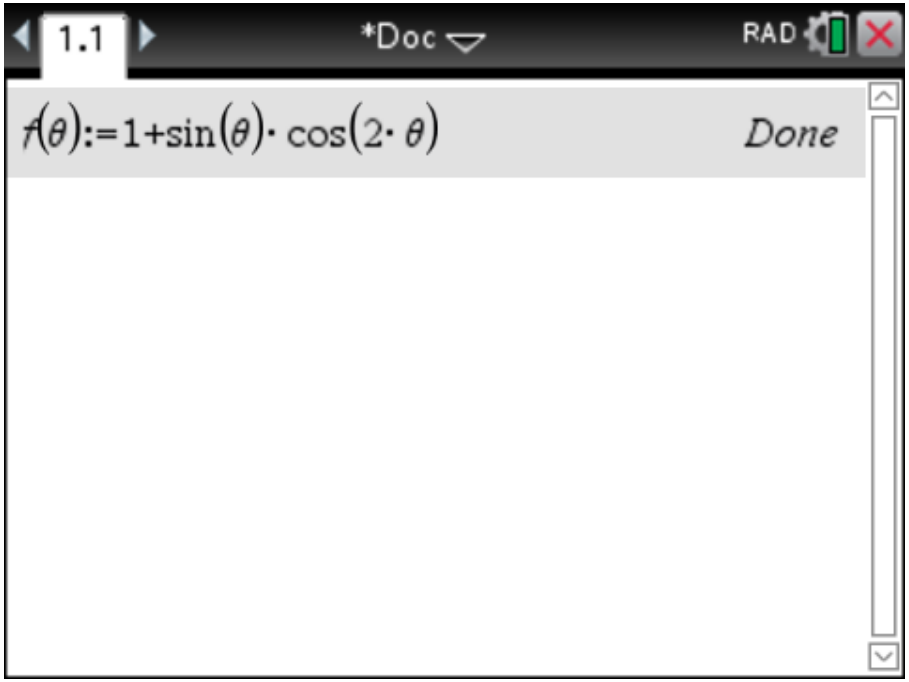
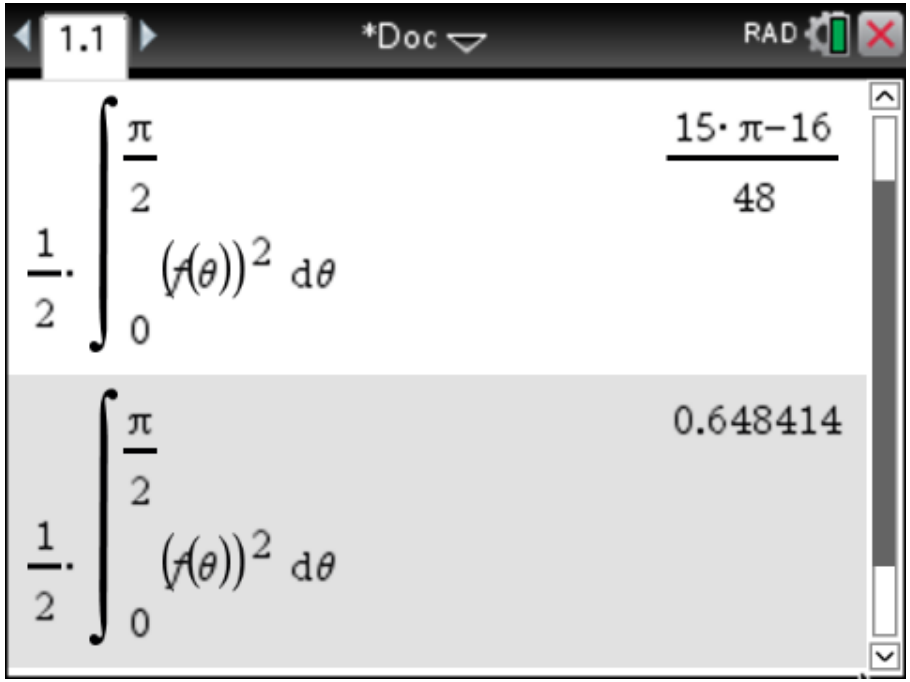
Part (a)

$$\text{Area} = \frac{1}{2}\int_0^{\pi/2} [f(\theta)]^2\, d\theta = 0.648$$





Example 1 More Areas

(a) Find the area of the region in the first quadrant bounded by the curve $r = g(\theta)$ and the x-axis.

