TI in Focus: AP® Calculus

2017 AP® Calculus Exam: AB-4/BC-4
Technology Solutions and Problem Extensions

Stephen Kokoska
Professor, Bloomsburg University
Former AP® Calculus Chief Reader

*AP is a registered trademark of the College Board, which was not involved in the production of, and does not endorse, this product.

Texas Instruments

Outline

(1) Free Response Question

(2) Scoring Guidelines

(3) Solutions using technology

(4) Problem extensions

4. At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91 degrees Celsius (°C) at time $t = 0$, and the internal temperature of the potato is greater than 27°C for all times $t > 0$. The internal temperature of the potato at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.

(a) Write an equation for the line tangent to the graph of H at $t = 0$. Use this equation to approximate the internal temperature of the potato at time $t = 3$.

(b) Use $\frac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the internal temperature of the potato at time $t = 3$.

(c) For $t < 10$, an alternate model for the internal temperature of the potato at time t minutes is the function G that satisfies the differential equation $\frac{dG}{dt} = -(G - 27)^{2/3}$, where $G(t)$ is measured in degrees Celsius and $G(0) = 91$. Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$?

Source: 2017 AP Calculus AB and BC Free-Response Questions
Copyright © 2017 The College Board.
Reproduced with permission. http://apcentral.collegeboard.com.

(a) $H'(0) = -\frac{1}{4}(91 - 27) = -16$

$H(0) = 91$

An equation for the tangent line is $y = 91 - 16t$.

The internal temperature of the potato at time $t = 3$ minutes is approximately $91 - 16 \cdot 3 = 43$ degrees Celsius.

$3 : \begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line} \\ 1 : \text{approximation} \end{cases}$

(b) $\frac{d^2H}{dt^2} = -\frac{1}{4}\frac{dH}{dt} = \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)(H - 27) = \frac{1}{16}(H - 27)$

$H > 27$ for $t > 0 \Rightarrow \frac{d^2H}{dt^2} = \frac{1}{16}(H - 27) > 0$ for $t > 0$

Therefore, the graph of H is concave up for $t > 0$. Thus, the answer in part (a) is an underestimate.

1 : underestimate with reason

Source: 2017 AP Calculus AB and BC Free-Response Questions
Copyright © 2017 The College Board.
Reproduced with permission. http://apcentral.collegeboard.com.

(c) $\dfrac{dG}{(G-27)^{2/3}} = -dt$

$$\int \frac{dG}{(G-27)^{2/3}} = \int (-1)\, dt$$

$$3(G-27)^{1/3} = -t + C$$

$$3(91-27)^{1/3} = 0 + C \Rightarrow C = 12$$

$$3(G-27)^{1/3} = 12 - t$$

$$G(t) = 27 + \left(\frac{12-t}{3}\right)^3 \text{ for } 0 \le t < 10$$

The internal temperature of the potato at time $t = 3$ minutes is $27 + \left(\dfrac{12-3}{3}\right)^3 = 54$ degrees Celsius.

$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration and uses initial condition} \\ 1 : \text{equation involving } G \text{ and } t \\ 1 : G(t) \text{ and } G(3) \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

Source: 2017 AP Calculus AB and BC Free-Response Questions
Copyright © 2017 The College Board.
Reproduced with permission. http://apcentral.collegeboard.com.

Part (a) and Part (b)

$$H'(0) = -\frac{1}{4}(91 - 27) = -16 \qquad H(0) = 91$$

An equation for the tangent line:

$$y - 91 = -16(t - 0) \implies y = -16t + 91 \implies y(0) = 43$$

Find and use the second derivative of H:

