TI in Focus: AP® Calculus

2017 AP® Calculus Exam: AB-1/BC-1
Riemann Sums

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Outline

(1) Background and assumptions

(2) Summation notation, common formulas

(3) Estimating area; graphically, technology

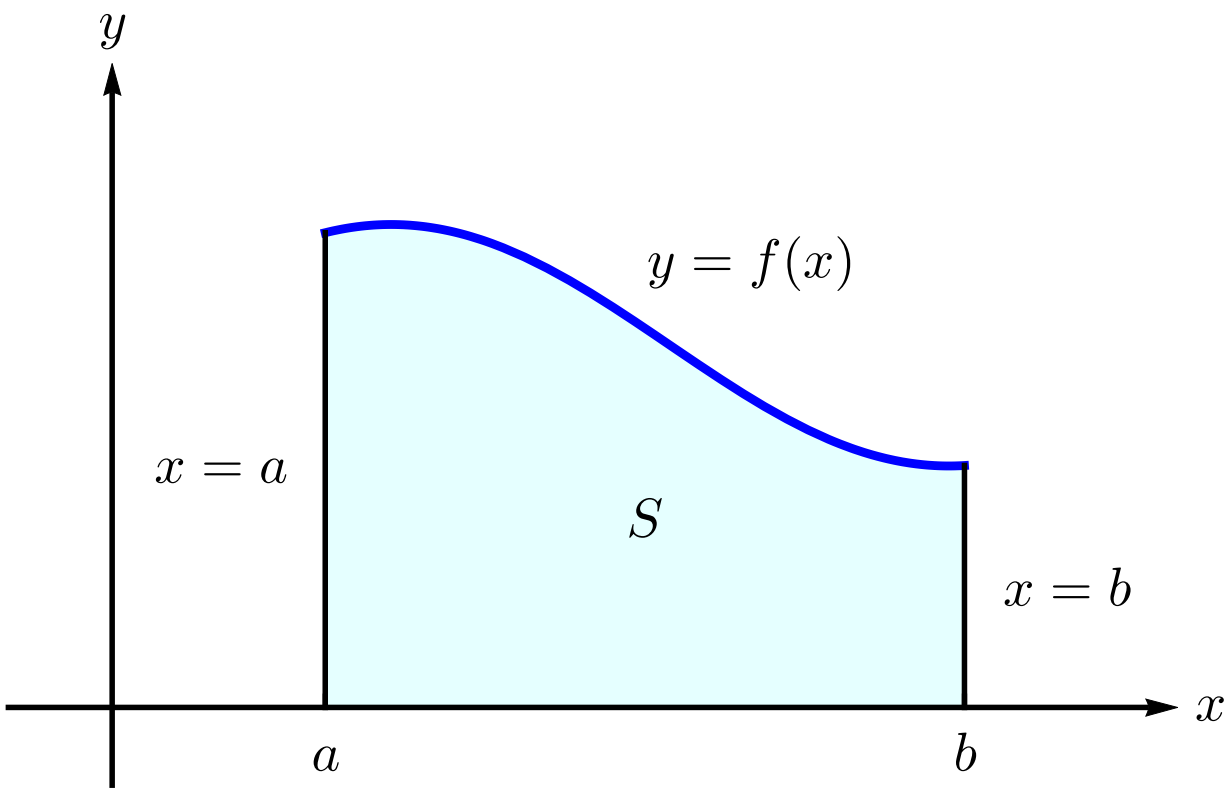
(4) Formal definitions

(5) Examples

Background

- One of the most important, unifying problems in calculus: the area problem.
- Suppose f is a continuous function and $f(x) \geq 0$.

  Find the area of the region S bounded by the graph of $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$.



Area Assumptions

(1) Area is positive; the area of a plane region is a positive number.

(2) Area is additive.

(3) If region 1 is contained in region 2, then the area of region 1 is less than or equal to the area of region 2.

(4) We know the area of a rectangle: area $=$ length $\times$ width

These assumptions suggest we will need to add the areas of a lot of rectangles.

Therefore, we need a compact way to write long sums.

Sigma Notation

The sum of n terms $a_1, a_2, a_3, \ldots, a_n$ is written as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

Properties of Summations

(1) $\displaystyle\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$

Constants pass freely thorough summation symbols.