(b) Find the area of the region in the first quadrant inside the graph of $r = f(\theta)$ and outside the circle $r = 1$.

Solution

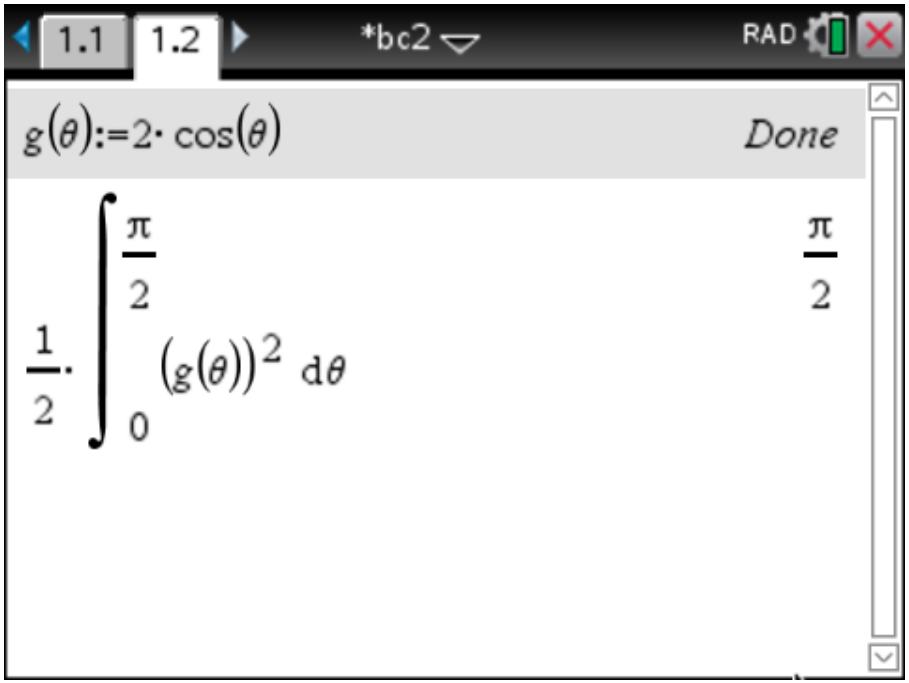
(a) Area $= \dfrac{1}{2}\displaystyle\int_0^{\pi/2} [g(\theta)]^2\, d\theta = \dfrac{1}{2}\int_0^{\pi/2} 4\cos^2\theta\, d\theta$

$$= 2\int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta = \theta + \frac{\sin 2\theta}{2}\Big]_0^{\pi/2}$$