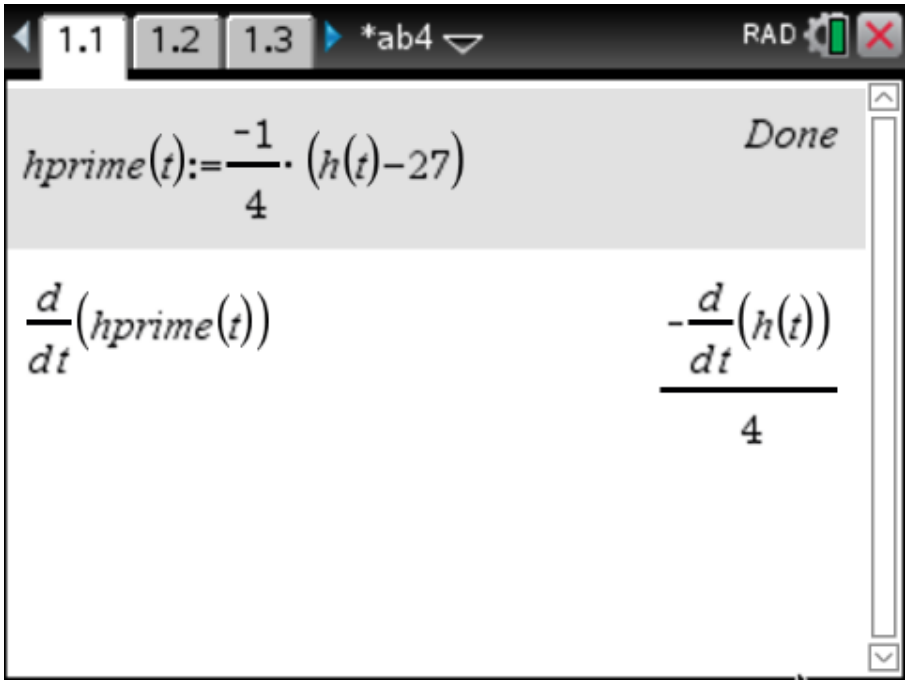
$$\frac{d^2H}{dt^2} = -\frac{1}{4}\,\frac{dH}{dt} = \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)(H - 27) = \frac{1}{16}(H - 27)$$

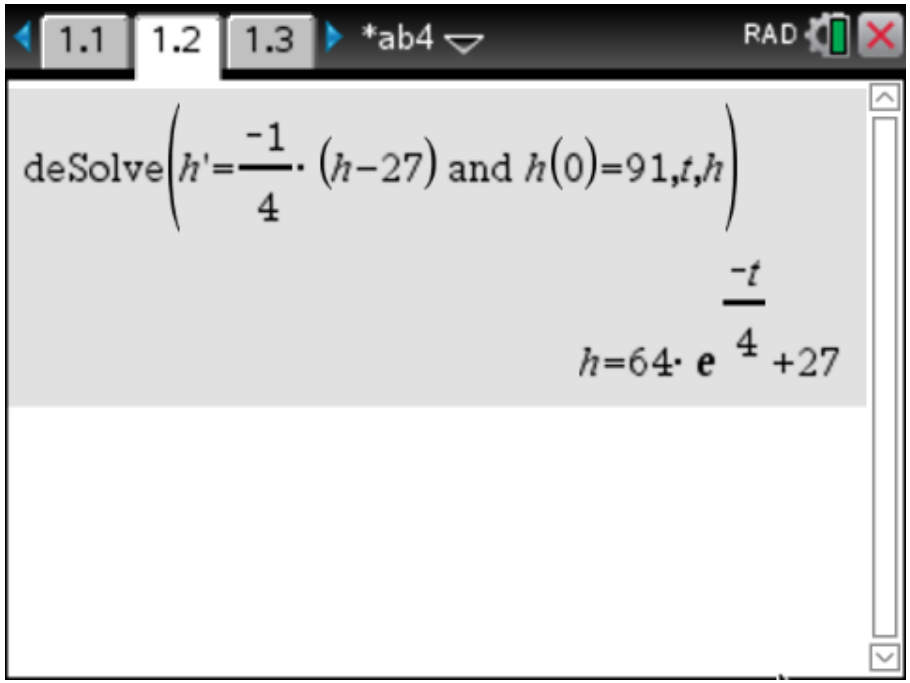
$$H > 27 \text{ for } t > 0 \implies \frac{d^2H}{dt^2} = \frac{1}{16}(H - 27) > 0 \text{ for } t > 0$$

The graph of H is concave up for $t > 0$.
$y(3) = 43$ is an underestimate.

