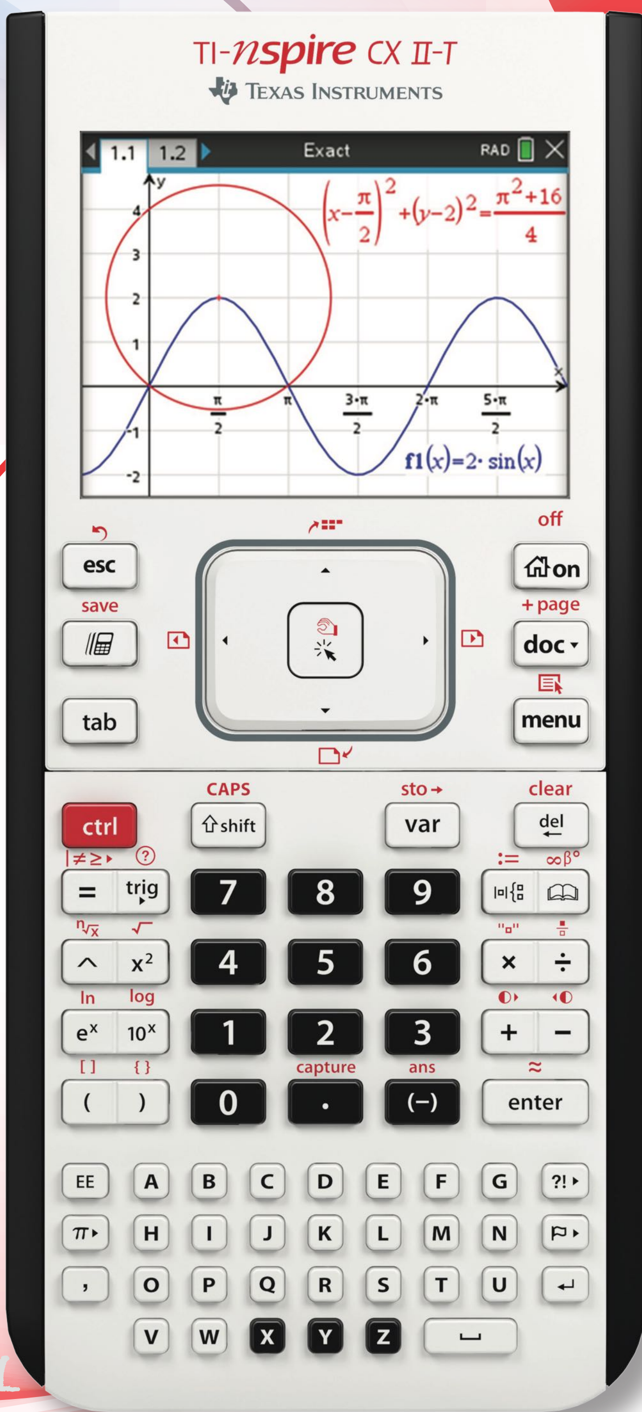
QCE Specialist Mathematics Teacher Resource Book for

TI-Nspire™ CX II-T graphing calculator

Teachers Teaching with Technology™
Professional Development from Texas Instruments

Authored by:

Peter Flynn
Frank Moya
David Tynan



Texas Instruments

Teachers Teaching with Technology™
Professional Development from Texas Instruments

The Teachers Teaching with Technology™ (T³™) Australia professional learning organization is comprised of some of the most creative and innovative mathematics and STEM teachers in the world. They are dynamic and passionate educators who share their knowledge and expertise with secondary teachers and students through professional development events and resource creation.

Table of Contents

1. Unit 1: Combinatorics, proof, vectors and matrices ... 3

1.1. Topic 1: Combinatorics ... 3
1.1.1. Introduction to counting techniques ... 3
1.1.2. Permutations and combinations ... 3
1.2. Topic 2: Introduction to proof ... 8
1.2.1. The nature of proof ... 8
1.2.2. Rational and irrational numbers ... 12
1.3. Topic 3: Vectors in the plane ... 16
1.3.1. Representing vectors in the plane by directed line segments ... 16
1.3.2. Vectors in two dimensions ... 21
1.4. Topic 4: Algebra of vectors in two dimensions ... 24
1.4.1. Algebra of vectors in two dimensions ... 24
1.5. Topic 5: Matrices ... 29
1.5.1. Matrix arithmetic and algebra ... 29

2. Unit 2: Complex numbers, further proof, trigonometry, functions and transformations ... 42

2.1. Topic 1: Complex numbers ... 42
2.1.1. Introduction to complex numbers ... 42
2.1.2. The complex plane ... 45
2.2. Topic 2: Complex arithmetic and algebra ... 49
2.2.1. Complex arithmetic using polar form ... 49
2.2.2. Subsets of the complex plane ... 52
2.2.3. Roots of real quadratic equations ... 55
2.3. Topic 3: Circle and geometric proofs ... 57
2.3.1. Circle properties and their proofs ... 57
2.3.2. Geometric proofs using vectors ... 60
2.4. Topic 4: Trigonometry and functions ... 63
2.4.1. Sketching graphs ... 63
2.4.2. The reciprocal trigonometric functions, secant, cosecant and cotangent ... 65
2.5. Topic 5: Matrices and transformations ... 69
2.5.1. Transformations in the plane ... 69

3. Unit 3: Further complex numbers, proof, vectors and matrices ... 75

3.1. Topic 1: Further complex numbers ... 75
3.1.1. Complex arithmetic using polar form ... 75
3.1.2. Roots of complex numbers ... 77
3.1.3. Factorisation of polynomials ... 82
3.2. Topic 2: Mathematical induction and trigonometric proofs ... 84
3.2.1. Mathematical induction ... 84
3.2.2. Trigonometric proofs using De Moivre’s theorem ... 86
3.3. Topic 3: Vectors in two and three dimensions ... 87
3.3.1. Vectors in three dimensions ... 87
3.3.2. Algebra of vectors in three dimensions ... 88
3.3.3. Vector and Cartesian equations ... 96
3.4. Topic 4: Vector calculus ... 106
3.4.1. Vector calculus ... 106
3.5. Topic 5: Further matrices ... 113
3.5.1. Matrix algebra and systems of equations ... 113
3.5.2. Applications of matrices ... 122

4. Unit 4: Further calculus and statistical inference ... 125

4.1. Topic 1: Integration techniques ... 125
4.1.1. Integration techniques ... 125
4.2. Topic 2: Applications of integral calculus ... 127
4.2.1. Applications of integral calculus ... 127
4.3. Topic 3: Rates of change and differential equations ... 138
4.3.1. Rates of change ... 138
4.3.2. Differential equations ... 142
4.4. Topic 4: Modelling motion ... 149
4.4.1. Modelling motion ... 149
4.5. Topic 5: Statistical inference ... 156
4.5.1. Sample means ... 156
4.5.2. Confidence intervals for means ... 164

Introduction

This publication, *QCE Specialist Mathematics Teacher Resource Book for the TI-Nspire™ CX II-T*, is intended to support senior secondary school mathematics teachers in Queensland as they seek to implement the new *QCAA Specialist Mathematics syllabus*.

Specifically, the publication highlights ways in which *TI-Nspire* technology might be used to assist in the teaching, learning and assessment of *QCE Specialist Mathematics Units 1 to 4*.

It is not a complete manual for using this technology, rather it tries to look at each syllabus dot point and make suggestions for possible classroom use.

It has been developed by experienced educators and reviewed by senior mathematics teachers from Queensland schools. We hope you find this to be a useful and supportive publication.

[Note: A digital version of this publication can be found at https://education.ti.com/aus/QLD].

Notes for teachers

To maximise the usefulness of *QCE Specialist Mathematics Teacher Resource Book for the TI-Nspire™ CX II-T* to teachers, the authors have provided the following explanatory notes.

- It is assumed that the user of this teacher resource book has a basic familiarity with navigating calculator documents and pages. Readers requiring an introduction to this are referred to tutorials at https://education.ti.com/en-au and https://www.youtube.com/@TIAustralia.
- Throughout this publication, unless otherwise specified, the default calculator document settings have been used. The calculator user interface language has been set to *English (U.K.).*
- For each example task, it is desirable to start a new calculator document. Alternatively, insert a new problem.
- When working with functions, use of the **assign** command has been privileged over the **define** command. While both commands essentially perform the same role, the **assign** command is a more natural command to use in the **Notes** application.
- Implied multiplication has been assumed when working with products such as $6x$. However, where it is necessary to use the multiplication key when entering the product $(x-2)(x-4)$, for example, the symbol '$*$' is used to denote multiplication.
- In some instances in this publication, space has been added to the syntax of commands to improve readability, even though in general spaces should **not** be used in calculator commands without a clear reason. For example, when entering a function, the authors may have expressed this as **$f(x) := ax^2+bx+c$,** but on the calculator, it will appear as $f(x){:=}a{\cdot}x^2+b{\cdot}x+c$.
- There will be some variation in the formatting of commands and text to be entered, but the authors have attempted to use bold formatting when referring to commands to be entered or accessed via the calculator.
- For screenshots from the **Graphs** Application, grid and label settings will vary. Use [menu] commands (or [ctrl][menu]) to modify these settings for an open document. The default settings (for all documents) for grids and labels can be edited by pressing [menu] and then select **Settings**.
- When catalogue commands are mentioned, pressing [📖] and then [1] will display catalog commands in alphabetical order. Pressing the first letter of the desired command will locate it more quickly.
- To make this publication as practical and concise as possible, mathematical problems considered have been restricted to those that can be attempted by teachers and students without using pre-prepared files. For more interactive digital resources aligned to *QCE Specialist Mathematics Units 1 to 4*, go to the following web page.

  https://education.ti.com/en-au/seniornspiredcurriculum/aus-nz/qld-nspire

Unit 1: Combinatorics, proof, vectors and matrices

1.1. Topic 1: Combinatorics

1.1.1. Introduction to counting techniques

Using factorial notation

Find the value of each of the following.

(a) $10!$ **(b)** $\dfrac{100!}{98!}$ **(c)** $6\times4!-7\times3!$

Text in the form of comments can be added to a **Calculator** page.

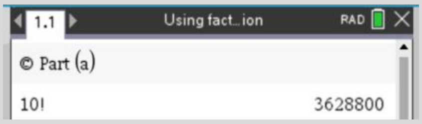
To add a comment such as '© Part (a)':

- Press menu > **Actions** > **Insert Comment**.

To access the factorial symbol:

- Press menu > **Probability** > **Factorial (!)**.

To evaluate parts **(a)**, **(b)** and **(c)**, enter as shown.

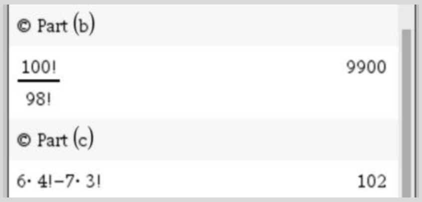


(a) $10!=3628800$ where $10!=10\times9\times8\times...\times3\times2\times1$

- Press ctrl ÷ to access the **Fraction** template.

(b) $\dfrac{100!}{98!}=9900$ where $\dfrac{100!}{98!}=\dfrac{100\times99\times98!}{98!}=100\times99$

(c) $6\times4!-7\times3!=102$



1.1.2. Permutations and combinations

Defining and using permutations

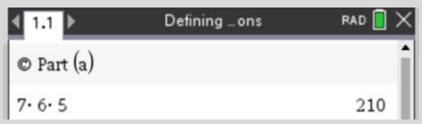
Adele has seven different books but there is only room for three of these books on her bookshelf. Find the number of ways Adele can randomly select the books and arrange them on her bookshelf using

(a) the multiplication principle **(b)** $\dfrac{n!}{(n-r)!}$ **(c)** ${}^nP_r$

To add a comment to a **Calculator** page:

- Press menu > **Actions** > **Insert Comment**.

(a) 7 books can be placed in the first position, 6 remain for the second and 5 for the third. So $7\times6\times5=210$.



To access the factorial symbol:

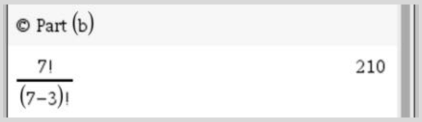
- Press menu > **Probability** > **Factorial (!)**.

... continued

To access the **Fraction** template:

- Press [ctrl] [÷].

(b) $\dfrac{7!}{(7-3)!} = 210$ where $\dfrac{7!}{4!} = \dfrac{7\times 6\times 5\times 4!}{4!} = 210$



$\dfrac{n!}{(n-r)!}$ can be interpreted as

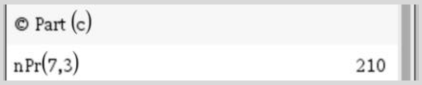
$$\frac{(\text{total number of objects})!}{(\text{total number of objects} - \text{number of objects to be arranged})!}$$

The number of ways to arrange r objects from a total of n objects is ${}^nP_r$ where ${}^nP_r = \dfrac{n!}{(n-r)!}$.

To access the **Permutations** command:

- Press [menu] > **Probability** > **Permutations**.

(c) ${}^7P_3 = 210$



Note: *${}^nP_r$ represents the number of ways of selecting r objects from n distinct objects where order is important.*

Solving problems involving permutations

In how many ways can 4 cats and 3 dogs be arranged in a row if

(a) they are placed randomly?

(b) the 4 cats are kept together and the 3 dogs are kept together?

(c) no cat is next to another cat?

To add a comment to a **Calculator** page:

- Press [menu] > **Actions** > **Insert Comment**.

To access the factorial symbol:

- Press [menu] > **Probability** > **Factorial (!)**.

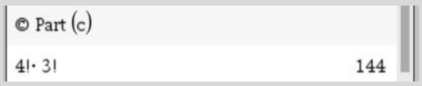
(a) There are $7! = 5040$ arrangements.

(b) There are 4! ways of keeping the 4 cats together and for each of these, 3! ways of keeping the dogs together. Also, there are 2 ways of arranging the group of cats and the group of dogs. The number of ways is $2\times 4!\times 3! = 288$.

(c) If no cat is next to another cat, the arrangement must be CDCDCDC.

There are 4! ways of arranging the 4 cats and for each of these, 3! ways of arranging the 3 dogs.

The number of ways is $4!\times 3! = 144$.



Evaluating ^nC_r

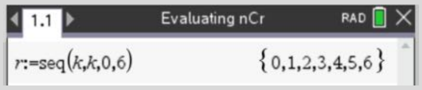
Evaluate ^6C_r for $r = 0,1,2,3,4,5,6$.

One way to evaluate ^6C_r for $r = 0,1,2,3,4,5,6$ is to use the sequence command.

On a **Calculator** page, assign the values of r as a sequence.

To enter $\boldsymbol{r := \text{seq}(k, k, 0, 6)}$:

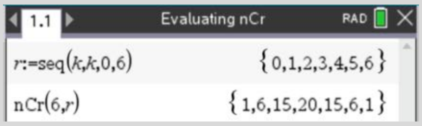
- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [menu] > **Statistics** > **List Operations** > **Sequence**.
- Enter as shown.



Note:** The syntax for expressing a sequence as a list is* ***seq(Expression, Variable, Low, High[,Step])*.*
*The default value for **Step** is 1.*

To access the **Combinations** command:

- Press [menu] > **Probability** > **Combinations**.
- Enter as shown.