(2) $\displaystyle\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

Split a sum or difference into two separate summations.

Summation Formulas

Let c be a constant and n a positive integer.

(a) $\displaystyle\sum_{i=1}^{n} 1 = n$

(b) $\displaystyle\sum_{i=1}^{n} c = cn$

(c) $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

(d) $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

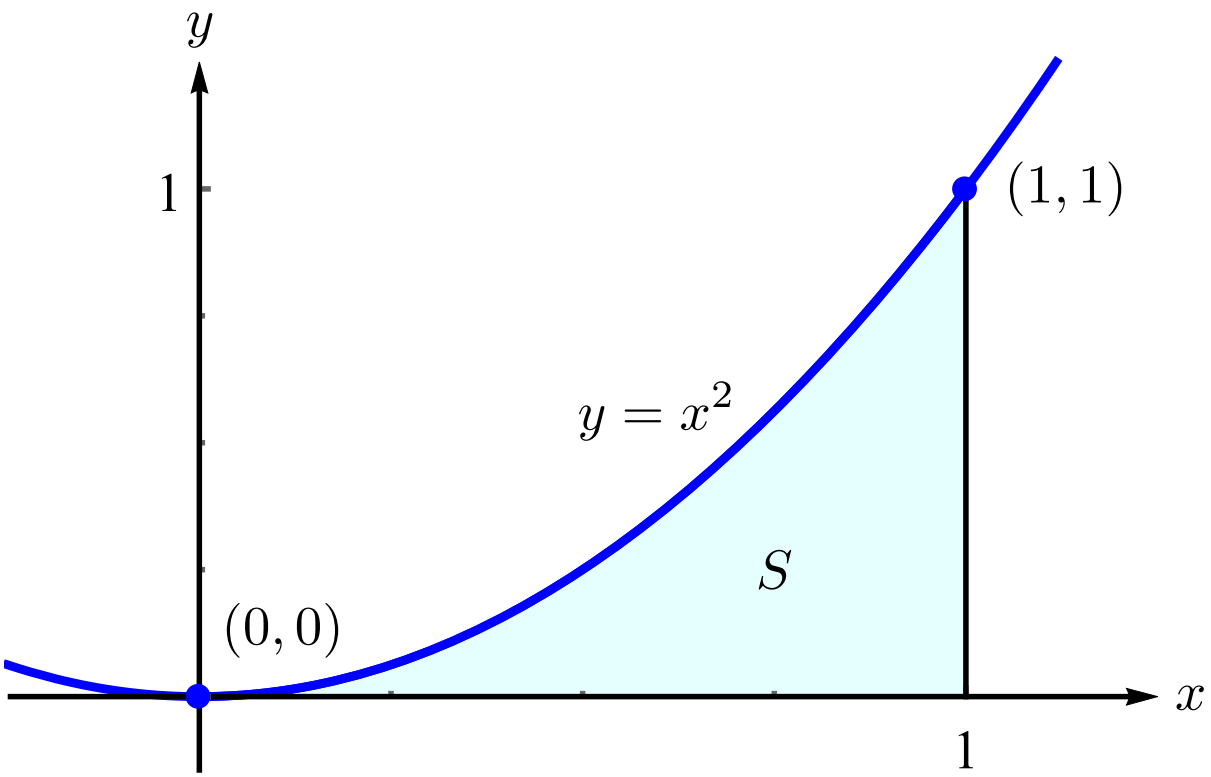
(e) $\displaystyle\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