Technology Enhanced Solutions

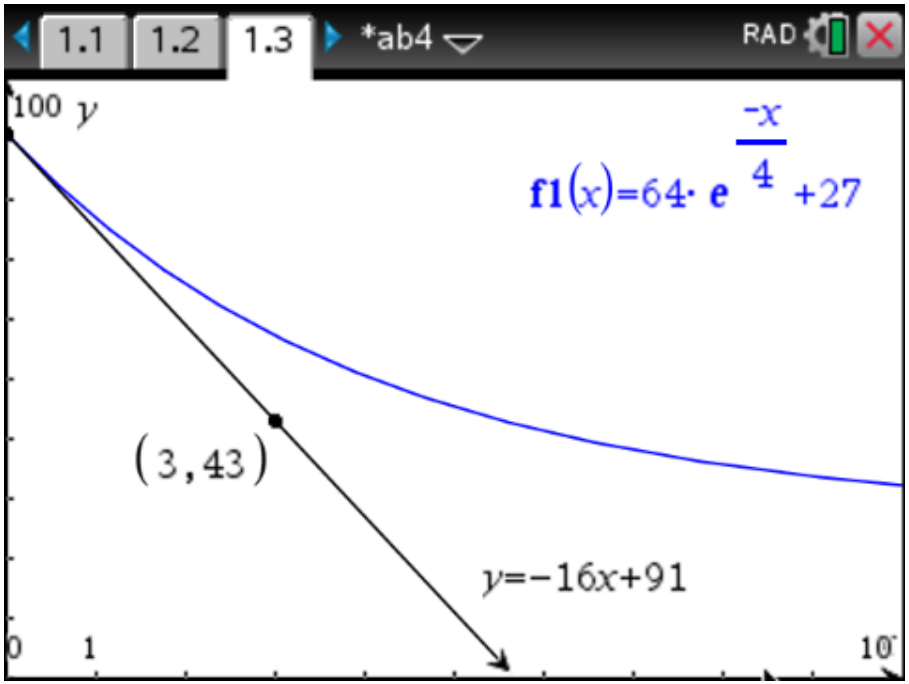
Find $\dfrac{d^2H}{dt^2}$ and solve the differential equation.