$^6C_0 = 1,\ ^6C_1 = 6,\ ^6C_2 = 15,\ ^6C_3 = 20,\ ^6C_4 = 15,\ ^6C_5 = 6,\ ^6C_6 = 1$

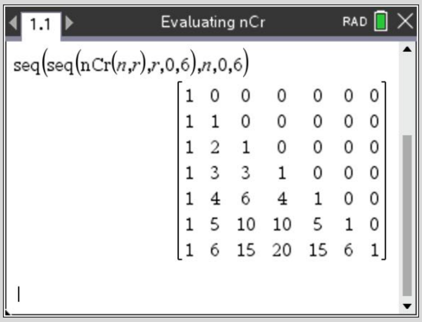
***Note:** This is the $n = 6$ row of Pascal's triangle.*
The $n = 0$ row of Pascal's triangle is the first row.
***Note:** Alternatively, enter on a **Calculator** page as shown.*
To access "{}", press [ctrl] [ )].

Extension:

A way to generate rows of Pascal's triangle is to write a command for "nested" sequences as shown on the **Calculator** page at right.

The first 7 rows of Pascal's triangle are displayed.

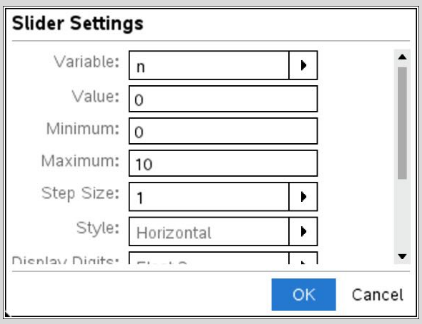
Can you see how it works?



Alternatively, on a **Notes** page:

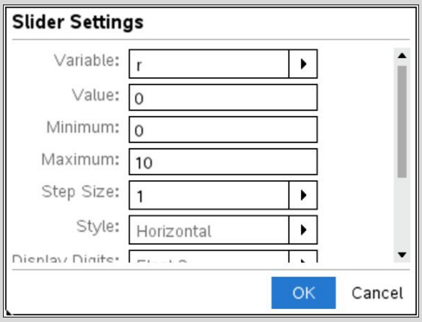
Insert a **Slider** to control the value of n as follows:

- Press [menu] > **Insert** > **Slider**.
- Set the **Slider Settings** as shown.
- Ensure to check the Minimised box.

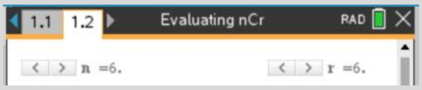


... continued

Repeat the above instructions to insert a slider for r.



Position the sliders as shown at right.



Insert a **Maths Box** as follows:

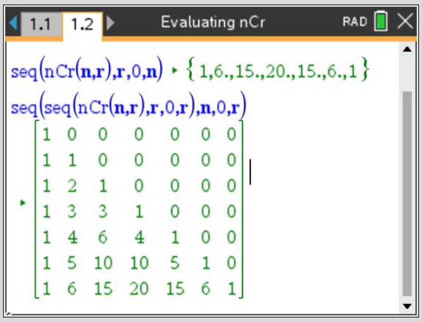
- Press [menu] **> Insert** > **Maths Box**.

Note: *Alternatively, to insert a* ***Maths Box****, press* [ctrl] [M].

- Enter **seq(nCr(*n*,*r*),*r*,0,*n*)** into the **Maths Box** as shown.

Insert another **Maths Box** as follows:

- Press [menu] **> Insert** > **Maths Box**.
- Enter **seq(seq(nCr(*n*,*r*),*r*,0,*r*),*n*,0,*r*)** into this second **Maths Box** as shown.



Click on the sliders to change the value of n and r.

The screenshot at right displays the $n = 6$ row of Pascal's triangle and the first 7 rows of Pascal's triangle.

Solving equations involving ^nC_r

Solve $3 \times {}^nC_6 = 11 \times {}^nC_4$ for n where n is a positive integer.

Note that ${}^nC_6 \geq 1$ and ${}^nC_4 > 1$ for $n \in \mathbb{Z}^+, n \geq 6$.

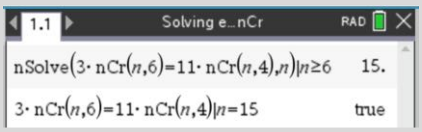
On a **Calculator** page:

- Press [menu] **> Algebra > Numerical Solve**.
- Press [menu] **> Probability > Combinations**.

Complete as shown.

To add the constraint $n \geq 6$:

- Press [ctrl] [=] to access the 'with' or 'given' symbol | and the $\geq$ symbol.



Solving $3 \times {}^nC_6 = 11 \times {}^nC_4$ for n with $n \geq 6$ gives $n = 15$.

Note: *Entering the equation with $n = 15$ gives the output* ***'true'***.

Solving problems involving combinations

A committee of three must be chosen from a cricket team of 11 players.

How many different committees are possible if:

(a) there are no restrictions?

(b) the captain of the team must be on the committee?

To add a comment to a **Calculator** page:

- Press [menu] > **Actions** > **Insert Comment**.

To access the **Combinations** command:

- Press [menu] > **Probability** > **Combinations**.

(a) There are ${}^{11}C_3 = 165$ possible committees.

(b) As the captain of the team must be on the committee, we simply need to select two of the remaining players.

There are ${}^{10}C_2 = 45$ possible committees.

Solving permutations and combinations problems including probability

Twenty balls numbered from 1 to 20 are placed in a barrel.

If two balls are randomly selected, what is the probability that they are both numbered under 10?

To access the **Fraction** template:

- Press [ctrl] [÷].

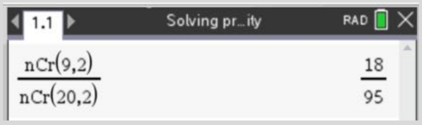
To access the **Combinations** command:

- Press [menu] > **Probability** > **Combinations**.

There are ${}^{9}C_2 = 36$ possibilities for the specified outcome since the two balls must come from those numbered from 1 to 9.

There are ${}^{20}C_2 = 190$ ways of selecting two objects from a set of 20 objects.

$$P(\text{both under } 10) = \frac{{}^{9}C_2}{{}^{20}C_2} = \frac{18}{95}$$



1.2. Topic 2: Introduction to proof

1.2.1. The nature of proof

Divisibility and modular arithmetic: Brief background

When an integer a is divided by another integer m then $a = km + r$ where $0 \le r < m$ and r is the remainder. For example, $39 = 5 \times 7 + 4$.

In terms of modular arithmetic (the study of the properties of remainders), a is said to be congruent to r modulo m if $m \mid (a - r)$. For example, $7 \mid (39 - 4)$.

For any integer a there exists a congruence $a \equiv r \pmod{m}$. For example, $39 \equiv 4 \pmod{7}$.

Congruence modulo m generalizes the notion of divisibility since $a \equiv 0 \pmod{m} \Leftrightarrow m \mid a$.

In summary, if $a = km + r$ then $a \equiv r \pmod{m}$, since $m \mid (a - r)$.

Proving by contradiction

Consider integers x and y such that $x^2 + y^2$ is exactly divisible by 4. Prove by contradiction that x and y cannot both be odd.

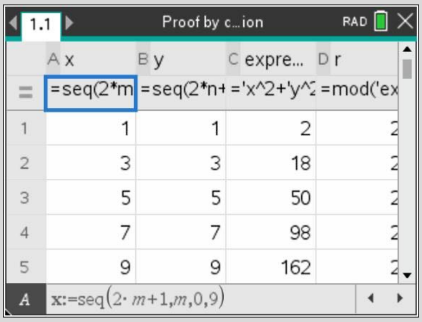
Start with an exploratory verification that both x and y cannot be odd when $x^2 + y^2$ is exactly divisible by 4.

To explore cases where both x and y are both odd, on a **Lists & Spreadsheet** page:

- In the column A heading cell, enter the variable $\boldsymbol{x}$.
- In the column B heading cell, enter the variable $\boldsymbol{y}$.
- In the column C heading cell, enter the variable ***expression***.
- In the column D heading cell, enter the variable $\boldsymbol{r}$.

Generate the required sequences of values as follows:

To enter $\boldsymbol{x := \text{seq}(2m+1, m, 0, 9)}$ in the column A formula cell:



- Press [book] [S], scroll down and select **seq(**.
- Enter as shown.

> ***Note:*** *The syntax for expressing a sequence as a list is* ***seq(Expression, Variable, Low, High[,Step])****.*
> *The default value for* ***Step*** *is 1.*

To enter $\boldsymbol{y := \text{seq}(2n+1, n, 0, 9)}$ in the column B formula cell:

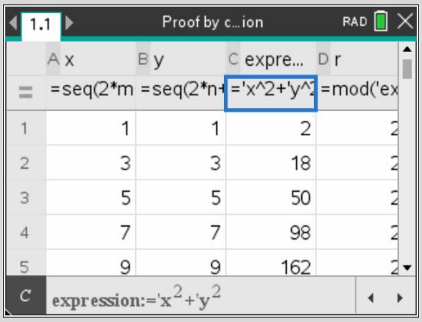
- Press [book] [S], scroll down and select **seq(**.
- Enter as shown.

> ***Note:*** *Press* [ctrl] [1] *to go to the last entry in a column.*
> *Press* [ctrl] [7] *to go to the first entry in a column.*
> *Press* [ctrl] [3] *to go down a page and* [ctrl] [9] *to go up a page.*
> *To go to a specific cell, press* [ctrl] [G] *and type in the cell reference.*

... continued

To enter $\boldsymbol{expression} := {}'x^2 + {}'y^2$ in the column C formula cell:

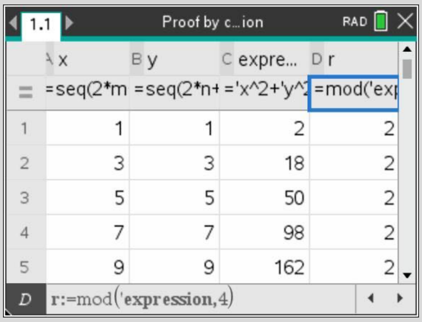
- Press [=]
- Press [?!▸] to access the ' symbol.
- Enter as shown.



Note: *Press* [var] *to access assigned/stored variables.*
Note: *The symbol ' in 'x specifies x as a variable reference. Otherwise, TI-Nspire CX II-T will consider x as a column reference. If the ' symbol is omitted, a prompt may appear asking whether the variable being used refers to a variable or to a spreadsheet column.*

To enter $\boldsymbol{r} := \mathbf{mod}({}'\boldsymbol{expression}, \mathbf{4})$ in the column D formula cell:

- Press [📖][M], scroll down and select **mod(**.
- Press [?!▸] to access the ' symbol.
- Press [var] to access ***expression***.
- Enter as shown.



Note: *The* ***remain(*** *command, accessed from the* ***Catalog****, can be used instead of* ***mod(****.*

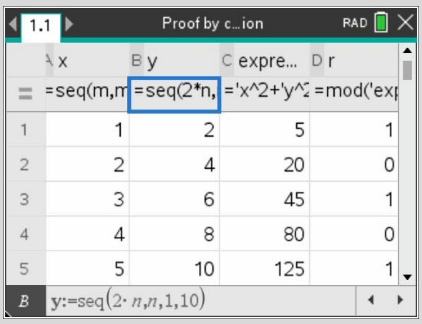
Column D of the spreadsheet indicates that when x and y are both odd, $x^2 + y^2$ is not exactly divisible by 4. In each case, the remainder is 2. This result forms an important part of the proof.

Note: *To access* ***mod(*** *on a* ***Calculator*** *page, press* [menu] > ***Number*** > ***Number Tools****.*

To explore the case where *y* is even:

- Enter $\boldsymbol{x} := \mathbf{seq}(\boldsymbol{m}, \boldsymbol{m}, \mathbf{1}, \mathbf{10})$ in the column A formula cell.
- Enter $\boldsymbol{y} := \mathbf{seq}(\mathbf{2} \cdot \boldsymbol{n}, \boldsymbol{n}, \mathbf{1}, \mathbf{10})$ in the column B formula cell.

Note: *Alternatively in the column B formula cell, enter* $\boldsymbol{y} := \mathbf{seqn}(\mathbf{2} \cdot \boldsymbol{n}, \mathbf{10})$ *where* ***seqn(****, found in the* ***Catalog****, generates a list beginning with n=1.*



Column D of the spreadsheet now indicates that when x and y are both even, $x^2 + y^2$ is exactly divisible by 4. In each case, the remainder is 0. When x is even and y is odd (or vice versa), $x^2 + y^2$ is not exactly divisible by 4. In each case, the remainder is 1.

... continued

Proof:

Assume that x and y are both odd.

Then $x = 2m+1$ and $y = 2n+1$ where $m, n \in \mathbb{Z}$.

$$\begin{aligned} x^2 + y^2 &= (2m+1)^2 + (2n+1)^2 \\ &= 4m^2 + 4m + 1 + 4n^2 + 4n + 1 \\ &= 4(m^2 + m + n^2 + n) + 2 \end{aligned}$$

$4(m^2 + m + n^2 + n)$ is always exactly divisible by 4.

However, 2 is not exactly divisible by 4.

So $x^2 + y^2$ is not exactly divisible by 4, which is a contradiction.

Hence x and y cannot both be odd.

***Note**: A number is exactly divisible by 4 if the number formed by its last two digits is divisible by 4.*

Using examples and counterexamples

Consider the set of numbers S of the form $n^2 - n + 41$ where $n \in \mathbb{Z}^+$.

(a) Prove that all elements of S are odd.

The first five elements of S are $\{41, 43, 47, 53, 61\}$. These are all prime numbers.

(b) Prove by use of a counterexample that not all elements of S are prime.

(a) $(n^2 - n) + 41 = n(n-1) + 41$

Either n is even or $n-1$ is even so $n^2 - n = n(n-1)$ is even (the product of an even number and an odd number is even).

Adding 41 to an even number gives an odd number.

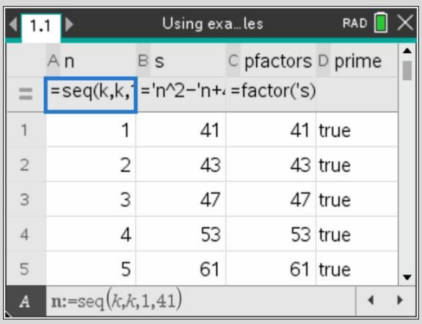
So $n^2 - n + 41$ is odd.

(b) Legendre (1798) discovered the prime-generating polynomial $n^2 - n + 41$. Note that $n^2 - n + 41$ is also prime for $n = 0$.

An exploratory verification can help in finding a counterexample that proves that not all elements of S are prime.

On a **Lists & Spreadsheet** page:

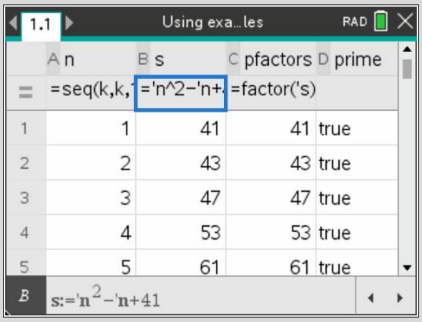
- In the column A heading cell, enter the variable $\boldsymbol{n}$.
- In the column B heading cell, enter the variable $\boldsymbol{s}$.
- In the column C heading cell, enter the variable ***pfactors***.
- In the column D heading cell, enter the variable ***prime***.