(f) $\displaystyle\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

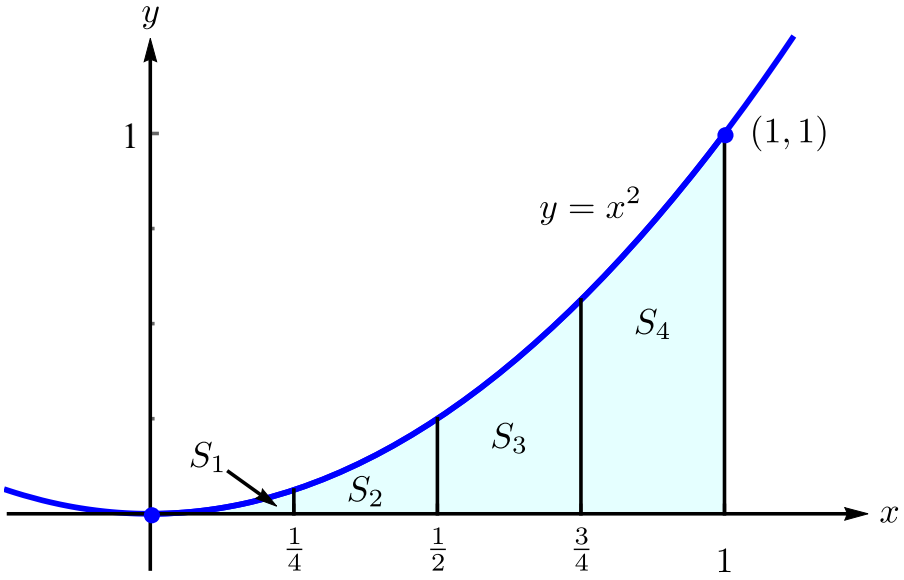
These formulas allow us to simplify lengthy summations.

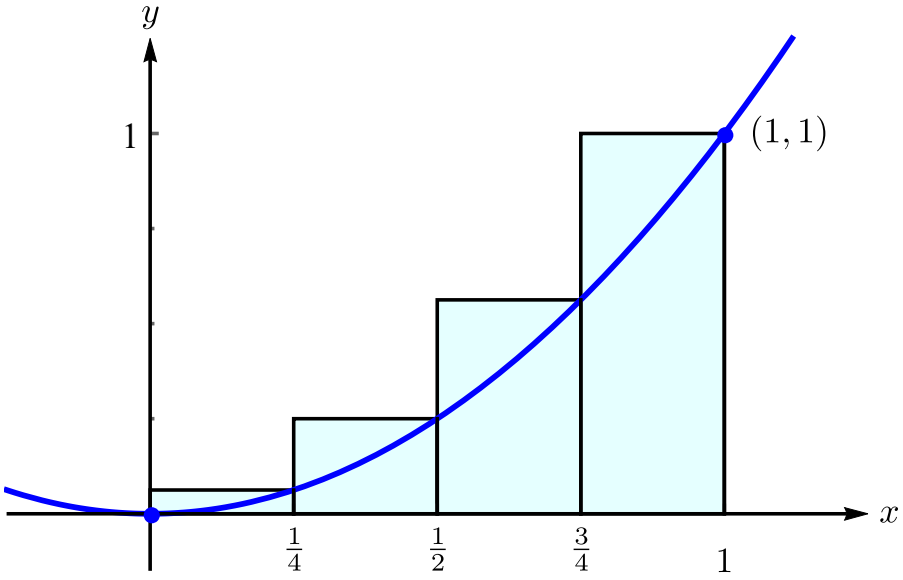
Example 1 Estimate the Area

Estimate the area of the region S bounded by the graph of $y = x^2$, the x-axis, and the lines $x = 0$ and $x = 1$.



- Divide the region S into four pieces.
- Approximate the area of each piece by using a rectangle.



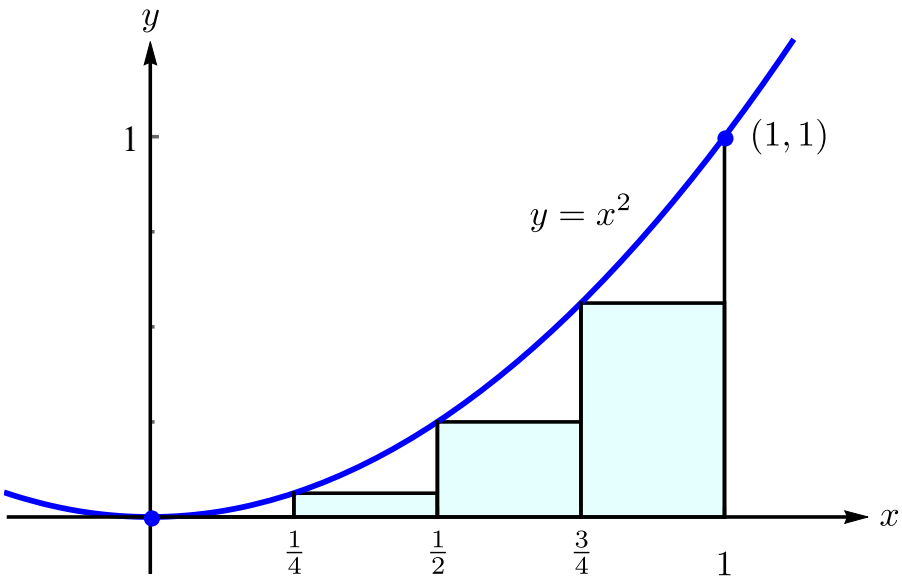


$R_4 =$ sum of the areas of these approximating rectangles.