$$= \frac{\pi}{2} + \frac{\sin\pi}{2} = \frac{\pi}{2}$$

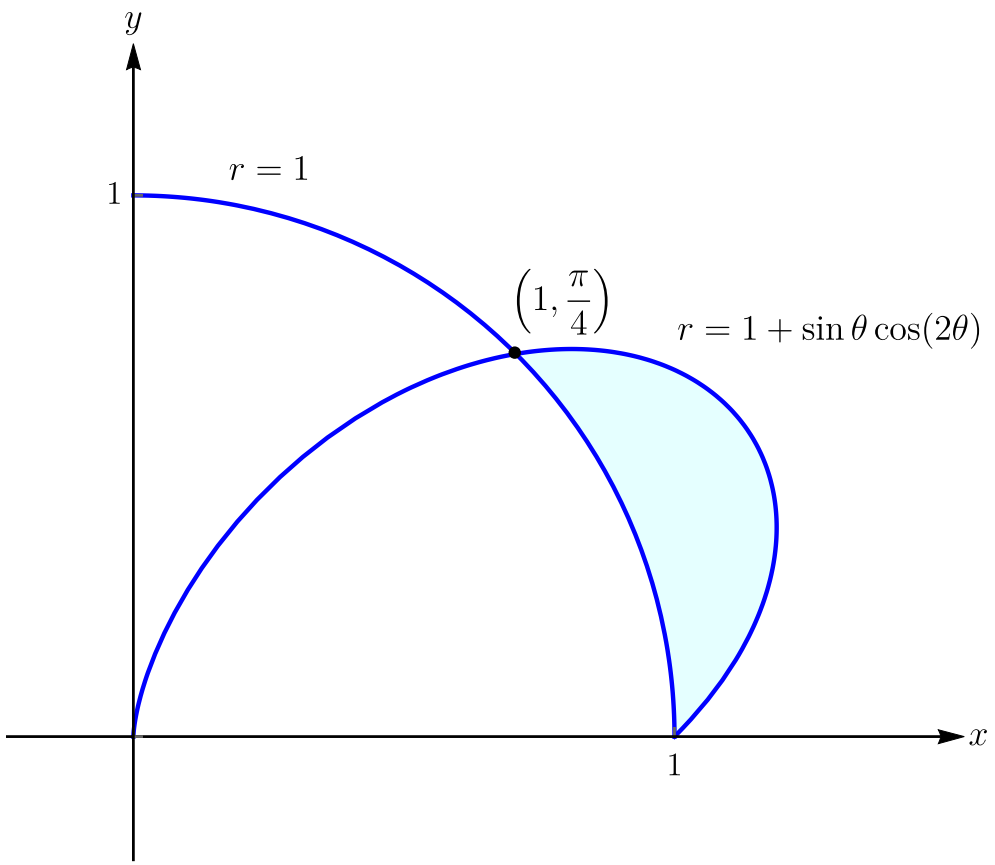
Solution

(a)


1.1
1.2
*bc2
RAD
$g(\theta):=2\cdot\cos(\theta)$
Done
$\frac{1}{2}\cdot\int_0^{\frac{\pi}{2}}(g(\theta))^2\,d\theta$
$\frac{\pi}{2}$

Solution

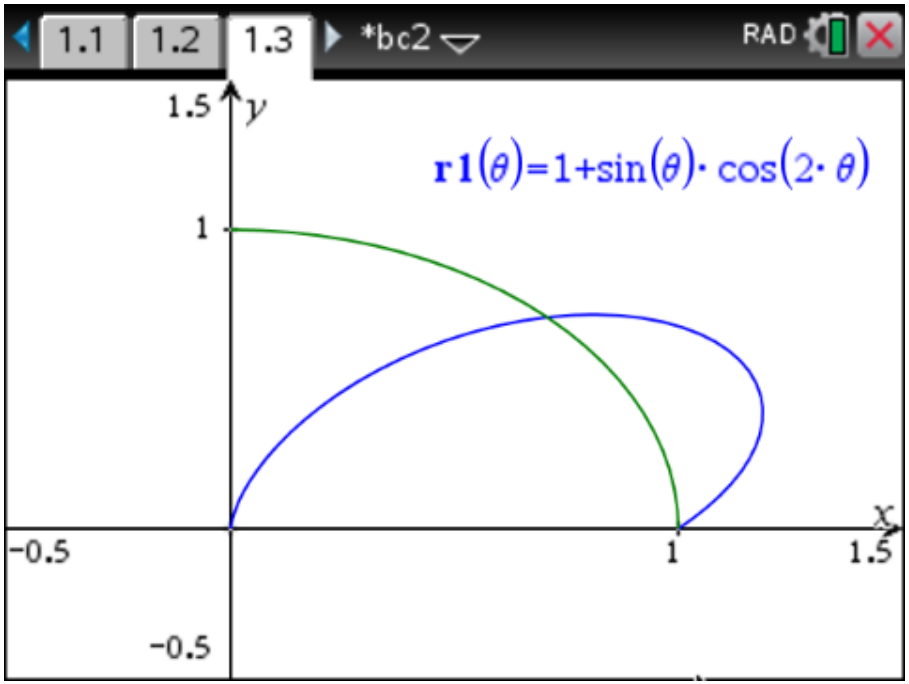
(b)