... continued

Generate the required sequences of values as follows:

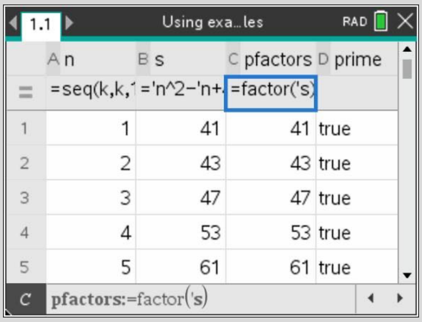
To enter $\boldsymbol{n := \mathbf{seq}(k,k,1,41)}$ in the column A formula cell:



- Press [📖] [S], scroll down and select **seq(**.
- Enter as shown.

Note: *The syntax for expressing a sequence as a list is* ***seq(Expression, Variable, Low, High[,Step])****. The default value for* ***Step*** *is 1.*

To enter $\boldsymbol{s := 'n^2 - 'n + 41}$ in the column B formula cell:



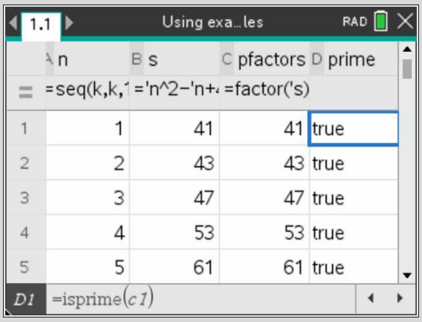
- Press [?!▸] to access the **'** symbol.
- Enter as shown.

Note: *The symbol* **'** *in* **'*n*** *specifies n as a variable reference. Otherwise, TI-Nspire CX II-T will consider n as a column reference. If the* **'** *symbol is omitted, a prompt may appear asking whether the variable being used refers to a variable or to a spreadsheet column.*

To enter $\boldsymbol{pfactors := \mathbf{factor}('s)}$ in the column C formula cell:

- Press [📖] [F] and select **factor(**.
- Enter as shown.

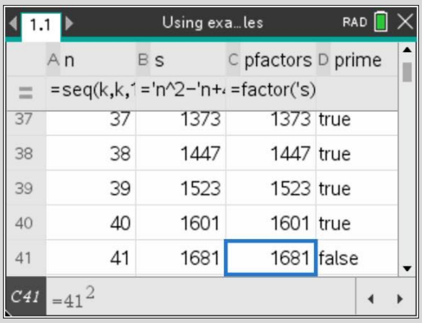
In cell D1:



- Press [=] [📖] [I], scroll down and select **isPrime(**.
- Enter $= \mathbf{isprime(c1)}$ where **c1** denotes the cell reference.

To fill down to cell D41:

- Press [menu] > **Data** > **Fill**.
- Press ▼ to extend a rectangular box down to and including cell D41.
- Press [enter].



Note: *Alternatively, press* [ctrl] [menu] *>* ***Fill****.*

The cells D1 through to D41 will be filled with either the output **'true'** or the output **'false'**.

Cell C41 shows that when $n = 41$, $n^2 - n + 41 = 1681 = 41^2$.

The output **'false'** in cell D41 confirms that $n^2 - n + 41$ is not prime when $n = 41$.

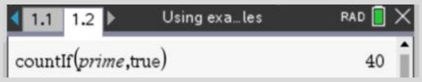
The key to finding a composite number (a number that is not prime) is to consider the constant term, 41.

Substituting $n = 41$ into $n^2 - n + 41$, for example, gives $41^2 - 41 + 41 = 41^2$ which has 41 as a factor and hence is not prime.

... continued

Alternatively on a **Calculator** page:

- Press [catalog][C], scroll down and select **countIf(**.
- Press [var] and select **prime**.
- Press [,] [catalog][T], scroll down and select **true**.



The output indicates that the first 40 integers are prime.

- Press [catalog][C], scroll down and select **countIf(**.
- Press [var] and select **prime**.
- Press [,] [catalog][F], scroll down and select **false**.

The output indicates that one integer (the 41st integer) is prime.

- Press [menu] > **Number** > **Factor**.
- Press [var] and select **s**.

This generates the list of integers in factored form where possible.

The last integer in the list, 1681, can be expressed as 41^2.

To see this integer in the list without scrolling:

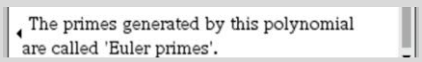
- Press [catalog][R], scroll down and select **right(**.
- Press [menu] > **Number** > **Factor**.
- Press [var] and select **s**.
- Press [,][1].

This confirms that $1681 = 41^2$.

Note: *The syntax for returning the rightmost elements in a list is* ***right(List[,Num])****.*

To add a comment to a **Calculator** page:

- Press [menu] > **Actions** > **Insert Comment**.



The primes generated by this polynomial are called ‘Euler primes’.

1.2.2. Rational and irrational numbers

Expressing rational numbers as recurring decimals

When a rational number, $\frac{m}{n}$, where m, n are relatively prime (i.e. m, n have no common factor other than 1), is expanded, the periodic decimal expansion begins after s digits and has length t digits, where s and t are the smallest numbers that satisfy $10^s \equiv 10^{s+t} \left(\bmod n\right)$.

Consider the decimal expansions of fractions with denominator 84.

(a) Find the period, s.

(b) Find the length, t.

... continued

To find the smallest values of s and t that satisfy $10^s \equiv 10^{s+t} \pmod{84}$, find the remainders of the powers of $10 \pmod{84}$.

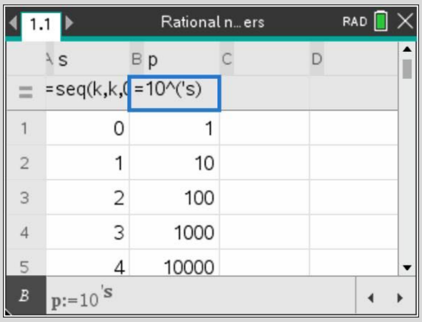
On a **Lists & Spreadsheet** page:

- In the column A heading cell, enter the variable $\boldsymbol{s}$.
- In the column B heading cell, enter the variable $\boldsymbol{p}$.
- In the column C heading cell, enter the variable $\boldsymbol{r}$.

Generate the required sequences of values as follows:

To enter $\boldsymbol{s := \mathrm{seq}(k, k, 0, 10)}$ in the column A formula cell:

- Press [📖] [S], scroll down and select **seq(**.
- Enter as shown.



Note: *The syntax for expressing a sequence as a list is* ***seq(Expression, Variable, Low, High[,Step])****. The default value for* ***Step*** *is 1.*

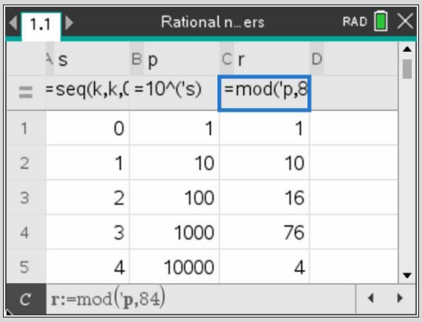
To enter $\boldsymbol{p := 10^{'s}}$ in the column B formula cell:

- Press [?!▸] to access the **'** symbol.
- Enter as shown.

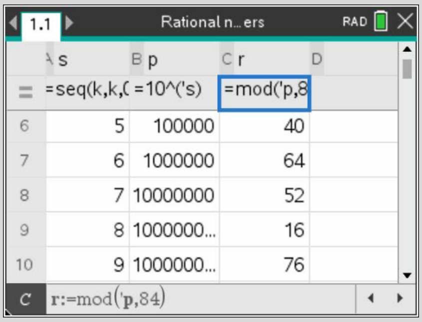
Note: *The symbol* **'** *in* **'s** *specifies s as a variable reference. Otherwise, TI-Nspire CX II-T will consider s as a column reference. If the* **'** *symbol is omitted, a prompt may appear asking whether the variable being used refers to a variable or to a spreadsheet column.*

To enter $\boldsymbol{r := \mathrm{mod}('p, 84)}$ in the column C formula cell:

- Press [📖] [M], scroll down and select **mod(**.
- Enter as shown.



Note: *The* ***remain(*** *command, accessed from the* ***Catalog****, can be used instead of* ***mod(****.*
Note*: To access* ***mod(*** *on a Calculator page, press* [menu] *>* ***Number*** *>* ***Number Tools****.*



Column C gives the remainders of the powers of $10 \pmod{84}$ for $s = 0,1,2,3,4,5,6,7,8,9,10$.

The remainders are $1,10,16,76,4,40,64,52,16,76,4$.

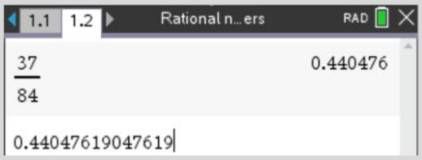
From these remainders:

(a) $s = 2$ i.e. the period begins after two digits.

(b) $t = 6$ i.e. the length is 6.

... continued

On a **Calculator** page:



- Enter the fraction $\frac{37}{84}$ (press ctrl ÷ to access the **Fraction** template) and press ctrl enter.
- Press ▲ enter.

This gives the output 0.44047619047619.

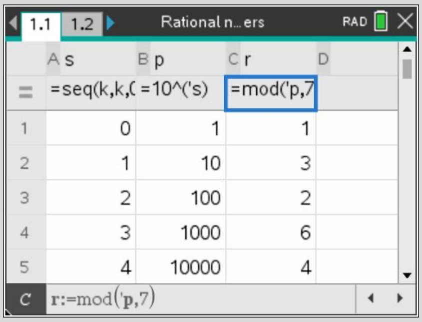
This decimal expansion confirms that $s = 2$ and $t = 6$.

$$\frac{37}{84} = 0.44\overline{047619}$$

Now consider when the denominator of the fraction, n, to be expanded does not have factors 2 or 5.

Since n is relatively prime to 10, then $10^t \equiv 1(\text{mod}\, n)$.

So $s = 0$ and the period starts with the first digit.



This is a purely periodic expansion.

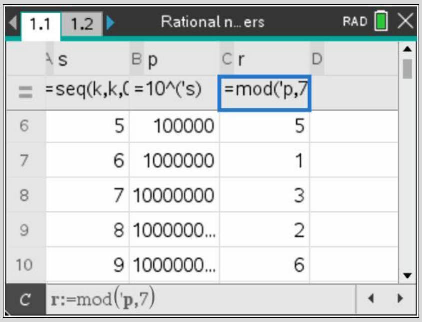
Repeat the above instructions with $n = 7$:

- In the Column C formula cell, replace 84 with 7.

Column C now gives the remainders of the powers of $10(\text{mod } 7)$ for $s = 0,1,2,3,4,5,6,7,8,9,10$.

The remainders are $1,3,2,6,4,5,1,3,2,6,4$.

From these remainders:



(a) $s = 0$ i.e. the period begins immediately.

(b) $t = 6$ i.e. the length is 6.

For example, $\frac{2}{7} = 0.\overline{285714}\ldots$.

When the denominator of $\frac{m}{n}$ has the form $n = n_0 \cdot 2^\alpha \cdot 5^\beta$ and $(n_0, 10) = 1$, the period begins after μ digits, where μ is the larger of α and β. The length of the period is the exponent to which 10 belongs $(\text{mod}\, n_0)$.

So when $n = 84 = 2^2 \times 21$, the period starts after the second digit and has length 6 since 10 belongs to the exponent $6(\text{mod}\, 21)$.

Proving results involving integers using the contrapositive

By proving the contrapositive, prove that if n^2-6n+5 is even then n is odd, $\forall n\in\mathbb{Z}$.

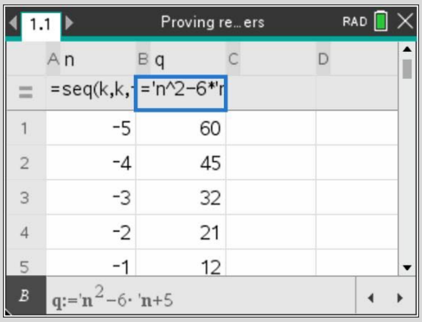
Start with an exploratory verification that n^2-6n+5 is even when n is odd, $\forall n\in\mathbb{Z}$.

On a **Lists & Spreadsheet** page:

- In the column A heading cell, enter the variable $\boldsymbol{n}$.
- In the column B heading cell, enter the variable $\boldsymbol{q}$.

Generate the required sequences of values as follows:

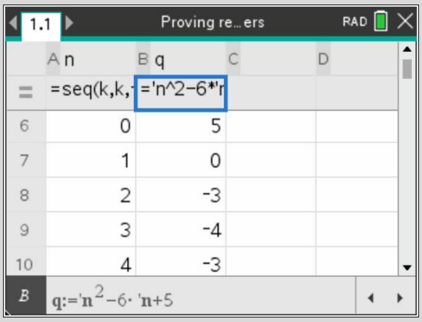
To enter $\boldsymbol{n:=\mathbf{seq}(k,k,-5,5)}$ in the column A formula cell:



- Press [book] [S], scroll down and select **seq(**.
- Enter as shown.

Note: *The syntax for expressing a sequence as a list is **seq(Expression, Variable, Low, High[,Step])**. The default value for **Step** is 1.*

To enter $\boldsymbol{q:='n^2-6\cdot'n+5}$ in the column B formula cell:



- Press [?!▸] to access the **'** symbol.
- Enter as shown.

Note: *The symbol* **'** *in* **'n** *specifies n as a variable reference. Otherwise, TI-Nspire CX II-T will consider n as a column reference. If the* **'** *symbol is omitted, a prompt may appear asking whether the variable being used refers to a variable or to a spreadsheet column.*

This numerical exploration suggests that when n is odd, n^2-6n+5 is even and when n is even, n^2-6n+5 is odd.

Proof:

The contrapositive statement is:

If n is even, then n^2-6n+5 is odd.

Let $n=2k$ for $k\in\mathbb{Z}$.

$$\begin{aligned}n^2-6n+5&=(2k)^2-6(2k)+5\\&=4k^2-12k+5\\&=4k^2-12k+4+1\\&=2\left(2k^2-6k+2\right)+1\end{aligned}$$

So $n^2-6n+5=2a+1$ where a is the integer $2k^2-6k+2$.

Thus n^2-6n+5 is odd.

Hence, by the contrapositive, if n^2-6n+5 is even then n is odd, $\forall n\in\mathbb{Z}$.

1.3. Topic 3: Vectors in the plane

1.3.1. Representing vectors in the plane by directed line segments

Representing vector addition and subtraction with the triangle rule

It is important to understand and use the vector notation $\overrightarrow{AB}$, $\underset{\sim}{c}$, $\boldsymbol{d}$ and $\hat{\boldsymbol{n}}$.

In Section 1.3, the vector notation $\overrightarrow{AB}$, $\boldsymbol{d}$ and $\hat{\boldsymbol{n}}$ is used.

A vector is a set of equivalent directed line segments.

- If $\boldsymbol{u} = \overrightarrow{AB}$ and $\boldsymbol{v} = \overrightarrow{BC}$, then $\boldsymbol{u} + \boldsymbol{v} = \overrightarrow{AB} + \overrightarrow{BC}$.
- Subtraction of vectors: $\boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u} + (-\boldsymbol{v})$.