$$R_4 = \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 + \frac{1}{4} \cdot 1^2 = 0.46875$$

$A < 0.46875$

- We could construct rectangles in another way to obtain another approximation of the area.
- Choose rectangles with heights equal to the values of f at the left endpoint of each subinterval.

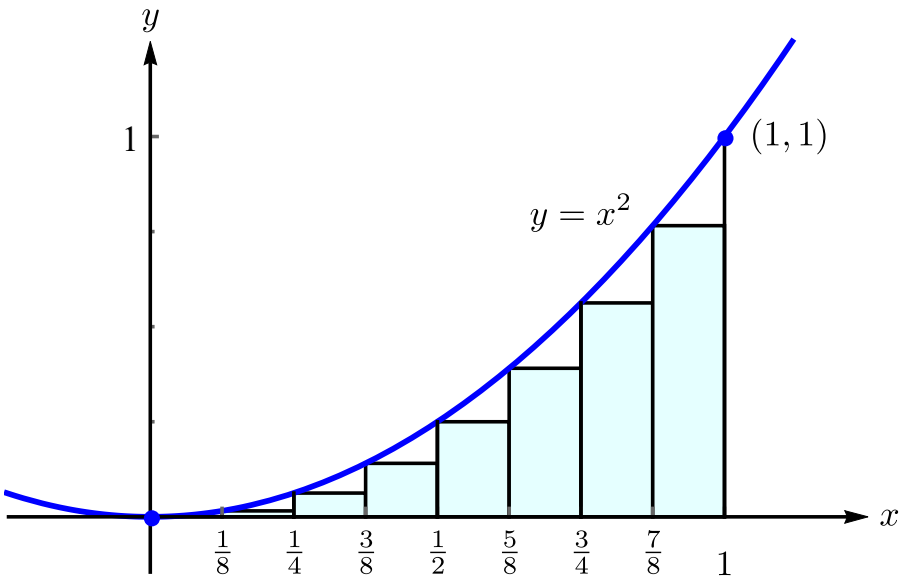


$$L_4 = \frac{1}{4} \cdot 0^2 + \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 = 0.21875$$

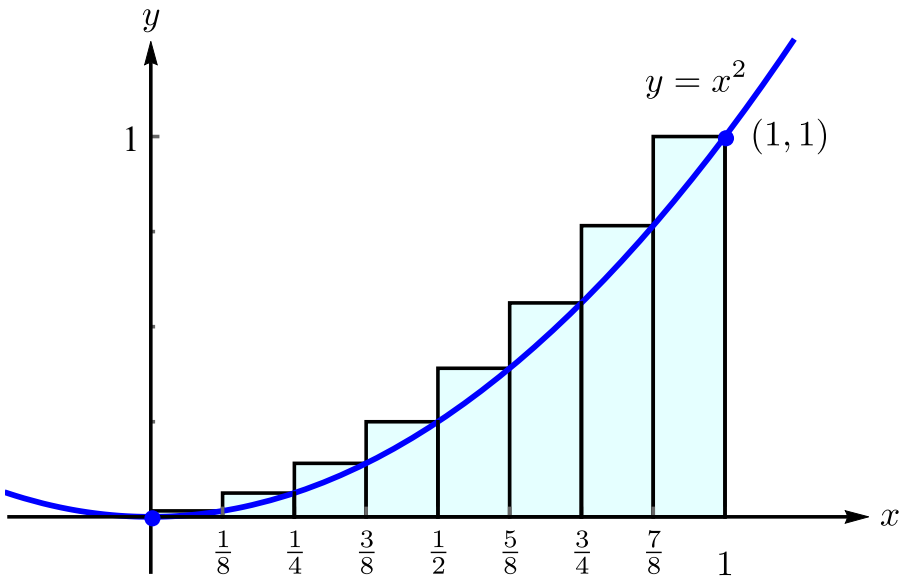
Note: $0.21875 < A < 0.46875$

- To obtain a better estimate of the area of S:

  Divide the region into smaller pieces; use smaller (thinner) rectangles.