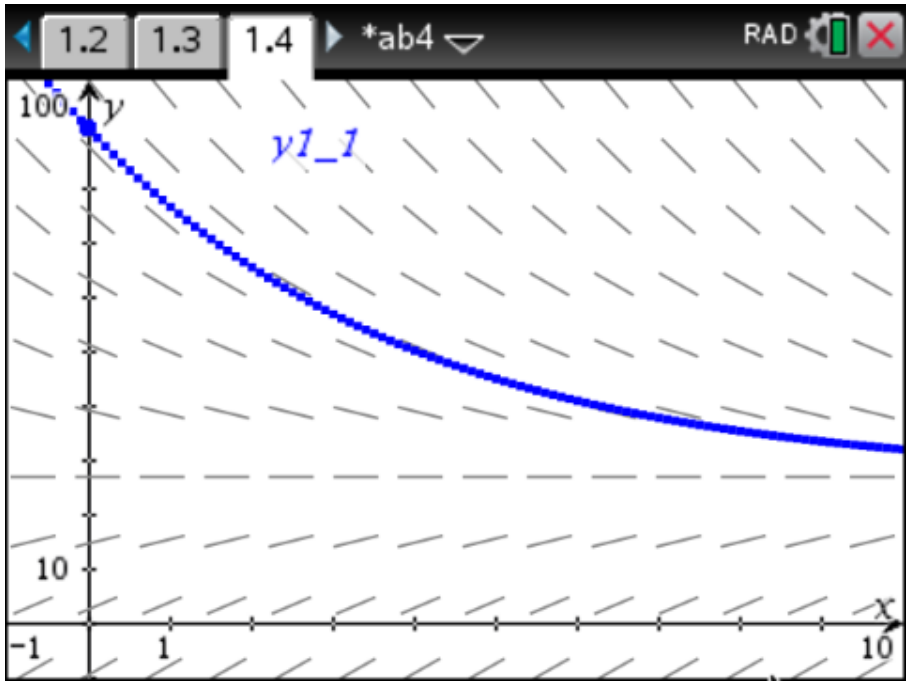




Technology Enhanced Solutions

Find an equation of the tangent line, and consider the slope field.





Example 1 Cooling Rate

Is the potato cooling faster when its temperature is $40°$C or when it is $70°$C? Explain your reasoning.

Solution

$$\left.\frac{dH}{dt}\right|_{H=40} = -\frac{1}{4}(40-27) = -\frac{13}{4} = -3.25$$

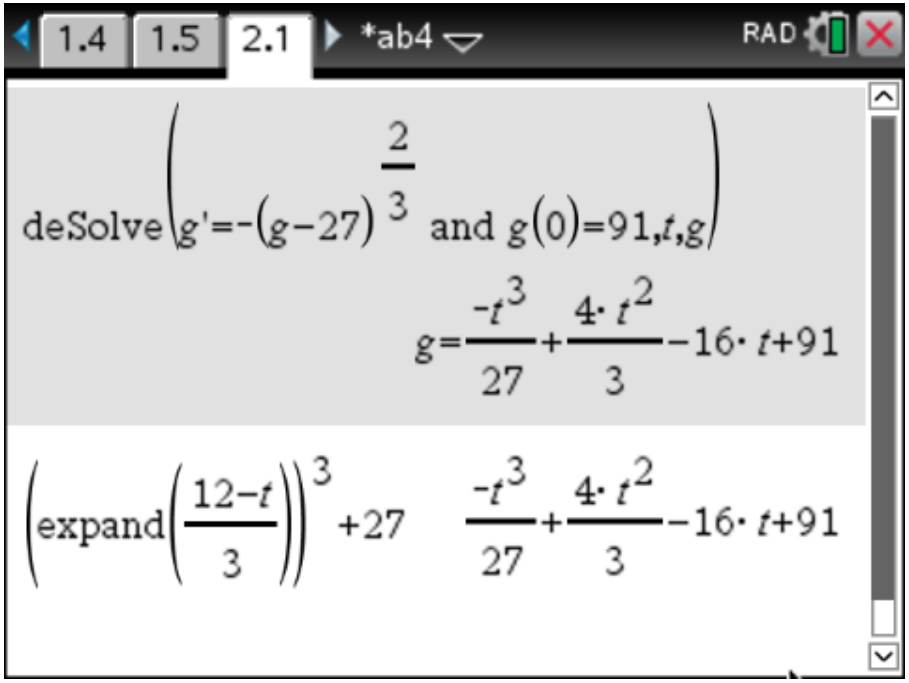
$$\left.\frac{dH}{dt}\right|_{H=70} = -\frac{1}{4}(70-27) = -\frac{43}{4} = -10.75$$