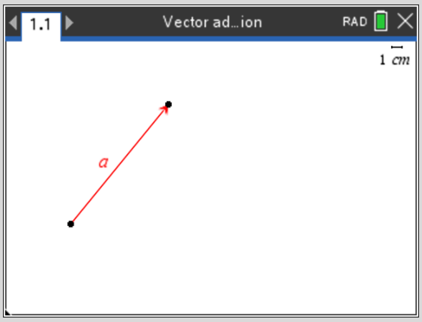
The triangle rule represents the resultant vector from the sum and difference of two vectors.

The following describes how to construct a simple geometric demonstration of the triangle rule.

Note: *This construction is best attempted using the TI-Nspire CX-II T Teacher Software rather than on the handheld device.*

On a **Geometry** page, add a vector $\boldsymbol{a}$ as follows:

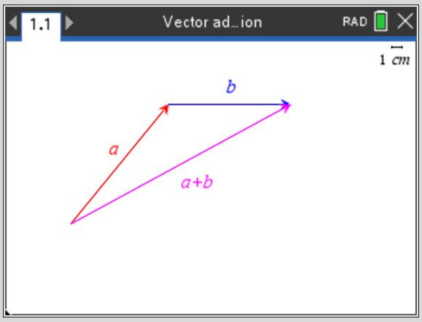
- Press [menu] > **Points & Lines** > **Vector**.
- Decide on the vector's starting point and press [enter].
- Decide on the vector's end point and press [enter].
- Complete as shown and press [esc].



To give a vector a label:

- Hover over the vector.
- Press [ctrl][menu] > **Label**.
- Label as shown.

Note: *To change the line colour of a vector, hover over the vector, press [ctrl][menu] >* ***Colour*** *>* ***Line Colour*** *and change the colour of the vector as desired.*
Note: *To change the attributes of a vector, hover over the vector, press [ctrl][menu] >* ***Attributes*** *and change the appearance of the vector as desired.*



Repeat the above instructions to add a vector $\boldsymbol{b}$ and a vector $\boldsymbol{a} + \boldsymbol{b}$ with labels and line colours as shown.

Note: *Press [tab] when you want to select an object from a set of objects that are close to each other on a page. A [tab] icon will appear next to the cursor in these situations and acts as a fine motor control.*

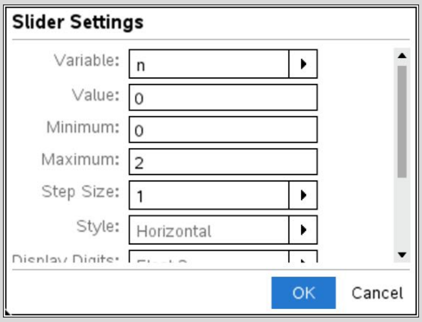
To show the triangle rule for addition dynamically, a slider with conditional attributes can be used.

This slider, set up for values of a variable $\boldsymbol{n}$, controls when vector $\boldsymbol{b}$ and vector $\boldsymbol{a} + \boldsymbol{b}$ are displayed on the page.

... continued

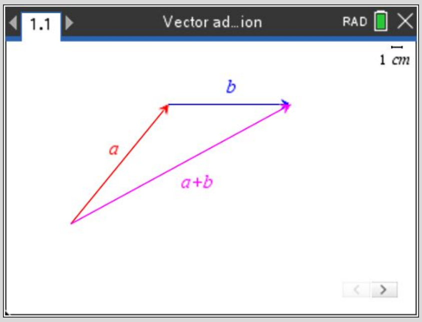
Insert a **Slider** to control the value of $\boldsymbol{n}$ as follows:

- Press menu > **Actions** > **Insert Slider**.
- Set the **Slider Settings** as shown.
- Check the **Minimised** box.
- Uncheck the **Show Variable** and **Show Scale** boxes.



To move the **Slider**:

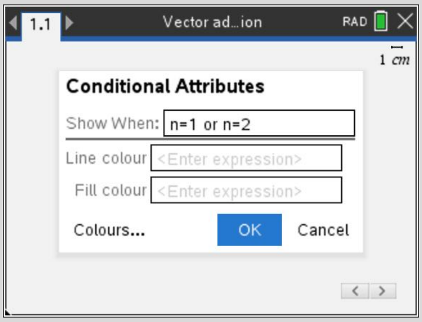
- Press ctrl menu and move it to the bottom right-hand corner as shown.



To set vector $\boldsymbol{b}$ to display when $\boldsymbol{n} = \mathbf{1}$ or $\boldsymbol{n} = \mathbf{2}$, complete as follows:

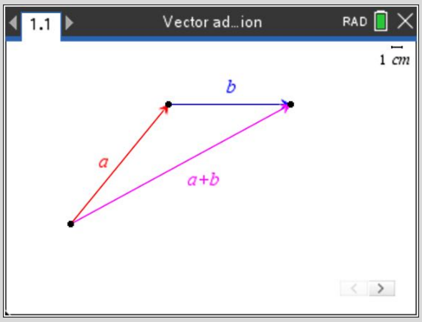
- Hover over the vector $\boldsymbol{b}$ and press tab to select the vector.
- Press ctrl menu > **Conditions**.
- Complete the **Conditional Attributes** settings as shown.

Note: *Alternatively, to set a condition press* menu > ***Actions*** *>* ***Set Conditions****.*



To set vector $\boldsymbol{a}+\boldsymbol{b}$ to display when $\boldsymbol{n} = \mathbf{2}$, complete as follows:

- Hover over the vector $\boldsymbol{a}+\boldsymbol{b}$ and press tab to select the vector.
- Press ctrl menu > **Conditions**.
- Complete the **Conditional Attributes** settings.



To hide all points except the starting point for vector $\boldsymbol{a}$:

- Hover over each point and press ctrl menu > **Hide.**

The screenshots right show the page when $\boldsymbol{n} = \mathbf{2}$ (for $\boldsymbol{a}+\boldsymbol{b}$).

Note*: The above construction can be adapted to show vector* ***a****, vector* **–*b*** *and the resulting vector* ***a*** *–* ***b****.*

Representing scalar multiplication

Multiplication by a real number (scalar) changes the length of a vector.

- The vector $k\boldsymbol{u}$, where $k \in \mathbb{R}^+$, has the same direction as $\boldsymbol{u}$, but its length is multiplied by a factor of k
- If $\boldsymbol{u} = \overrightarrow{AB}$, then $-\boldsymbol{u} = -\overrightarrow{AB} = \overrightarrow{BA}$.
- Two non-zero vectors, $\boldsymbol{u}$ and $\boldsymbol{v}$, are parallel if $\boldsymbol{u} = k\boldsymbol{v}$ where $k \in \mathbb{R} \setminus \{0\}$.

The following describes how to construct a simple geometric demonstration of scalar multiplication.

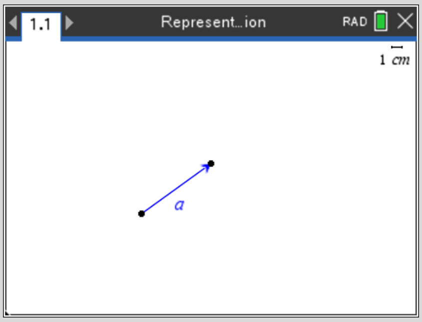
***Note:** This construction is best attempted using the TI-Nspire CX-II T Teacher Software rather than on the handheld device.*

On a **Geometry** page, add a vector $\boldsymbol{a}$ as follows:

- Press [menu] > **Points & Lines** > **Vector**.
- Decide on the vector's starting point and press [enter].
- Decide on the vector's end point and press [enter].
- Complete as shown and press [esc].

To give a vector a label:

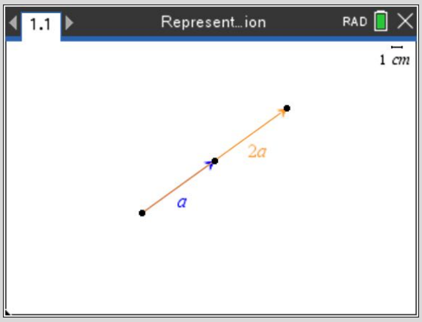
- Hover over the vector.
- Press [ctrl] [menu] > **Label**.
- Label as shown.



***Note:** To change the line colour of a vector, hover over the vector, press [ctrl] [menu] > **Colour** > **Line Colour** and change the colour of the vector as desired.*
***Note:** To change the attributes of a vector, hover over the vector, press [ctrl] [menu] > **Attributes** and change the appearance of the vector as desired.*

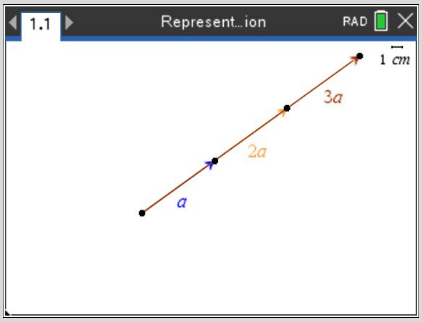
The **Transformation** menu provides a dilation tool.

The **Dilation** command provides the image of an object with a point that is the centre of dilation and a number specifying the dilation factor.



Add a vector $2\boldsymbol{a}$ with label and line colour as follows:

- Press [menu] > **Transformation** > **Dilation**.
- Click (press [click]) on the vector's starting point.
- Click (press [click]) on the vector.
- Press [2] to set the dilation factor and press [enter] [esc].
- Set the vector with label and line colour as shown.
- Hover over the 2 and press [ctrl] [menu] > **Hide**.



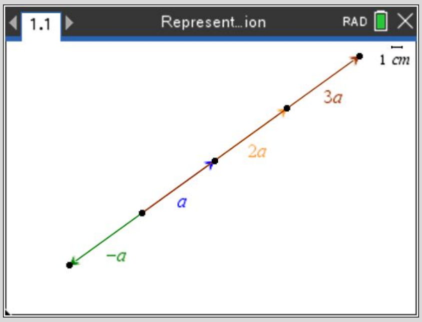
Repeat the above to display the vector $3\boldsymbol{a}$ with label and line colour as shown.

***Note:** For vector $3\boldsymbol{a}$, press [3] to set the dilation factor.*

... continued

Repeat the above to display the vector $-\boldsymbol{a}$ with label and line colour as shown.

***Note:** For vector $-\boldsymbol{a}$, press* (−) 1 *to set the dilation factor.*

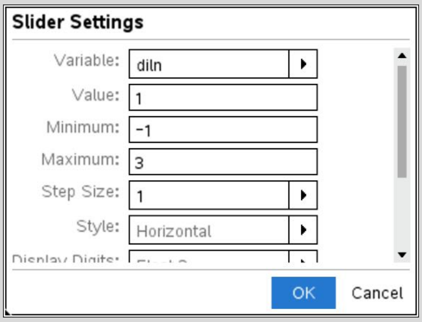


To show scalar multiplication of a vector dynamically, a slider with conditional attributes can be used.

This slider, set up for values of a variable ***diln***, controls when vector $\boldsymbol{a}$, vector $2\boldsymbol{a}$, vector $3\boldsymbol{a}$ and vector $-\boldsymbol{a}$ are displayed on the page.

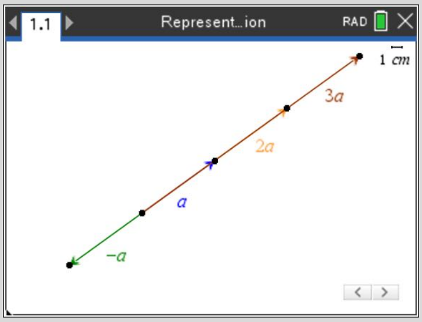
Insert a **Slider** to control the value of ***diln*** as follows:

- Press menu > **Actions** > **Insert Slider**.
- Set the **Slider Settings** as shown.
- Check the **Minimised** box.
- Uncheck the **Show Variable** and **Show Scale** boxes.



To move the **Slider**:

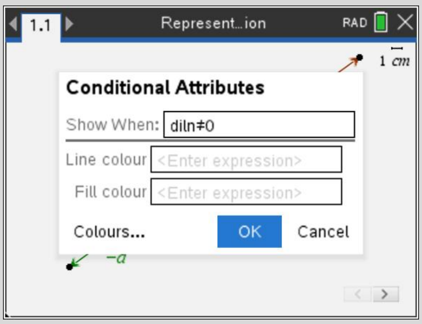
- Press ctrl menu and move it to the bottom right-hand corner as shown.



To set vector $\boldsymbol{a}$ to display when $\boldsymbol{diln} \neq \mathbf{0}$, complete as follows:

- Hover over the vector $\boldsymbol{a}$ and press tab to select the vector.
- Press ctrl menu > **Conditions**.
- Complete the **Conditional Attributes** settings as shown.

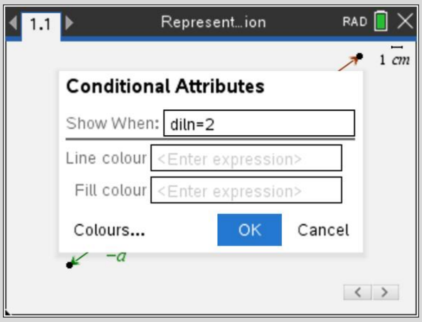
***Note:** To access* $\neq$ *, press* ctrl = (|≠≥▸) *and select as required.*
Note:** Alternatively, to set a condition press* menu *>* ***Actions > Set Conditions*.*



... continued

To set vector $2\boldsymbol{a}$ to display when $\boldsymbol{diln}=\mathbf{2}$, complete as follows:

- Hover over the vector $2\boldsymbol{a}$ and press tab to select the vector.
- Press ctrl menu > **Conditions**.
- Complete the **Conditional Attributes** to display vector $2\boldsymbol{a}$ when $\boldsymbol{diln}=\mathbf{2}$.



Repeat the above instructions to display vector $3\boldsymbol{a}$ with the following conditional attribute:

- Display vector $3\boldsymbol{a}$ when $\boldsymbol{diln}=\mathbf{3}$.

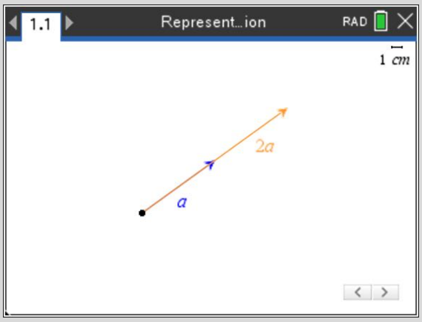
Repeat the above instructions for vector $-\boldsymbol{a}$ with the following conditional attribute:

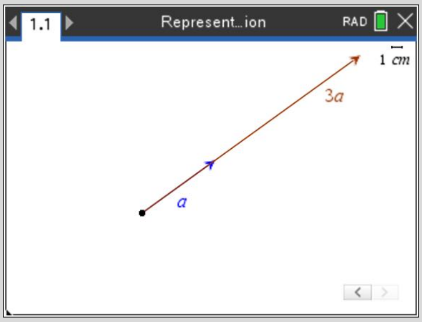
Display vector $-\boldsymbol{a}$ when $\boldsymbol{diln}=\mathbf{-1}$.