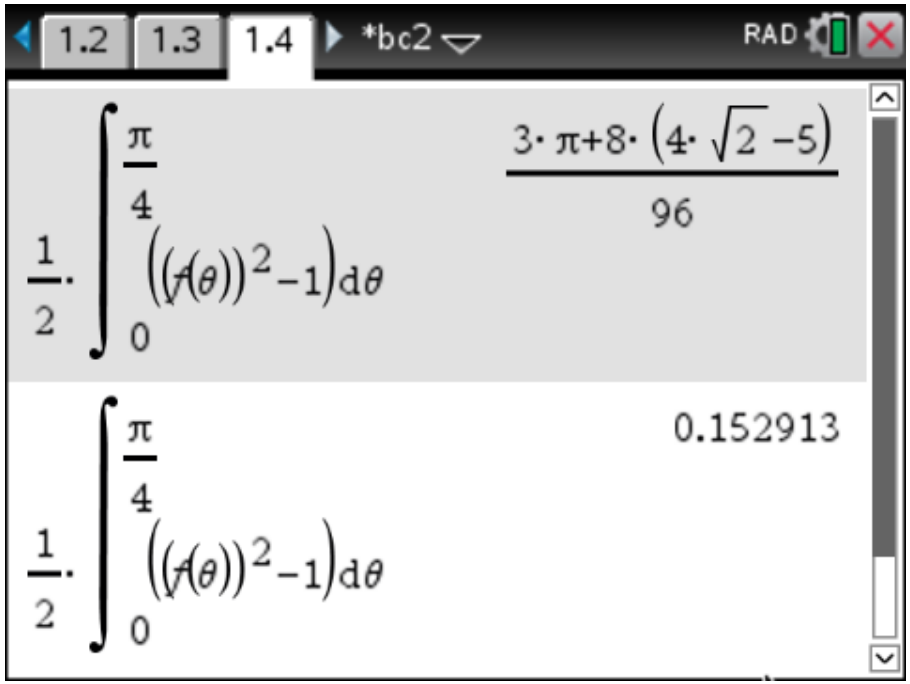


$$1 + \sin\theta\cos(2\theta) = 1 \implies \sin\theta\cos(2\theta) = 0 \implies \theta = \frac{\pi}{4}$$

Solution

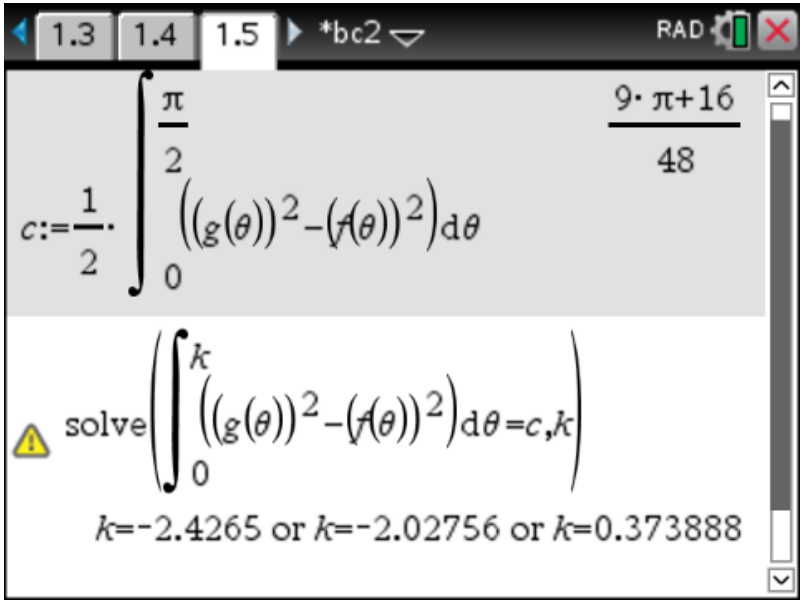
(b) Area $= \dfrac{1}{2}\displaystyle\int_0^{\pi/4} [f(\theta)]^2\, d\theta - \dfrac{1}{2}\int_0^{\pi/4} 1^2\, d\theta = 0.153$