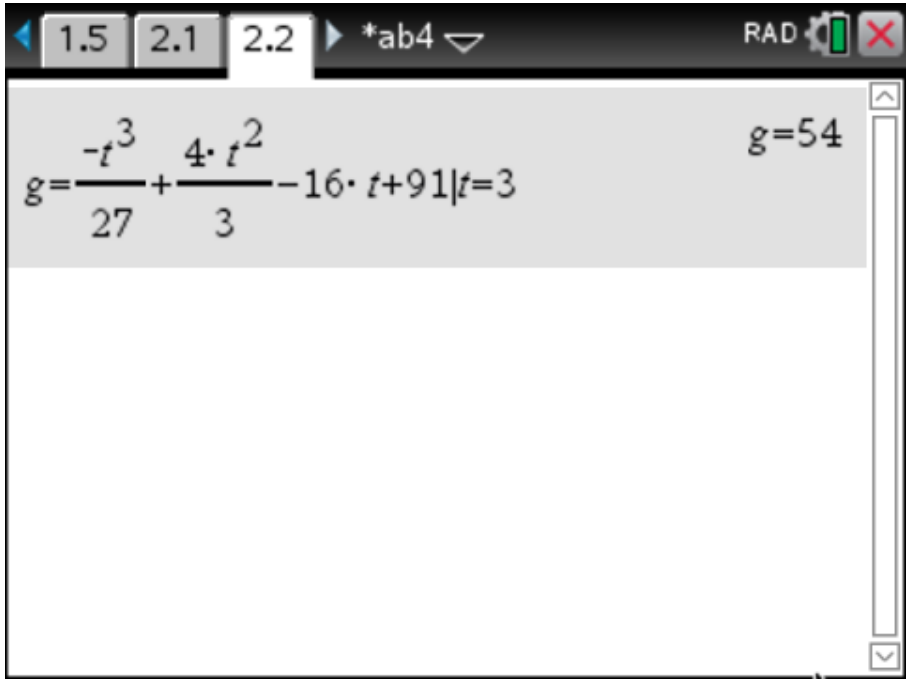
$$\left.\frac{dH}{dt}\right|_{H=70} < \left.\frac{dH}{dt}\right|_{H=40} < 0$$

Therefore, the potato is cooling faster when it is $70°$C.

Part (c)

Solve the differential equation, and find the temperature of the potato at time $t = 3$.





Example 2 Almost Original Boiled Potato

For $t > 0$, an alternate model for the internal temperature of the potato at time t minutes is the function T that satisfies the differential equation

$$\frac{dT}{dt} = k(T - 20)$$

where $T(t)$ is measured in degrees Celsius, the internal temperature of the potato is $91°$C at time $t = 0$, $T(5) = 60$, and k is a constant.

(a) Find an expression for $T(t)$.

(b) When will the internal temperature of the potato be $40°$C?

(c) Find $\lim_{t \to \infty} T(t)$ and explain the meaning of this answer in the context of the problem.

(d) Sketch the graph of T and the slope field on the same coordinate axes.

Solution

(a) Note

- Newton's Law of Cooling
- Separation of variables

Change of variables:

Let $y = T - 20 \implies \dfrac{dy}{dt} = \dfrac{dT}{dt}; \quad y(0) = T(0) - 20 = 91 - 20 = 71$

$$\frac{dy}{dt} = ky; \qquad y(0) = 71$$

Therefore: $y(t) = y(0)e^{kt} = 71e^{kt}$

Find k.

Solution

(a) $y(5) = T(5) - 20 = 60 - 20 = 40 = 71e^{kt}$

$$k = \frac{\ln\left(\frac{40}{71}\right)}{5} \approx -0.11476$$

Therefore:

$$y(t) = 71e^{-0.11476t}$$

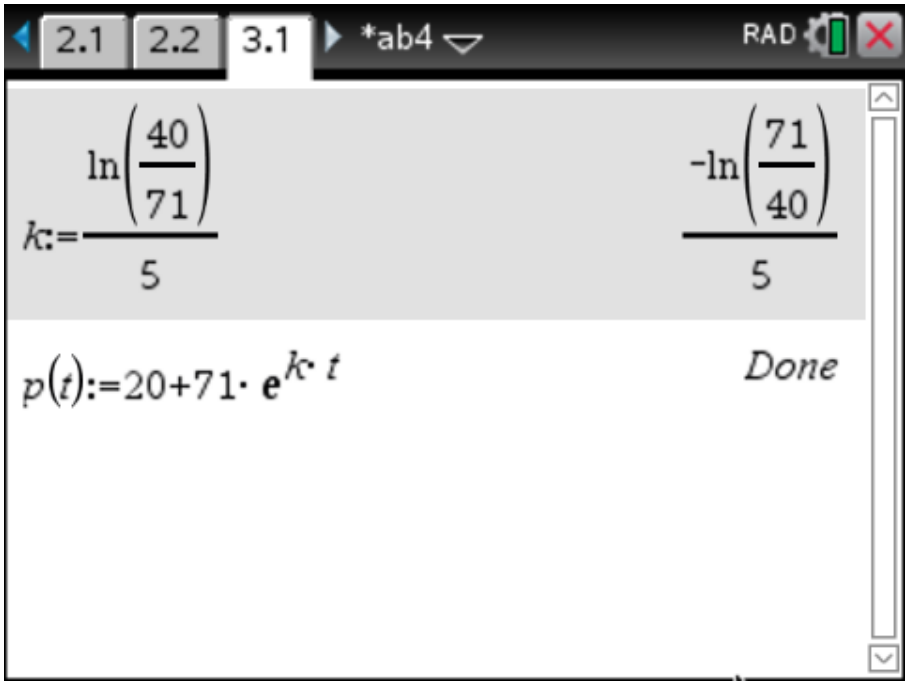
$$T(t) = 20 + 71e^{-0.11476t}$$

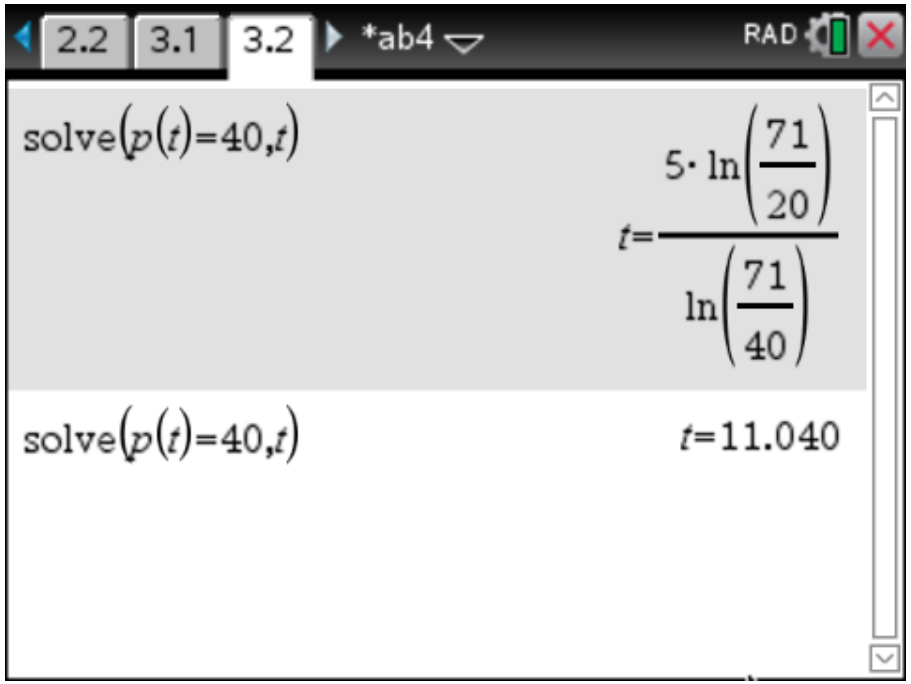
(b) $T(t) = 20 + 71e^{-0.11476t} = 40$

$$t = 11.04$$

Solution

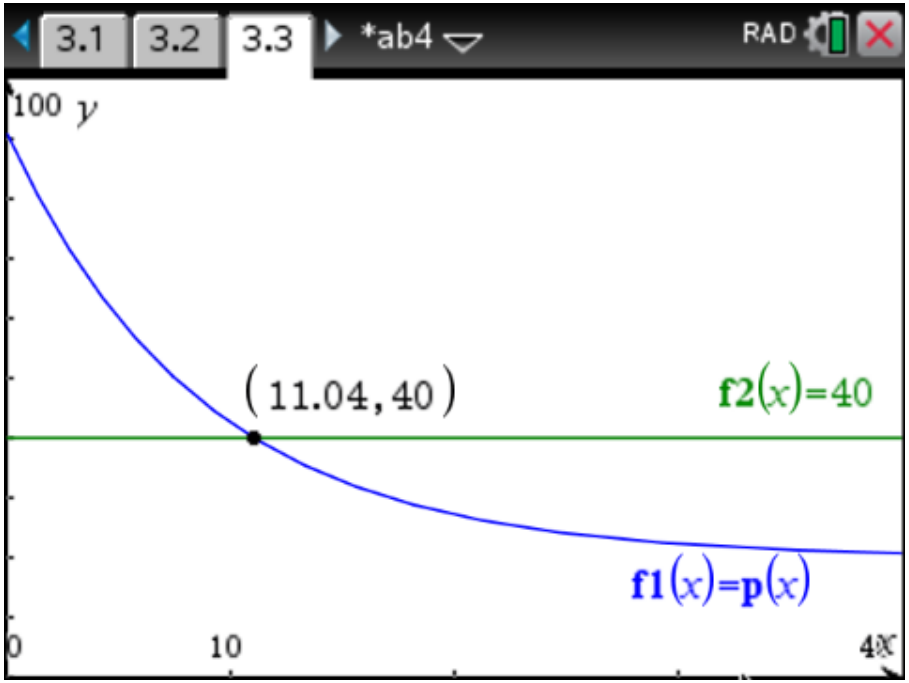
Exact symbolic answer and an approximation.





Solution

A solution on the graph screen.