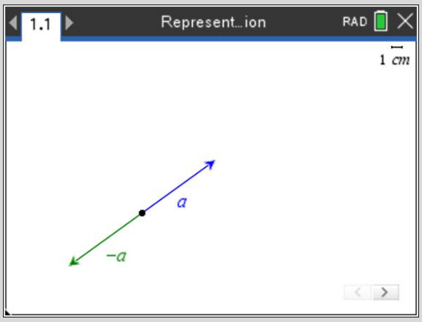
To hide all points except the starting point for $\boldsymbol{a}$:

- Hover over each point and press ctrl menu > **Hide**.

The following screenshots show the page when $\boldsymbol{diln}=\mathbf{2}$, $\boldsymbol{diln}=\mathbf{3}$ and $\boldsymbol{diln}=\mathbf{-1}$ respectively.







1.3.2. Vectors in two dimensions

Calculating the magnitude and direction of a vector

A position vector in two dimensions can be represented using ordered pair notation (x, y) and column vector notation $\begin{pmatrix} x \\ y \end{pmatrix}$.

The magnitude and direction of a vector are defined respectively as:

- $|\boldsymbol{a}| = \left| \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \right| = \sqrt{a_1^{\,2} + a_2^{\,2}}$
- $\tan(\theta) = \dfrac{y}{x},\ x \neq 0$

Raghu cycled 28 km north from A to B, then 19 km east from B to C and finally 12 km south from C to D.

(a) Find $\overrightarrow{AD}$.

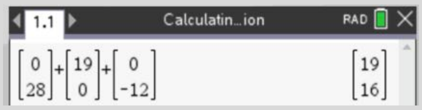
(b) Find $\left|\overrightarrow{AD}\right|$, giving your answer correct to the nearest tenth of a km.

(c) Find the direction of $\overrightarrow{AD}$, giving your answer as a true bearing correct to the nearest degree.

On a **Calculator** page:

- Press [5], select the **2-by-1 Matrix** template and enter as shown. Alternatively, this template can be found via the key.

(a) $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$.

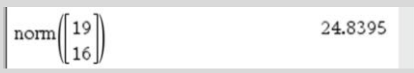


$$\overrightarrow{AD} = \begin{pmatrix} 0 \\ 28 \end{pmatrix} + \begin{pmatrix} 19 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -12 \end{pmatrix} = \begin{pmatrix} 19 \\ 16 \end{pmatrix}$$

***Note:** Here, column vectors are used. This is because there is enough space on the screen to display all the calculations. In subsequent examples where this is not possible, row vectors will be used.*

To find $\left|\overrightarrow{AD}\right|$:

- Press [menu] > **Matrix & Vector** > **Norms** > **Norm**.
- Press ▲ to select the column vector and press [enter].
- Press [ctrl] [enter] to obtain a decimal magnitude.



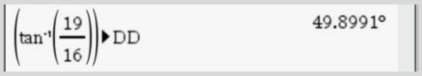
(b) $\left|\overrightarrow{AD}\right| = \left| \begin{pmatrix} 19 \\ 16 \end{pmatrix} \right| = \sqrt{19^2 + 16^2} = 24.8394...$

$\left|\overrightarrow{AD}\right| = 24.8\ \text{(km)}$

... continued

To find the direction of $\overrightarrow{AD}$:

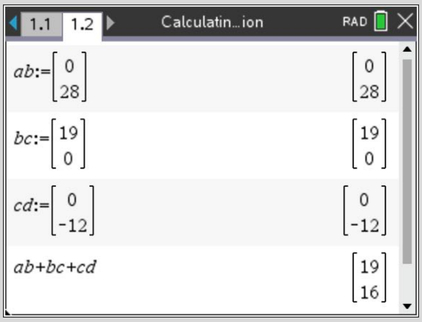
- Press [trig] and enter as shown.
- Press [📖][D], scroll down and select ▶DD.



(c) $\tan^{-1}\left(\frac{19}{16}\right) = 49.8990...°$

The bearing is $050°\text{T}$.

Note: *The* ***Assign*** *command can also be used to assign vectors as shown at right.*



Using vectors in Cartesian form

A unit vector, $\hat{\boldsymbol{n}}$, in the plane is given by $\hat{\boldsymbol{n}} = \frac{\boldsymbol{n}}{|\boldsymbol{n}|}$.

Vectors in Cartesian form are expressed using the unit perpendicular vectors $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$.

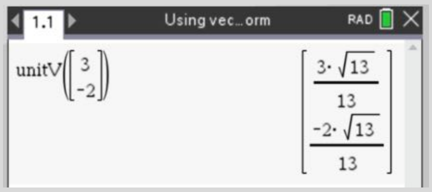
A unit vector in the direction of $3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$ is equal to

(A) $\frac{1}{13}\left(3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}\right)$ **(B)** $\frac{1}{5}\left(3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}\right)$

(C) $\frac{1}{\sqrt{13}}\left(3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}\right)$ **(D)** $-\frac{1}{\sqrt{13}}\left(3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}\right)$

On a **Calculator** page:

- Press [menu] > **Matrix & Vector** > **Vector** > **Unit Vector**.
- Press [📖][5], select the **2-by-1 Matrix** template and enter as shown.



A unit vector in the direction of $3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$ is $\frac{1}{\sqrt{13}}\left(3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}\right)$.

Answer: Option **C**.

Option **D**, $-\frac{1}{\sqrt{13}}\left(3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}\right)$ is a unit vector in the opposite direction.

Using vectors in Cartesian form and polar form

In polar form, a vector in the plane is expressed using the notation (r,θ).

- $\boldsymbol{u} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$ where $x = r\cos(\theta)$ and $y = r\sin(\theta)$
- $r = \sqrt{x^2 + y^2}$ and $\tan(\theta) = \dfrac{y}{x},\ x \neq 0$

(a) Convert $\boldsymbol{u} = (5, 30°)$ to Cartesian form.

(b) Convert $\boldsymbol{v} = \hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$ to polar form, giving the angle correct to the nearest tenth of a degree.

***Note:** In **Document Settings > Real or Complex** (accessed by pressing [on]), there is a choice to set the TI-Nspire CX II-T to either **Real** or **Rectangular** or **Polar** mode. In this example, TI-Nspire CX II-T was set to **Rectangular** mode and **Radian** mode.*

On a **Calculator** page:

- Press [book] [5] to select the **2-by-1 Matrix** template.
- Press [book] [Z], scroll down and select the angle symbol.
- Press [π▸] to access the degree symbol.
- Enter as shown.

(a) $\boldsymbol{u} = (5, 30°)$

$\boldsymbol{u} = 5\cos(30°)\hat{\boldsymbol{i}} + 5\sin(30°)\hat{\boldsymbol{j}}$

$= \dfrac{5\sqrt{3}}{2}\hat{\boldsymbol{i}} + \dfrac{5}{2}\hat{\boldsymbol{j}}$

On a **Calculator** page:

- Press [book] [5], select the **2-by-1 Matrix** template and enter as shown.
- Press [menu] **> Number > Complex Number Tools > Convert to Polar**.
- Copy and paste the exact angle (in radians) to a new entry line.
- Press [book] [D], scroll down and select ▶DD.
- Press [ctrl] [enter] to obtain a decimal angle in degrees.

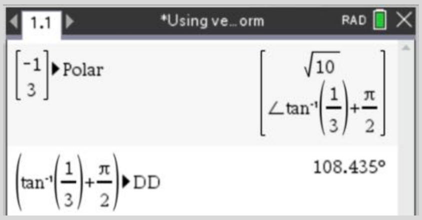
(b) $r = \sqrt{1^2 + (-3)^2} = \sqrt{10}$

$\theta = \tan^{-1}(-3)$

$= 180° - \tan^{-1}(3)$

$= 180° - 71.5650...°$

$= 108.434...°$



In polar form, $\boldsymbol{v} = \left(\sqrt{10}, 108.4°\right)$, where θ is correct to the nearest tenth of a degree.

1.4. Topic 4: Algebra of vectors in two dimensions

1.4.1. Algebra of vectors in two dimensions

Using position vectors in Cartesian form

In Section 1.4, the vector notation $\overrightarrow{AB}$, $\boldsymbol{d}$ and $\hat{\boldsymbol{n}}$ is used.

The position vectors of points A and B are given by $\overrightarrow{OA} = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ and $\overrightarrow{OB} = 5\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$.

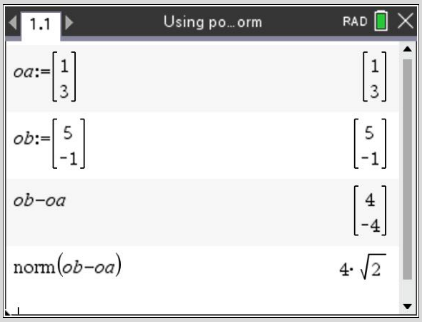
Find the exact distance between points A and B.

On a **Calculator** page, assign $\overrightarrow{OA}$ and $\overrightarrow{OB}$ as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [catalog] [5], select the **2-by-1 Matrix** template and enter as shown.

To find the exact distance between points A and B:

- Press [menu] **> Matrix & Vector > Norms > Norm**.
- Enter as shown.



$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \left(5\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}\right) - \left(\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}\right)$

$= 4\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$

$\left|\overrightarrow{AB}\right| = \sqrt{4^2 + (-4)^2}$

$= 4\sqrt{2}$

The exact distance between A and B is $4\sqrt{2}$.

***Note:** Press [var] to access assigned/stored variables.*

Using the scalar (dot) product to find the angle between two vectors

The scalar (dot) product is defined as:

- $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos(\theta)$
- $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1 b_1 + a_2 b_2$

Points A, B and C are defined by the position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively, where $\boldsymbol{a} = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$, $\boldsymbol{b} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ and $\boldsymbol{c} = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$.

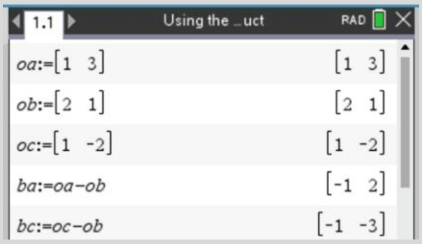
Find the angle between $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

On a **Calculator** page, assign $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ as row vectors as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.

... continued

- Press [templates] [5], select the **1-by-2 Matrix** template and enter as shown.



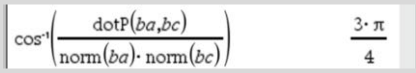
Assign $\overrightarrow{BA}$ and $\overrightarrow{BC}$ as row vectors as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [templates] [5], select the **1-by-2 Matrix** template and enter as shown.

Note: *Press* [var] *to access assigned/stored variables.*

To determine the angle, θ, between $\overrightarrow{BA}$ and $\overrightarrow{BC}$:

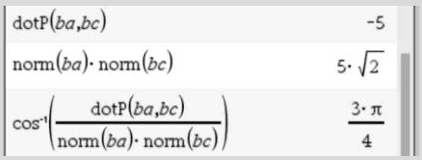
- Press [trig] and select **cos**$^{-1}$.
- Press [ctrl] [÷] to access the **Fraction** template.
- Press [menu] > **Matrix & Vector** > **Vector** > **Dot Product**.
- Enter the numerator as shown.
- Press [menu] > **Matrix & Vector** > **Norms** > **Norm**.
- Enter the denominator as shown.



$$\overrightarrow{BA}\cdot\overrightarrow{BC}=\left|\overrightarrow{BA}\right|\left|\overrightarrow{BC}\right|\cos(\theta)$$

$$\begin{aligned}\theta&=\cos^{-1}\left(\frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{\left|\overrightarrow{BA}\right|\left|\overrightarrow{BC}\right|}\right)\\&=\cos^{-1}\left(\frac{\left(-\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}\right)\cdot\left(-\hat{\boldsymbol{i}}-3\hat{\boldsymbol{j}}\right)}{\left|-\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}\right|\left|-\hat{\boldsymbol{i}}-3\hat{\boldsymbol{j}}\right|}\right)\\&=\cos^{-1}\left(\frac{1-6}{\sqrt{5}\times\sqrt{10}}\right)\\&=\cos^{-1}\left(-\frac{5}{5\sqrt{2}}\right)\\&=\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)\\&=\frac{3\pi}{4}\end{aligned}$$

Note: *Instead of performing all the steps at once on TI-Nspire CX II-T, it is a good teaching idea to show the required steps one at a time as shown at right.*



Using the scalar (dot) product to determine when two vectors are perpendicular

- $\boldsymbol{a}\cdot\boldsymbol{b}=0$ if and only if $\boldsymbol{a}=\boldsymbol{0}$ or $\boldsymbol{b}=\boldsymbol{0}$ or $\boldsymbol{a}\perp\boldsymbol{b}$

Find the value(s) of p for which the vectors $\boldsymbol{u}=p^2\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}$ and $\boldsymbol{v}=3\hat{\boldsymbol{i}}-(2+2p)\hat{\boldsymbol{j}}$ are perpendicular.

Find $\boldsymbol{u}\cdot\boldsymbol{v}$:

$$\begin{aligned}\boldsymbol{u}\cdot\boldsymbol{v}&=\left(p^2\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}\right)\cdot\left(3\hat{\boldsymbol{i}}-(2+2p)\hat{\boldsymbol{j}}\right)\\&=3p^2-2(2+2p)\\&=3p^2-4p-4\end{aligned}$$

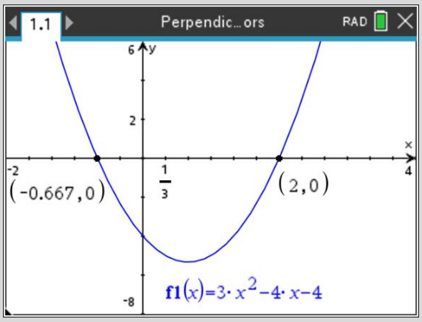
Find the values of p for which $\boldsymbol{u}\cdot\boldsymbol{v}=0$.

On a **Graphs** page:

- Enter $\boldsymbol{f1(x)=3x^2-8x-8}$.
- Press [menu] > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin = –2 Xmax = 4 XScale = 1/3
  YMin = –8 YMax = 6 YScale = 2

To determine the x-intercepts:

- Press [menu] **> Geometry > Points & Lines > Intersection Point(s)**.
- Click (press [click]) on the graph and click (press [click]) on the x-axis.
- On each point of intersection, press [ctrl] [menu] > **Coordinates and Equations**.
- In **Float 3** graph settings, the coordinates $(-0.667,0)$ and $(2,0)$ are now pasted on the screen.



So $p=-\dfrac{2}{3}$ or 2.

This can be verified algebraically as follows:

$$\begin{aligned}3p^2-4p-4&=0\\(3p+2)(p-2)&=0\\p&=-\frac{2}{3},2\end{aligned}$$

Finding the vector projection of one vector onto another

The scalar projection of $\boldsymbol{a}$ on $\boldsymbol{b}$ is defined as:

- $|\boldsymbol{a}|\cos(\theta) = \boldsymbol{a} \cdot \hat{\boldsymbol{b}}$

The vector projection of $\boldsymbol{a}$ on $\boldsymbol{b}$ is defined as:

- $|\boldsymbol{a}|\cos(\theta)\hat{\boldsymbol{b}} = (\boldsymbol{a} \cdot \hat{\boldsymbol{b}})\hat{\boldsymbol{b}} = \left(\dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\right)\boldsymbol{b}$

Find the vector projection of $\boldsymbol{a} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ onto $\boldsymbol{b} = -\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$.