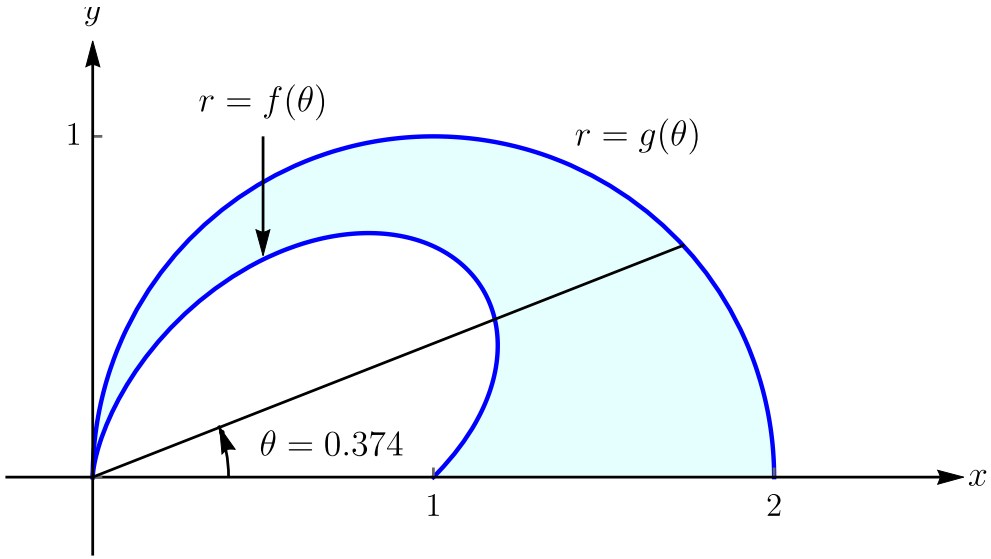




Part (b)

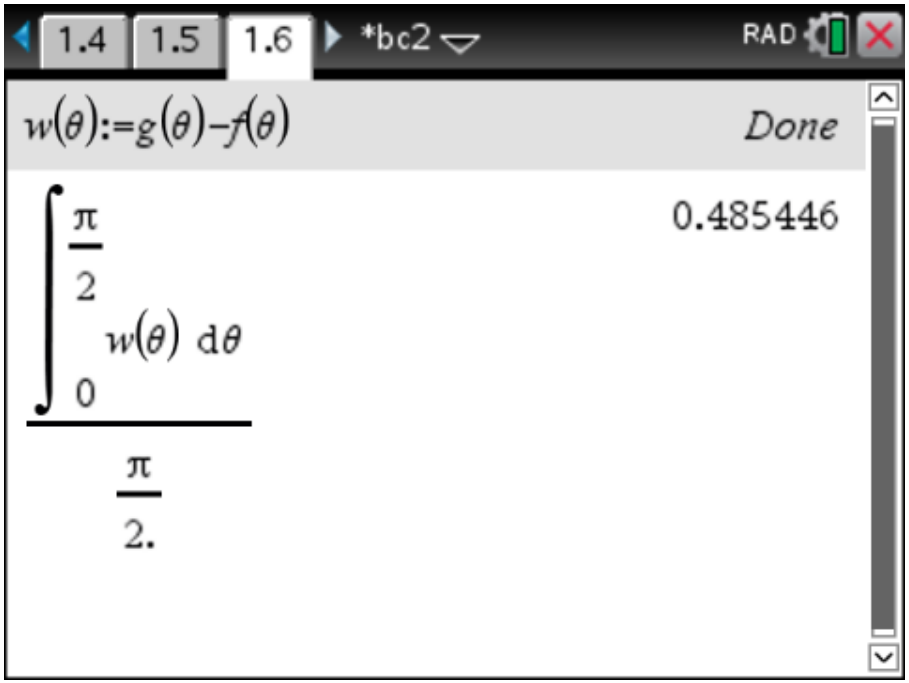
$$\int_0^k [(g(\theta))^2 - (f(\theta))^2]\, d\theta = \frac{1}{2}\int_0^{\pi/2} [(g(\theta))^2 - (f(\theta))^2]\, d\theta$$