Approximate the area of S with 8 rectangles using left endpoints.



Approximate the area of S with 8 rectangles using right endpoints.

$$L_8 = 0.2734375 < A < 0.3984375 = R_8$$

- Better estimates: increase the number of rectangles.
- Here is a table of estimates.

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	A n	B l_n	C r_n	D
=				
1	20	0.30875	0.35875	
2	30	0.316852	0.350185	
3	50	0.3234	0.3434	
4	100	0.32835	0.33835	
5	1000	0.332834	0.333834	

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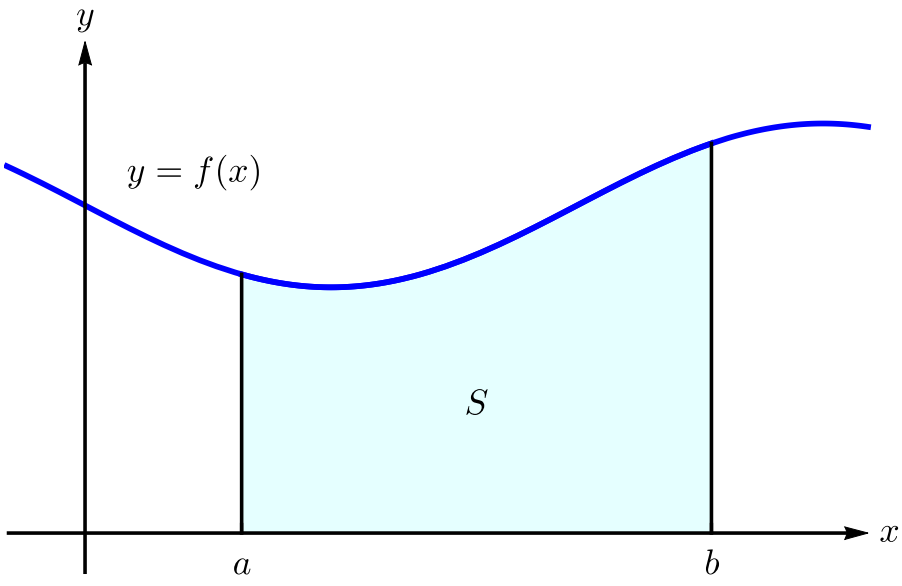
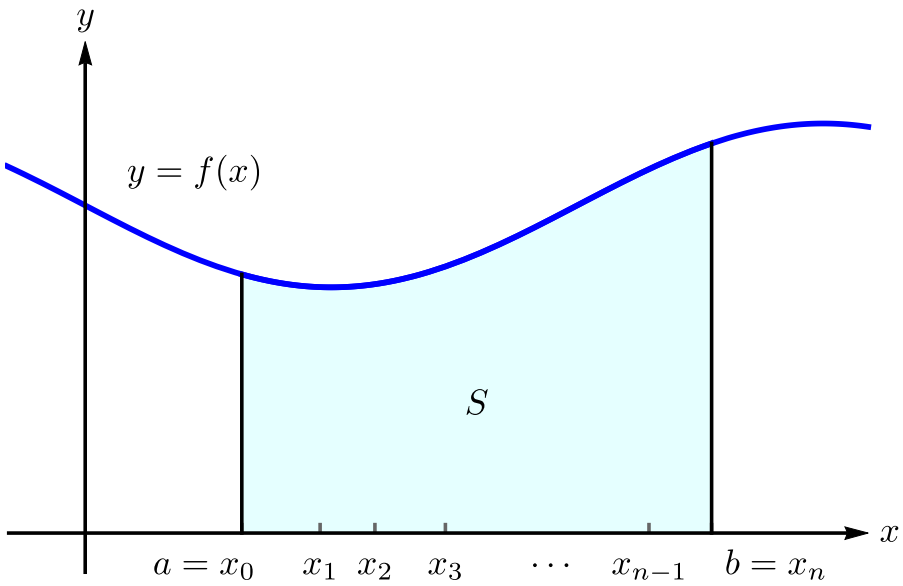
- This table suggests that R_n and L_n are approaching $\frac{1}{3}$.