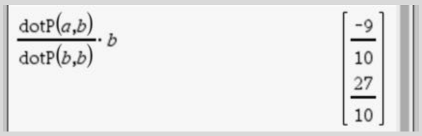
On a **Calculator** page, assign $\boldsymbol{a}$ and $\boldsymbol{b}$ as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [catalog] [5], select the **2-by-1 Matrix** template and enter as shown.

To determine the vector projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$:

- Press [(].
- Press [ctrl] [÷] to access the **Fraction** template.
- Press [menu] > **Matrix & Vector** > **Vector** > **Dot Product**.
- Enter the numerator as shown.
- Press [menu] > **Matrix & Vector** > **Vector** > **Dot Product**.
- Enter the denominator as shown.
- Press ▶ [×] and enter as shown.

The vector projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ is $\begin{pmatrix} -\dfrac{9}{10} \\ \dfrac{27}{10} \end{pmatrix}$.



This can be verified as follows:

Let $\boldsymbol{u}$ be the vector projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$.

$$\begin{aligned}
\boldsymbol{u} &= \left(\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\right)\boldsymbol{b} \\
&= \left(\frac{-3+12}{1+9}\right)(-\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}) \\
&= \frac{9}{10}(-\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}) \\
&= -\frac{9}{10}\hat{\boldsymbol{i}} + \frac{27}{10}\hat{\boldsymbol{j}}
\end{aligned}$$

Modelling and solving problems with vectors

Let $\hat{i}$ and $\hat{j}$ be unit vectors in the east and north directions respectively.

Po leaves her base camp at point O and walks on flat terrain for 4 km in a NE direction to point A.

She then walks a further 6 km on a true bearing of $300°$ to point B.

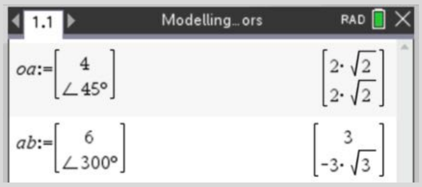
If Po then walks directly back to O, find an expression in exact form in terms of $\hat{i}$ and $\hat{j}$ for the vector that describes her final path.

Note: *In* ***Document Settings > Real or Complex*** *(accessed by pressing [on]), there is a choice to set the TI-Nspire CX II-T to either* ***Real*** *or* ***Rectangular*** *or* ***Polar*** *mode.*
In this example, TI-Nspire CX II-T was set to ***Rectangular*** *mode and* ***Radian*** *mode.*

$\overrightarrow{OA} = (4, 45°)$ and $\overrightarrow{AB} = (6, 300°)$

On a **Calculator** page, assign $\overrightarrow{OA}$ and $\overrightarrow{AB}$ as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [catalog] [5], select the **2-by-1 Matrix** template.
- Press [catalog] [Z], scroll down and select the angle symbol.
- Press [π▸] to access the degree symbol.
- Enter as shown.



$$\overrightarrow{OA} = \begin{pmatrix} 2\sqrt{2} \\ 2\sqrt{2} \end{pmatrix} \text{ and } \overrightarrow{AB} = \begin{pmatrix} 3 \\ -3\sqrt{3} \end{pmatrix}$$

Note: *Press [var] to access assigned/stored variables.*

Resolving into $\hat{i}$ and $\hat{j}$ components:

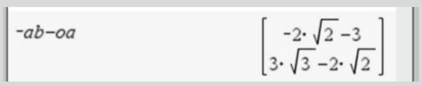
$$\begin{aligned} \overrightarrow{OA} &= 4\cos(45°)\hat{i} + 4\sin(45°)\hat{j} \\ &= 4\left(\frac{\sqrt{2}}{2}\hat{i} + \frac{\sqrt{2}}{2}\hat{j}\right) \\ &= 2\sqrt{2}\left(\hat{i} + \hat{j}\right) \end{aligned}$$

$$\begin{aligned} \overrightarrow{AB} &= 6\cos(300°)\hat{i} + 6\sin(300°)\hat{j} \\ &= 6\left(\frac{1}{2}\hat{i} - \frac{\sqrt{3}}{2}\hat{j}\right) \\ &= 3\left(\hat{i} - \sqrt{3}\hat{j}\right) \end{aligned}$$

Po's final path is described by the vector $\overrightarrow{BO}$.

$\overrightarrow{BO} = \overrightarrow{BA} + \overrightarrow{AO} = -\overrightarrow{AB} - \overrightarrow{OA}$

Enter $-\overrightarrow{AB} - \overrightarrow{OA}$ as shown:



$$\begin{aligned} \overrightarrow{BO} &= -3\left(\hat{i} - \sqrt{3}\hat{j}\right) - 2\sqrt{2}\left(\hat{i} + \hat{j}\right) \\ &= \left(-3 - 2\sqrt{2}\right)\hat{i} + \left(3\sqrt{3} - 2\sqrt{2}\right)\hat{j} \end{aligned}$$

1.5. Topic 5: Matrices

1.5.1. Matrix arithmetic and algebra

Understanding the matrix definition and notation

A matrix is a rectangular array of elements.

The order of a matrix is $r \times c$, where r is the number of rows and c is the number of columns.

Defining, adding and subtracting matrices

Two matrices can be added or subtracted if they have the same order.

Addition and subtraction are performed by adding or subtracting the corresponding elements in each matrix.

In general for 2×2 matrices, if $\boldsymbol{A}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\boldsymbol{B}=\begin{bmatrix} e & f \\ g & h \end{bmatrix}$, then $\boldsymbol{A}\pm\boldsymbol{B}=\begin{bmatrix} a\pm e & b\pm f \\ c\pm g & d\pm h \end{bmatrix}$.

A fruit and vegetable cooperative has three shops, S_1, S_2 and S_3.

On a particular Monday:

- S_1 sold 45 avocados, 18 lettuces and 11 watermelons.
- S_2 sold 35 avocados, 18 lettuces and 9 watermelons.
- S_3 sold 47 avocados, 29 lettuces and 10 watermelons.

On a particular Tuesday:

- S_1 sold 28 avocados, 13 lettuces and 16 watermelons.
- S_2 sold 31 avocados, 17 lettuces and 13 watermelons.
- S_3 sold 29 avocados, 28 lettuces and 19 watermelons.

Let the sales for the Monday be denoted by matrix $\boldsymbol{M}$ and the sales for the Tuesday be denoted by matrix $\boldsymbol{T}$.

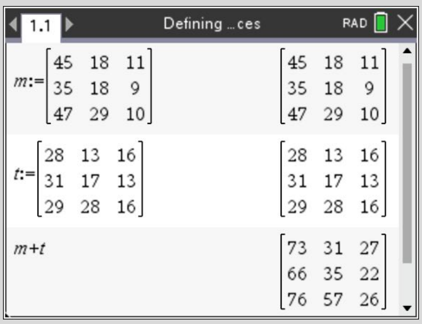
(a) Find the sales for the three shops over the two days.

(b) Find the number of lettuces sold by S_2 over the two days.

$$\boldsymbol{M}=\begin{bmatrix} 45 & 18 & 11 \\ 35 & 18 & 9 \\ 47 & 29 & 10 \end{bmatrix} \text{ and } \boldsymbol{T}=\begin{bmatrix} 28 & 13 & 16 \\ 31 & 17 & 13 \\ 29 & 28 & 19 \end{bmatrix}$$

On a **Calculator** page, assign $\boldsymbol{M}$ and $\boldsymbol{T}$ as follows:

- Press ctrl [templates key] to access the **Assign** := command.
- Press menu > **Matrix & Vector** > **Create** > **Matrix**.
- Set the number of rows to be 3 and the number of columns to be 3.
- Enter as shown and calculate $\boldsymbol{M}+\boldsymbol{T}$ as shown.



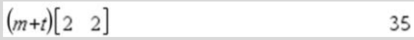
... continued

(a) $\boldsymbol{M}+\boldsymbol{T}=\begin{bmatrix}73 & 31 & 27\\ 66 & 35 & 22\\ 76 & 57 & 29\end{bmatrix}$

Note: *Alternatively, to create a 3 x 3 matrix, press [book] [5] (or [templates]), then select the **m-by-n Matrix** template and complete as above.*

The number of lettuces sold over the two days is given by the element $(2,2)$.

To access element $(2,2)$, enter **(*M*+*T*)[2,2]**.



(b) Element $(2,2)$ indicates that S_2 sold 35 lettuces on the Monday and the Tuesday.

Note: *The first element in $[2 \quad 2]$ indicates the row number and the second element indicates the column number.*
Note: *The **Define** command ([menu] > **Actions** > **Define**) and the **Store** command (press [ctrl] [var] to access [sto→]) can also be used.*

Verifying matrix addition properties

The following matrix properties are important:

- $\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{B}+\boldsymbol{A}$ (commutative law for addition)
- $\boldsymbol{A}+\boldsymbol{O}=\boldsymbol{A}$ (additive identity)
- $\boldsymbol{A}+(-\boldsymbol{A})=\boldsymbol{O}$ (additive inverse)

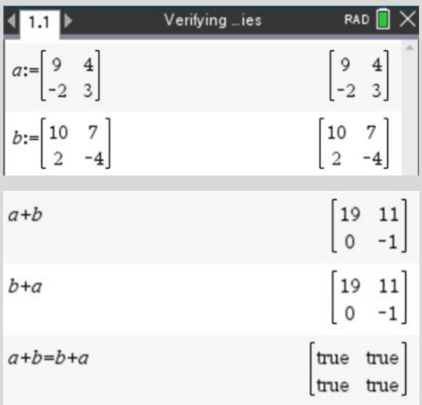
Let $\boldsymbol{A}=\begin{bmatrix}9 & 4\\ -2 & 3\end{bmatrix}$ and $\boldsymbol{B}=\begin{bmatrix}10 & 7\\ 2 & -4\end{bmatrix}$.

Verify the following results:

(a) $\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{B}+\boldsymbol{A}$ **(b)** $\boldsymbol{A}+\boldsymbol{O}=\boldsymbol{A}$ **(c)** $\boldsymbol{A}+(-\boldsymbol{A})=\boldsymbol{O}$

On a **Calculator** page, assign $\boldsymbol{A}$ and $\boldsymbol{B}$ as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [book] [5], select the **2-by-2 Matrix** template and enter as shown.



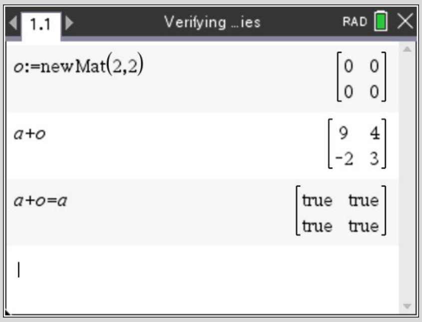
(a) $\boldsymbol{A}+\boldsymbol{B}=\begin{bmatrix}19 & 11\\ 0 & -1\end{bmatrix}$ and $\boldsymbol{B}+\boldsymbol{A}=\begin{bmatrix}19 & 11\\ 0 & -1\end{bmatrix}$.

Note: *Entering $\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{B}+\boldsymbol{A}$ gives the output* $\begin{bmatrix}\text{true} & \text{true}\\ \text{true} & \text{true}\end{bmatrix}$.

... continued

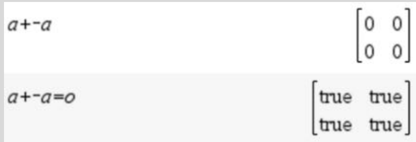
To create $\boldsymbol{O}$:

- Press ctrl [templates key] to access the **Assign** [:=] command.
- Press menu > **Matrix & Vector** > **Create** > **Zero Matrix**.
- Enter as shown.



(b) $\boldsymbol{A}+\boldsymbol{O}=\begin{bmatrix}9 & 4\\-2 & 3\end{bmatrix}=\boldsymbol{A}$.

Note: *Entering* $\boldsymbol{A}+\boldsymbol{O}=\boldsymbol{A}$ *gives the output* $\begin{bmatrix}\text{true} & \text{true}\\\text{true} & \text{true}\end{bmatrix}$.



(c) $\boldsymbol{A}+(-\boldsymbol{A})=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}=\boldsymbol{O}$.

Note: *Entering* $\boldsymbol{A}+(-\boldsymbol{A})=\boldsymbol{O}$ *gives the output* $\begin{bmatrix}\text{true} & \text{true}\\\text{true} & \text{true}\end{bmatrix}$.

Note: *When attempting to add two matrices of different order, a 'dimension mismatch' error message is displayed. An example of this is shown at right.*

To enter $\boldsymbol{C}$ *, press* [catalog] [5]*, select the* ***2-by-1 Matrix*** *template and enter as shown.*

Defining scalar and matrix multiplication

Matrices can be multiplied by scalar (real number) quantities.

In general for 2×2 matrices, if $\boldsymbol{A}=\begin{bmatrix}a & b\\c & d\end{bmatrix}$ then $k\boldsymbol{A}=\begin{bmatrix}ka & kb\\kc & kd\end{bmatrix}$ where k is a scalar.

Two matrices can be multiplied together if the number of columns in the first matrix equals the number of rows in the second matrix.

In general, if an $m\times n$ matrix is multiplied by an $n\times p$ matrix, the resulting matrix will be of order $m\times p$, that is, $(m\times \boldsymbol{n})\times(\boldsymbol{n}\times p)=m\times p$.

Note that, in general, for two matrices, $\boldsymbol{A}$ and $\boldsymbol{B}$, $\boldsymbol{AB}\neq\boldsymbol{BA}$.

In other words, matrix multiplication in general is not commutative.

The following matrix properties are important:

- $\boldsymbol{A}(\boldsymbol{B}+\boldsymbol{C})=\boldsymbol{AB}+\boldsymbol{AC}$ (left distributive law)
- $(\boldsymbol{B}+\boldsymbol{C})\boldsymbol{A}=\boldsymbol{BA}+\boldsymbol{CA}$ (right distributive law)

Let $\boldsymbol{A}=\begin{bmatrix}9 & 4\\-2 & 3\end{bmatrix}$, $\boldsymbol{B}=\begin{bmatrix}10 & 7\\2 & -4\end{bmatrix}$ and $\boldsymbol{C}=\begin{bmatrix}4 & 0\\-3 & 2\end{bmatrix}$.

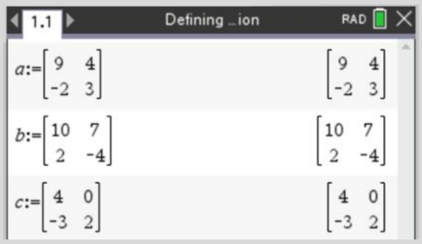
Verify the following results:

(a) $\boldsymbol{AB}\neq\boldsymbol{BA}$ **(b)** $\boldsymbol{A}(\boldsymbol{B}+\boldsymbol{C})=\boldsymbol{AB}+\boldsymbol{AC}$ **(c)** $(\boldsymbol{B}+\boldsymbol{C})\boldsymbol{A}=\boldsymbol{BA}+\boldsymbol{CA}$

... continued

On a **Calculator** page, assign A, B and C as follows:

- Press [ctrl] [⊞] to access the **Assign** [:=] command.
- Press [📖] [5], select the **2-by-2 Matrix** template and enter as shown.