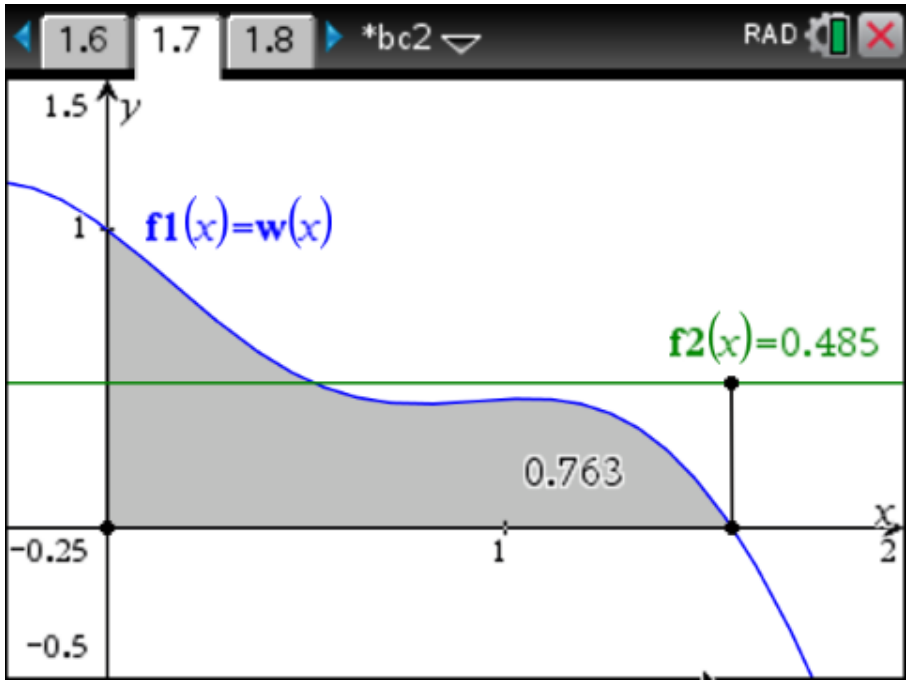




Part (c)

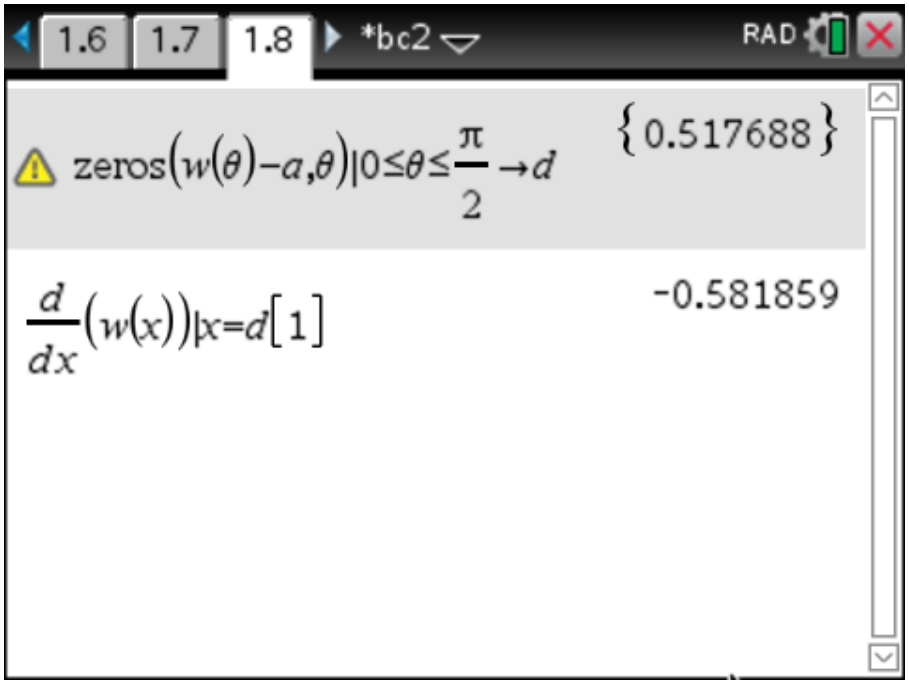
$$w_A = \frac{1}{\frac{\pi}{2} - 0} \int_0^{\pi/2} w(\theta)\, d\theta = 0.485$$





Part (d)

- Solve the equation $w(\theta) = w_A$ for $0 \le \theta \le \dfrac{\pi}{2} \implies \theta = 0.517688$
- $w'(0.518) < 0 \implies w(\theta)$ is decreasing at $\theta = 0.518$.