Generalization

- Consider other methods to approximate the area of each subregion, and add the areas to approximate the area of S.
- Consider a plane region S bounded above by the graph of a nonnegative, continuous function $y = f(x)$, below by the x-axis, on the left by $x = a$ and on the right by $x = b$.
- Divide the interval $[a, b]$ into n nonoverlapping subintervals as defined by $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

  This is called a partition of the interval $[a, b]$, denoted $P = \{x_0, x_1, x_2, \cdots, x_n\}$





Generalization

- Let Δx_i be the length of the ith subinterval: $\Delta x_i = x_i - x_{i-1}$.
- Let x_i^* be an arbitrary point in the ith subinterval: $x_i^* \in [x_{i-1}, x_i]$.
- A **Riemann sum** of the function f over the interval $[a, b]$ with partition P is

$$\sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}) = \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

- For a given partition P, different Riemann sums are obtained depending upon how the values x_i^* are selected.

Riemann Sums

(1) If x_i^* is the left endpoint of each subinterval, $x_i^* = x_{i-1}$ for all i, then the sum is called a **left Riemann sum**, denoted L_n.

$$L_n = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x_i$$

(2) If x_i^* is the right endpoint of each subinterval, $x_i^* = x_i$ for all i, then the sum is called a **right Riemann sum**, denoted R_n.

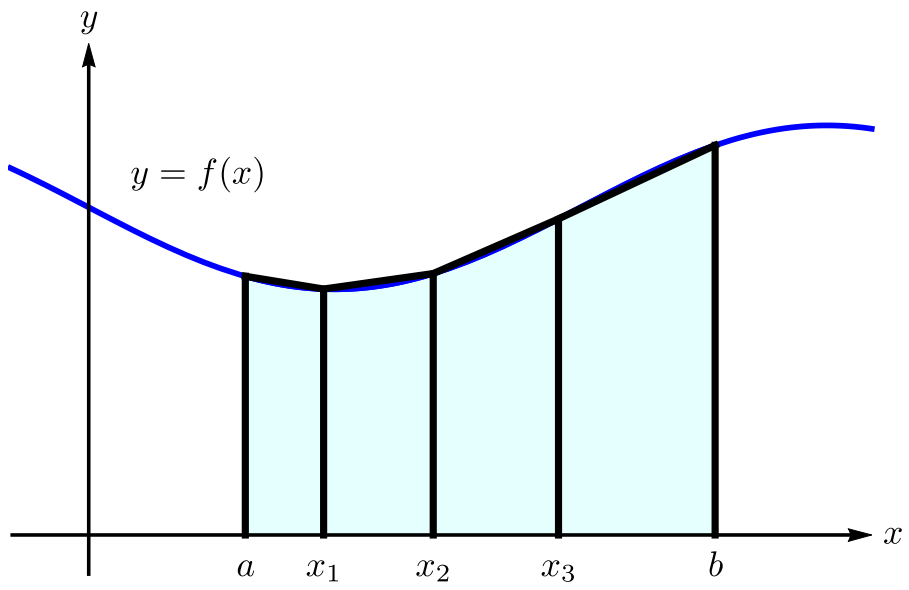
$$R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x_i$$

(3) If x_i^* is the midpoint of each subinterval, $x_i^* = \bar{x}_i = \frac{1}{2}(x_i + x_{i-1})$ for all i, then the sum is called a **midpoint Riemann sum**, denoted M_n.

$$M_n = \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x_i$$

Riemann Sums

(4) We can also use trapezoids to estimate the area of each subregion.



For a given partition the sum of the areas of the trapezoids is called a **trapezoidal sum**, denoted T_n.

$$T_n = \sum_{i=1}^{n} \frac{1}{2}(f(x_{i-1}) + f(x_i))\Delta x_i = \frac{1}{2}(L_n + R_n)$$

Regular Partition

If we divide the interval $[a, b]$ into equal subintervals, the partition is **regular**.

$$\Delta x_i = \Delta x = \frac{b-a}{n}$$

The points in the partition are $a, a+\Delta x, a+2\Delta x, \ldots, a+(n-1)\Delta x, b$

Left Riemann Sum

$$L_n = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

$$= \Delta x[f(a) + f(a+\Delta x) + f(a+2\Delta x) + \cdots + f(a+(n-1)\Delta x)]$$