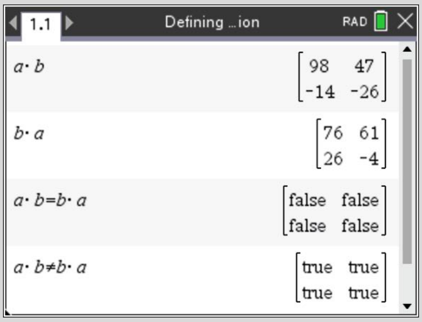


(a) $AB = \begin{bmatrix} 98 & 47 \\ -14 & -26 \end{bmatrix}$ and $BA = \begin{bmatrix} 76 & 61 \\ 26 & -4 \end{bmatrix}$.

__Note:__ Entering $AB = BA$ *gives the output* $\begin{bmatrix} \text{false} & \text{false} \\ \text{false} & \text{false} \end{bmatrix}$

and entering $AB \neq BA$ *gives the output* $\begin{bmatrix} \text{true} & \text{true} \\ \text{true} & \text{true} \end{bmatrix}$.

__Note:__ When performing matrix multiplication, always use the multiplication key, [×].

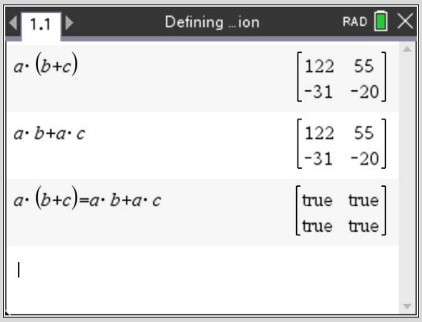


Enter $A(B+C)$ and $AB+AC$ as shown.

(b) $A(B+C) = \begin{bmatrix} 122 & 55 \\ -31 & -20 \end{bmatrix}$ and $AB + AC = \begin{bmatrix} 122 & 55 \\ -31 & -20 \end{bmatrix}$.

__Note:__ Entering $A(B+C) = AB + AC$ *gives the output*

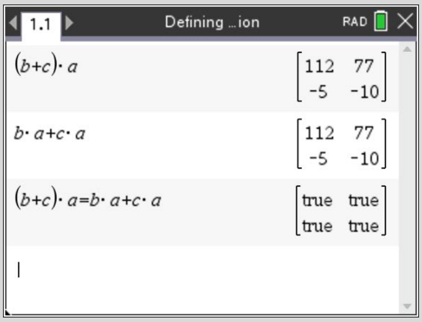
$\begin{bmatrix} \text{true} & \text{true} \\ \text{true} & \text{true} \end{bmatrix}$.



(c) $(B+C)A = \begin{bmatrix} 112 & 77 \\ -5 & -10 \end{bmatrix}$ and $BA + CA = \begin{bmatrix} 112 & 77 \\ -5 & -10 \end{bmatrix}$.

__Note:__ Entering $(B+C)A = BA + CA$ *gives the output*

$\begin{bmatrix} \text{true} & \text{true} \\ \text{true} & \text{true} \end{bmatrix}$.



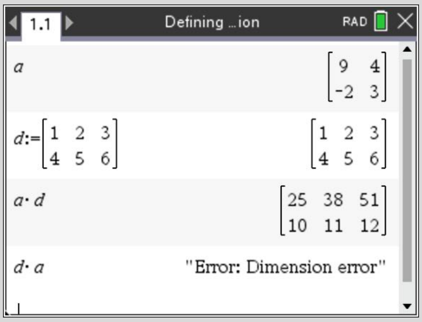
On a **Calculator** page, assign D as follows:

- Press [ctrl] [⊞] to access the **Assign** [:=] command.
- Press [📖] [5], select the **m-by-n Matrix** template, fix the dimensions as 2-by-3 and enter as shown.

AD can be found because A is a 2×2 matrix and D is a 2×3 matrix ($2=2$).

However, DA cannot be found because D is a 2×3 matrix and A is a 2×2 matrix ($3\neq2$).

__Note:__ When attempting to multiply two matrices of different order, a 'dimension error' message is displayed when the number of columns in the first matrix does not equal the number of rows in the second matrix. An example of this is shown at right.



Raising a matrix to a power

Consider $\boldsymbol{M} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$.

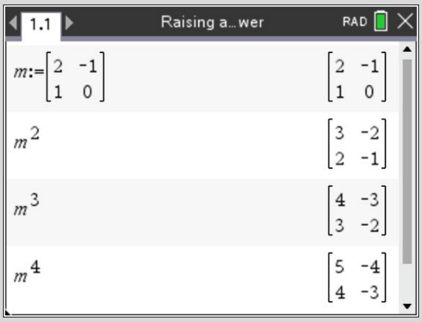
(a) Find $\boldsymbol{M}^2$, $\boldsymbol{M}^3$ and $\boldsymbol{M}^4$.

(b) Hence, infer a general result for $\boldsymbol{M}^n$ where $n \in \mathbb{Z}^+$.

(c) Use your result to determine $\boldsymbol{M}^{2025}$ and check your answer with *TI-Nspire CX II-T*.

On a **Calculator** page, assign $\boldsymbol{M}$ as follows:

- Press ctrl ⊞ to access the **Assign** := command.
- Press 📖 5, select the **2-by-2 Matrix** template and enter as shown.



(a) $\boldsymbol{M}^2 = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}$, $\boldsymbol{M}^3 = \begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}$ and $\boldsymbol{M}^4 = \begin{bmatrix} 5 & -4 \\ 4 & -3 \end{bmatrix}$.

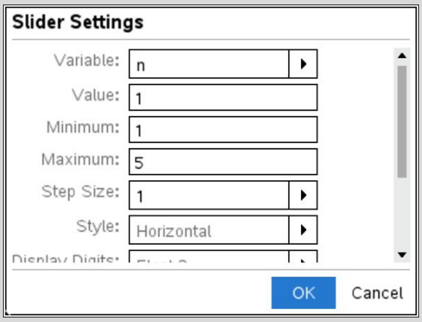
(b) $\boldsymbol{M}^n = \begin{bmatrix} n+1 & -n \\ n & 1-n \end{bmatrix}$.

(c) $\boldsymbol{M}^{2025} = \begin{bmatrix} 2025+1 & -2025 \\ 2025 & 1-2025 \end{bmatrix} = \begin{bmatrix} 2026 & -2025 \\ 2025 & -2024 \end{bmatrix}$.

Alternatively, on a **Notes** page:

Insert a **Slider** to control the value of n as follows:

- Press menu > **Insert** > **Slider**.
- Set the **Slider Settings** as shown.
- Ensure to check the **Minimised** box.



Insert a **Maths Box** as follows:

- Press menu > **Insert** > **Maths Box**.

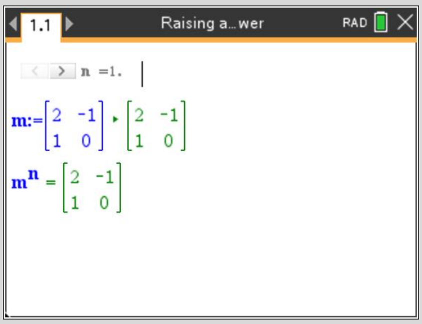
***Note:** Alternatively, to insert a **Maths Box**, press* ctrl M.

Assign $\boldsymbol{M}$ as follows:

- Press ctrl ⊞ to access the **Assign** := command.
- Press 📖 5, select the 2-by-2 matrix template and enter as shown.

Now:

- Insert another **Maths Box** and enter $\boldsymbol{m}^n$.

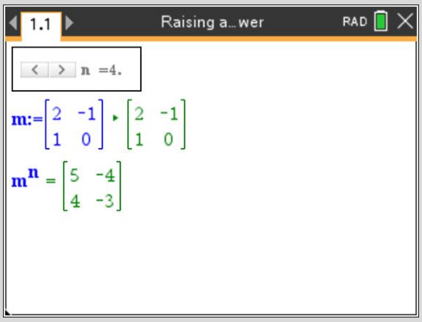


... *continued*

To change the display of a **Maths Box**, for example, to display an equals sign:

- Click on the **Maths Box**.
- Press menu > **Maths Box Options** > **Maths Box Attributes**.
- Press tab to highlight the **Insert Symbol** field.
- Press ▸ and select $=$.

***Note: Maths Box Attributes** can also be accessed within a **Maths Box** by pressing* ctrl menu.



Click on the slider to change the value of n.

To calculate M^n for $n = 2025$, manually change the value of n to 2025 in the slider box.

Introducing the multiplicative identity and multiplicative inverse

An identity matrix, $\boldsymbol{I}$, is a square matrix that has the number 1 for all elements on the main diagonal and 0 for all other elements.

The following matrix properties are important:

- $\boldsymbol{AI} = \boldsymbol{A} = \boldsymbol{IA}$ (multiplicative identity)
- $\boldsymbol{AA}^{-1} = \boldsymbol{I} = \boldsymbol{A}^{-1}\boldsymbol{A}$ (multiplicative inverse)

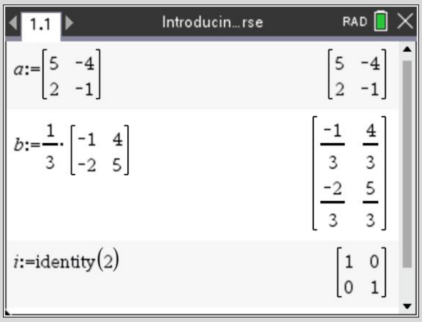
Let $\boldsymbol{A} = \begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix}$ and $\boldsymbol{B} = \frac{1}{3}\begin{bmatrix} -1 & 4 \\ -2 & 5 \end{bmatrix}$.

Verify the following results:

(a) $\boldsymbol{AI} = \boldsymbol{A} = \boldsymbol{IA}$ **(b)** $\boldsymbol{AB} = \boldsymbol{I} = \boldsymbol{BA}$ where $\boldsymbol{B} = \boldsymbol{A}^{-1}$

On a **Calculator** page, assign $\boldsymbol{A}$ and $\boldsymbol{B}$ as follows:

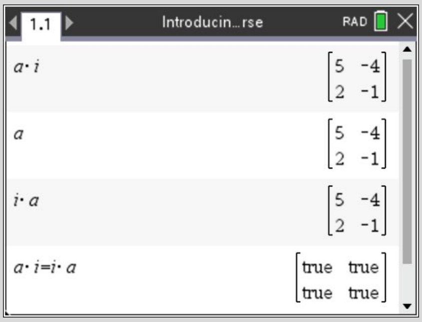
- Press ctrl [templates] to access the **Assign** [:=] command.
- Press [catalog] 5, select the 2-by-2 matrix template and enter as shown.



To create $\boldsymbol{I}$, where $\boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$:

- Press ctrl [templates] to access the **Assign** [:=] command.
- Press menu > **Matrix & Vector** > **Create** > **Identity**.
- Enter as shown.

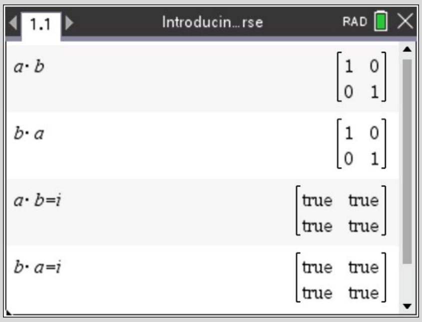
(a) $\boldsymbol{AI} = \begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix}$, $\boldsymbol{IA} = \begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix}$ and $\boldsymbol{AI} = \boldsymbol{A} = \boldsymbol{IA}$.



***Note:** Entering* $\boldsymbol{AI} = \boldsymbol{IA}$ *gives the output* $\begin{bmatrix} \text{true} & \text{true} \\ \text{true} & \text{true} \end{bmatrix}$.

... continued

Enter $\boldsymbol{AB}$ and $\boldsymbol{BA}$ as shown.



(b) $\boldsymbol{AB} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\boldsymbol{BA} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Note: *Entering* $\boldsymbol{AB} = \boldsymbol{I}$ *and* $\boldsymbol{BA} = \boldsymbol{I}$ *both give the output* $\begin{bmatrix} \text{true} & \text{true} \\ \text{true} & \text{true} \end{bmatrix}$.

From part (b), it can be concluded that $\boldsymbol{B} = \boldsymbol{A}^{-1}$.

Calculating the determinant of 2 x 2 matrices

If $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(\boldsymbol{A}) = ad - bc$.

The Fibonacci sequence is defined by $F_n = F_{n-1} + F_{n-2}$ where $F_1 = F_2 = 1$, $F_0 = 0$ and $n \ge 3$.

F_n is the nth term of the sequence.

(a) Determine F_3, F_4, F_5 and F_6.

Consider $\boldsymbol{P} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

(b) Find $\det(\boldsymbol{P})$.

(c) Find

(i) $\boldsymbol{P}^2$ and $\det(\boldsymbol{P}^2)$ **(ii)** $\boldsymbol{P}^3$ and $\det(\boldsymbol{P}^3)$ **(iii)** $\boldsymbol{P}^4$ and $\det(\boldsymbol{P}^4)$

(d) Hence infer a general result for $\boldsymbol{P}^n$ and $\det(\boldsymbol{P}^n)$.

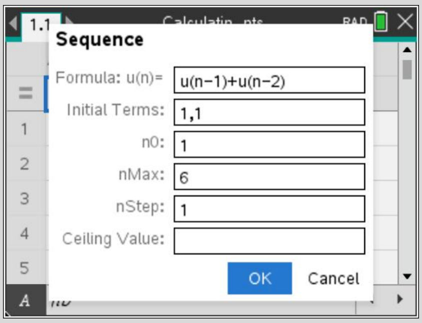
(e) Use the result from part **(d)** to find an expression for $F_{n+1}F_{n-1} - F_n^{\,2}$ in terms of n.

On a **Lists & Spreadsheet** page:

- In the column A heading cell, enter the variable ***fib***.

Generate terms of a Fibonacci sequence as follows:

- In the column A formula cell, press menu > **Data** > **Generate Sequence**.
- Complete the **Sequence** dialog box as shown.

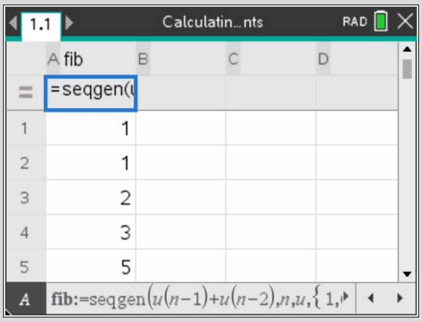


(a) $F_3 = 2$, $F_4 = 3$, $F_5 = 5$ and $F_6 = 8$.

(b) $\det(\boldsymbol{P}) = -1$.

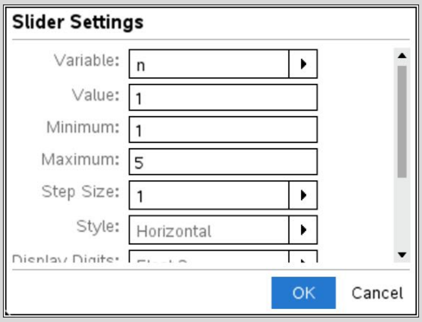
... continued

On a **Notes** page:



Insert a **Slider** to control the value of n as follows:

- Press [menu] > **Insert** > **Slider**.
- Set the **Slider Settings** as shown.
- Ensure to check the **Minimised** box.