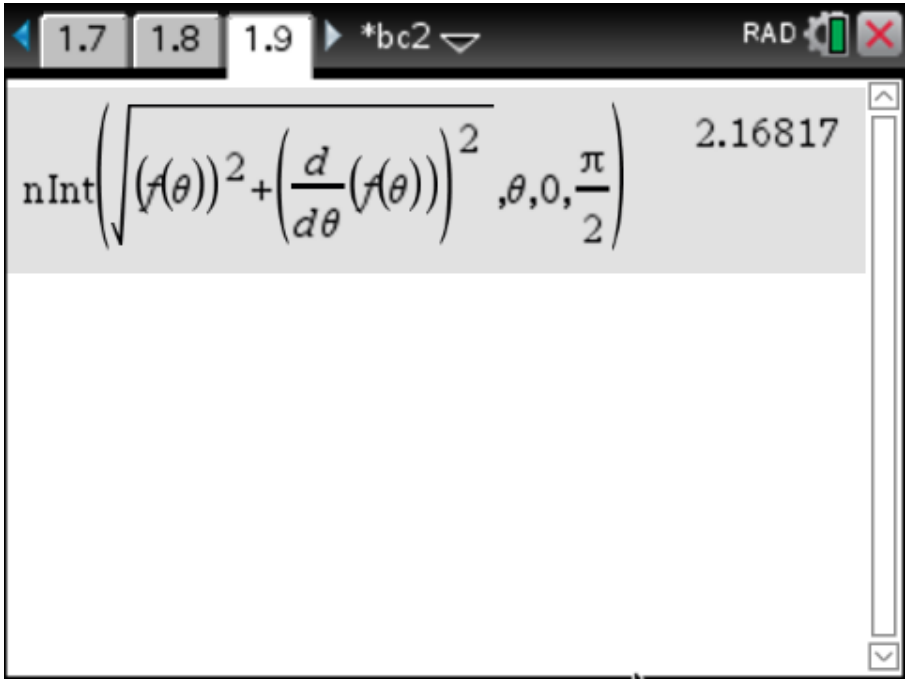
Example 2 Arc Length

Find the length of each polar curve.

Solution

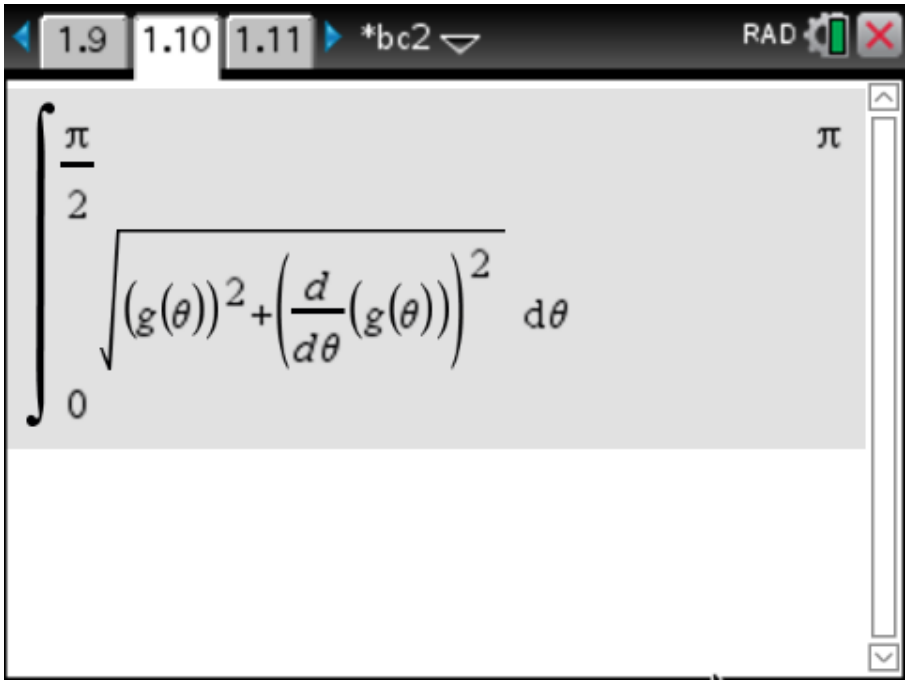
Recall: $L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

$r = 1 + \sin\theta\cos(2\theta); \qquad \frac{dr}{d\theta} = \cos\theta\cos(2\theta) - 2\sin\theta\sin(2\theta)$



Solution

$$r = 2\cos\theta \implies \frac{dr}{d\theta} = -2\sin\theta$$



Example 3 All My Life's a Circle

Consider the polar curve described by $r = g(\theta) = 2\cos\theta$.

(a) Show that the Cartesian equation for the polar curve described by $r = g(\theta)$ is a circle centered at $(1, 0)$ with radius 1.

(b) Find (analytically) the length of the polar curve $r = g(\theta)$ for $0 \le \theta \le \dfrac{\pi}{2}$.

(c) Find (analytically) the area of the region in the first quadrant bounded by the curve $r = g(\theta)$.

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