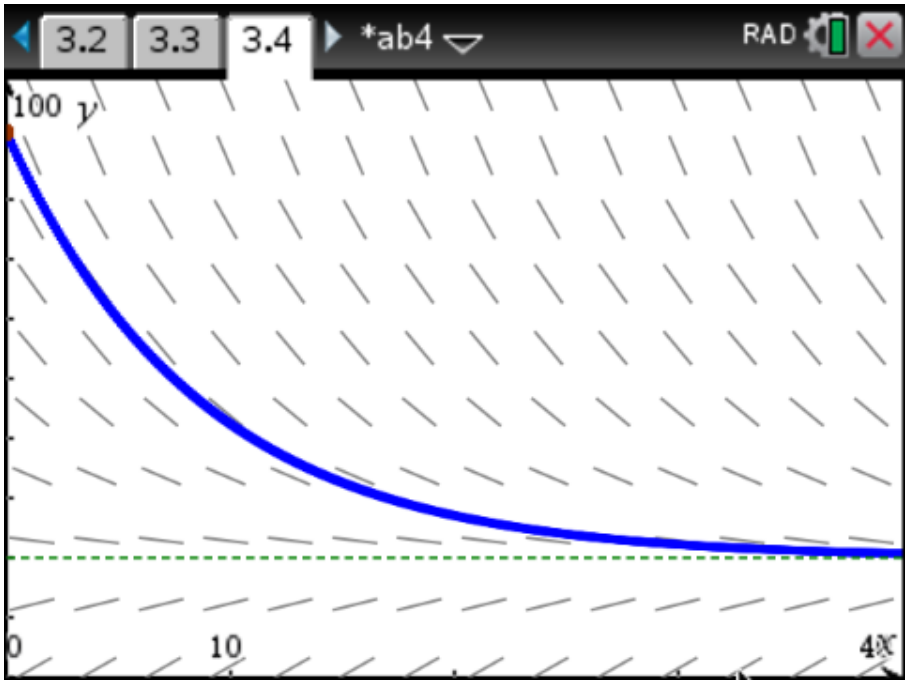


Solution

(c) $\lim_{t\to\infty} T(t) = \lim_{t\to\infty} (20 + 71e^{-0.11476t}) = 20$

The temperature of the potato approaches the temperature in the room, the temperature of the surroundings.

(d) Graph of the solution curve and the slope field.



education.ti.com