Right Riemann Sum

$$R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$= \Delta x[f(a+\Delta x) + f(a+2\Delta x) + \cdots + f(b)]$$

Midpoint Sum

$$M_n = \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x$$

$$= \Delta x \left[f\left(a + \frac{\Delta x}{2}\right) + f\left(a + \frac{3\Delta x}{2}\right) + \cdots + f\left(a + \frac{(2n-1)\Delta x}{2}\right) \right]$$

Trapezoidal Sum

$$T_n = \sum_{i=1}^{n} \frac{1}{2}[f(x_{i-1} + f(x_i)]\,\Delta x$$

$$= \frac{1}{2}\Delta x\,[(f(a) + f(x_1)) + (f(x_1) + f(x_2)) + \cdots + (f(x_{n-1} + f(b))]$$

$$= \frac{1}{2}\Delta x\,[f(a) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{i-1}) + f(b)]$$

$$= \frac{1}{2}\Delta x\,[f(a) + 2f(a + \Delta x) + 2f(a + 2\Delta x) + \cdots + 2f(a + (n-1)\Delta x) + f(b)]$$

Example 2 More Sums

For the region S in Example 1 and a regular partition, extend the table to include midpoint and trapezoidal sums.

Solution

n	L_n	R_n	M_n	T_n
10	0.2850000	0.3850000	0.3325000	0.3350000
20	0.3087500	0.3587500	0.3331250	0.3337500
30	0.3168519	0.3501852	0.3332407	0.3335185
50	0.3234000	0.3434000	0.3333333	0.3334000
100	0.3283500	0.3383500	0.3333250	0.3333500
1000	0.3328335	0.3338335	0.3333333	0.3333335

Observations

(1) Each of the sums appears to be zeroing in on, or approaching, the same value, $\frac{1}{3}$.

(2) The midpoint sum and the trapezoidal sum appear to be approaching this limiting value faster.

Example 3 The Limiting Value

For the region S in Example 1 and a regular partition, show that $\lim_{n\to\infty} R_n = \frac{1}{3}$.

Solution

Divide the interval $[0, 1]$ into n equal subintervals, each of width

$\Delta x = \frac{1-0}{n} = \frac{1}{n}$; The partition is $\left\{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{n-1}{n}, 1\right\}$

$$R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$= \left(\frac{1}{n}\right)^2 \frac{1}{n} + \left(\frac{2}{n}\right)^2 \frac{1}{n} + \left(\frac{3}{n}\right)^2 \frac{1}{n} + \cdots + \left(\frac{n}{n}\right)^2 \frac{1}{n}$$ Write out the summation.

$$= \frac{1}{n^3}(1^2 + 2^2 + 3^2 + \cdots + n^2)$$ Simplify.

$$= \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2}$$ Summation formula; simplify.

Solution

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2}$$

$$= \lim_{n\to\infty} \frac{1}{6}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)$$ Rewrite as a product.

$$= \lim_{n\to\infty} \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$ Rewrite each fraction as a sum; simplify.

$$= \frac{1}{6}\cdot 1\cdot 2 = \frac{1}{3}$$ Evaluate the limit.

The exact area of the region S is $\frac{1}{3}$.

A Closer Look

(1) We can define the area A to be the limit of these Riemann sums.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n = \frac{1}{3}$$

(2) We could approximate the area of a region S by forming lower or upper sums.

Select x_i^* so that $f(x_i^*)$ is the minimum, or maximum, value of f on the ith subinterval.

A is the unique number that is smaller than all the upper sums and bigger than all the lower sums.

The upper and lower sums sandwich, or squeeze, the area of the region S.

Example 4

Let A be the area of the region bounded above by the graph of $f(x) = \dfrac{1}{1+x^2}$, below by the x-axis, and between the lines $x = 0$ and $x = 3$.

(a) Using right endpoints and n equal subintervals, find an expression for A as a limit.

(b) Estimate the area using a midpoint sum and a trapezoidal sum with $n = 4$ and $n = 10$ subintervals. Carefully sketch a graph to illustrate each sum for $n = 4$.

(c) Use technology to construct a table of values for L_n, R_n, M_n, and T_n for various values of n. Use this table to estimate the exact value of A.

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