Insert a **Maths Box** as follows:

- Press [menu] > **Insert** > **Maths Box**.

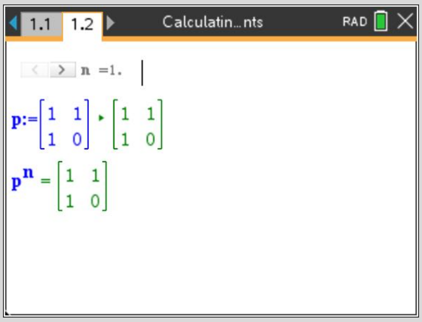
Note: *Alternatively, to insert a* ***Maths Box****, press* [ctrl] [M].

Assign $\boldsymbol{P}$ as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [catalog] [5], select the **2-by-2 Matrix** template and enter as shown.

Now:

- Insert a **Maths Box** and enter the expression $\boldsymbol{p}^n$.

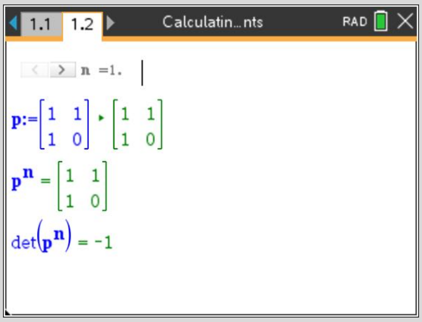


To display an equals sign in a **Maths Box**:

- Click on the **Maths Box**.
- Press [menu] > **Maths Box Options** > **Maths Box Attributes**.
- Press [tab] to highlight the **Insert Symbol** field.
- Press ▶ and select $=$.

Note: Maths Box Attributes *can also be accessed within a* ***Maths Box*** *by pressing* [ctrl] [menu].

To enter the expression $\mathbf{det}(\boldsymbol{p}^n)$ in a **Maths Box**:



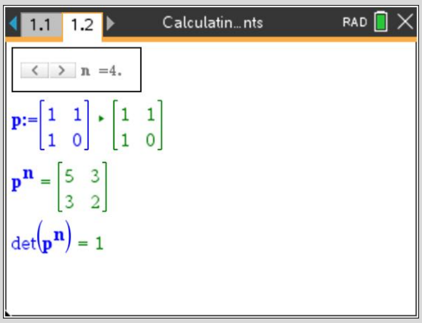
- Press [menu] > **Calculations** > **Matrix & Vector** > **Determinant**.
- Enter as shown and change the display to show an equals sign.

Click on the slider to change the value of n.

(c) (i) $\boldsymbol{P}^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $\det(\boldsymbol{P}^2) = 1$.

(ii) $\boldsymbol{P}^3 = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$ and $\det(\boldsymbol{P}^3) = -1$.

(iii) $\boldsymbol{P}^4 = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$ and $\det(\boldsymbol{P}^4) = 1$.



(d) $\boldsymbol{P}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$ and $\det(\boldsymbol{P}^n) = (-1)^n$.

(e) $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$.

Calculating the determinant and multiplicative inverse of 2 x 2 matrices

If $\boldsymbol{A}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(\boldsymbol{A}) = ad - bc$.

$\boldsymbol{A}^{-1}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}=\dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}=\dfrac{1}{\det(\boldsymbol{A})}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ where $\det(\boldsymbol{A}) \neq 0\ (ad-bc \neq 0)$.

If $\det(\boldsymbol{A}) = 0$, then $\boldsymbol{A}$ is a singular matrix and $\boldsymbol{A}^{-1}$ does not exist.

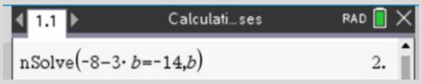
Consider $\boldsymbol{M}=\begin{bmatrix} -2 & b \\ 3 & 4 \end{bmatrix}$ where $\det(\boldsymbol{M}) = -14$.

(a) Find the value of b.

(b) Find $\boldsymbol{M}^{-1}$.

On a **Calculator** page, solve the linear equation $-8-3b=-14$ for b as follows:

- Press menu > **Algebra** > **Numerical Solve**.
- Enter as shown.



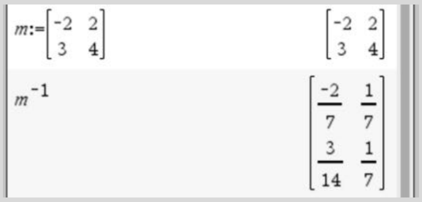
(a) $\det(\boldsymbol{M}) = -8-3b$

Solving $-8-3b=-14$ for b gives $b=2$

On a **Calculator** page, assign $\boldsymbol{M}$ as follows:

- Press ctrl [templates key] to access the **Assign** := command.
- Press [catalog key] 5, select the **2-by-2 Matrix** template and enter as shown.

To calculate $\boldsymbol{M}^{-1}$ enter as shown.



(b) $\boldsymbol{M}^{-1} = -\dfrac{1}{14}\begin{bmatrix} 4 & -2 \\ -3 & -2 \end{bmatrix}$.

Note: *To find* $\boldsymbol{M}^{-1}$ *without using TI-Nspire CX II-T, interchange the elements on the main diagonal, change the signs of the elements on the secondary diagonal and divide by* $\det(\boldsymbol{M})$.

To check your answer, calculate $\boldsymbol{M}\boldsymbol{M}^{-1}$ *or* $\boldsymbol{M}^{-1}\boldsymbol{M}$.

$\boldsymbol{M}$ *and* $\boldsymbol{M}^{-1}$ *should satisfy* $\boldsymbol{M}\boldsymbol{M}^{-1} = \boldsymbol{M}^{-1}\boldsymbol{M} = \boldsymbol{I}$.

TEXAS INSTRUMENTS

Solving matrix equations involving matrices of up to dimension 2 x 2

Inverses can be used to solve matrix equations.

If $\boldsymbol{AX}=\boldsymbol{B}$, where $\boldsymbol{A}$ is a square matrix and has inverse $\boldsymbol{A}^{-1}$ such that $\boldsymbol{A}^{-1}\boldsymbol{A}=\boldsymbol{I}$, then the solution is $\boldsymbol{X}=\boldsymbol{A}^{-1}\boldsymbol{B}$.

$\left(\boldsymbol{A}^{-1}\boldsymbol{A}\right)\boldsymbol{X}=\boldsymbol{A}^{-1}\boldsymbol{B}$ (pre-multiplying both sides by $\boldsymbol{A}^{-1}$)

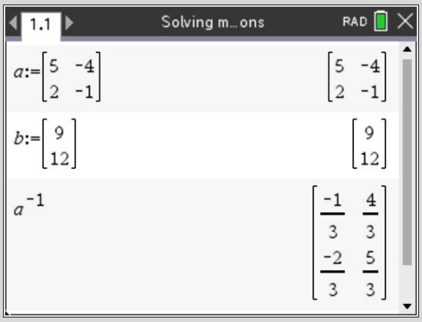
$\boldsymbol{X}=\boldsymbol{A}^{-1}\boldsymbol{B}$ (since $\boldsymbol{A}^{-1}\boldsymbol{A}=\boldsymbol{I}$)

If $\boldsymbol{A}=\begin{bmatrix}5 & -4\\ 2 & -1\end{bmatrix}$, $\boldsymbol{B}=\begin{bmatrix}9\\ 12\end{bmatrix}$ and $\boldsymbol{X}$ is a matrix such that $\boldsymbol{AX}=\boldsymbol{B}$, find:

(a) $\boldsymbol{A}^{-1}$. **(b)** $\boldsymbol{X}$.

On a **Calculator** page, assign $\boldsymbol{A}$ and $\boldsymbol{B}$ as follows:

- Press ctrl [templates] to access the **Assign** := command.
- Press [catalog] [5], select the **2-by-2 Matrix** template for $\boldsymbol{A}$ and enter as shown.
- Press [catalog] [5], select the **2-by-1 Matrix** template for $\boldsymbol{B}$ and enter as shown.



To calculate $\boldsymbol{A}^{-1}$ enter as shown.

(a) $\boldsymbol{A}^{-1}=\dfrac{1}{3}\begin{bmatrix}-1 & 4\\ -2 & 5\end{bmatrix}$.

Enter $\boldsymbol{A}^{-1}\boldsymbol{B}$ as shown.

(b) Since $\boldsymbol{AX}=\boldsymbol{B}$, then $\boldsymbol{X}=\boldsymbol{A}^{-1}\boldsymbol{B}$.

$$\boldsymbol{X}=\frac{1}{3}\begin{bmatrix}-1 & 4\\ -2 & 5\end{bmatrix}\begin{bmatrix}9\\ 12\end{bmatrix}$$

$$=\begin{bmatrix}13\\ 14\end{bmatrix}$$

Solving systems of linear equations involving matrices of up to dimension 2 x 2

Solve the following system of linear equations

$$-2x+3y=-19$$
$$5x-2y=20$$

(a) using the **Reduced Row-Echelon Form** (**rref**) command.

(b) using the **Simultaneous** command.

(c) using the **Row Operations** menu.

In matrix form , the system of equations can be expressed as

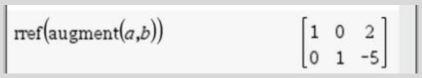
$\boldsymbol{AX}=\boldsymbol{B}$ where $\boldsymbol{A}=\begin{bmatrix}-2 & 3\\ 5 & -2\end{bmatrix}$, $\boldsymbol{X}=\begin{bmatrix}x\\ y\end{bmatrix}$ and $\boldsymbol{B}=\begin{bmatrix}-19\\ 20\end{bmatrix}$.

On a **Calculator** page, assign $\boldsymbol{A}$ and $\boldsymbol{B}$ as follows:

- Press [ctrl] [template] to access the **Assign** [:=] command.
- Press [book] [5], select the **2-by-2 Matrix** template for $\boldsymbol{A}$ and enter as shown.
- Press [book] [5], select the **2-by-1 Matrix** template for $\boldsymbol{B}$ and enter as shown.

Solve $\boldsymbol{AX}=\boldsymbol{B}$ using reduced row-echelon form as follows:

- Press [menu] > **Matrix & Vector** > **Reduced Row-Echelon Form**.
- Press [menu] > **Matrix & Vector** > **Create** > **Augment**.
- Enter as shown.



Note: *The* ***Augment*** *command is used to combine* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *so that the* ***Reduced Row-Echelon Form*** *command can be used directly without the need to create a 2 × 3 matrix from a template.*

The **Reduced Row-Echelon Form** command instructs the *TI-Nspire CX II-T* to solve the system of linear equations $-2x+3y=-19$ and $5x-2y=20$ (in the form of a 2×3 augmented matrix) using the method of elimination.

(a) The new augmented matrix, $\begin{bmatrix}1 & 0 & 2\\ 0 & 1 & -5\end{bmatrix}$, is a translation of the original augmented matrix $\begin{bmatrix}-2 & 3 & -19\\ 5 & -2 & 20\end{bmatrix}$.

The matrix $\begin{bmatrix}1 & 0 & 2\\ 0 & 1 & -5\end{bmatrix}$ can be interpreted as the two equivalent transformed linear equations:

$1x+0y=2$

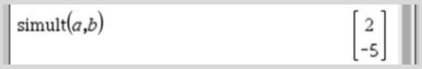
$0x+1y=-5$

Thus we have $x=2$ and $y=-5$.

... continued

To solve this system of linear equations using the **Simultaneous** command:

- Press [menu] > **Matrix & Vector** > **Simultaneous** and enter as shown.

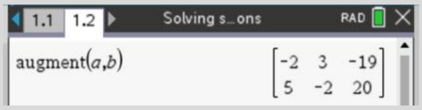


(b) So $x = 2$ and $y = -5$.

(c) The following shows Gaussian elimination using the **Row Operations** menu of commands.

On a **Calculator** page with $\boldsymbol{A}$ and $\boldsymbol{B}$ assigned as before:

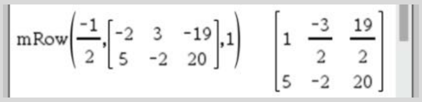
- Press [menu] > **Matrix & Vector** > **Create** > **Augment**.
- Enter as shown.



Step 1: $-\frac{1}{2}R_1 \to R_1$ (multiply row 1 by $-\frac{1}{2}$)

To perform **Step 1**:

- Press [menu] > **Matrix & Vector** > **Row Operations** > **Multiply Row**.
- Enter as shown.



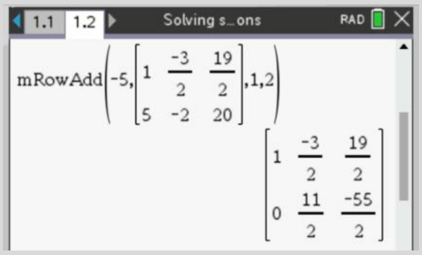
$$\begin{bmatrix} 1 & -\frac{3}{2} & \frac{19}{2} \\ 5 & -2 & 20 \end{bmatrix}$$

Note: *The syntax for the* ***Multiply Row*** *command is* ***mRow(Value, Matrix, Index)****.* ***Value*** *is the multiplier and* ***Index*** *is the row number.*

Step 2: $-5R_1 + R_2 \to R_2$ (multiply row 1 by -5 and add it to row 2)

To perform **Step 2**:

- Press [menu] > **Matrix & Vector** > **Row Operations** > **Multiply Row & Add**.
- Enter as shown.



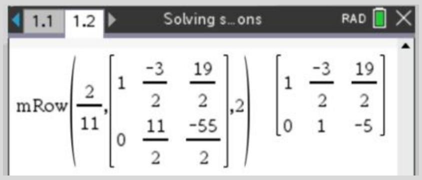
$$\begin{bmatrix} 1 & -\frac{3}{2} & \frac{19}{2} \\ 0 & \frac{11}{2} & -\frac{55}{2} \end{bmatrix}$$

Note: *The syntax for the* ***Multiply Row & Add*** *command is* ***mRowAdd(Value, Matrix, Index1, Index2)****.* ***Value*** *is the multiplier,* ***Index1*** *is the row being multiplied and* ***Index2*** *is the row being added to.*

... continued

Step 3: $\frac{2}{11}R_2 \rightarrow R_2$ (multiply row 2 by $\frac{2}{11}$)

To perform **Step 3**:

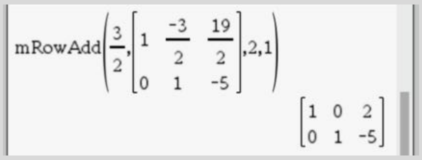


- Press [menu] > **Matrix & Vector** > **Row Operations** > **Multiply Row**.
- Enter as shown.

$$\begin{bmatrix} 1 & -\frac{3}{2} & \frac{19}{2} \\ 0 & 1 & -5 \end{bmatrix}$$

Step 4: $\frac{3}{2}R_2 + R_1 \rightarrow R_1$ (multiply row 2 by $\frac{3}{2}$ & add to row 1)

To perform **Step 4**:



- Press [menu] > **Matrix & Vector** > **Row Operations** > **Multiply Row & Add**.
- Enter as shown.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -5 \end{bmatrix}$$

So $x = 2$ and $y = -5$.

Unit 2: Complex numbers, further proof, trigonometry, functions and transformations

2.1. Topic 1: Complex numbers

2.1.1. Introduction to complex numbers

Exploring roots of a quadratic polynomial over $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$

Consider the following quadratic polynomials:

- $p_1(x) = x^2 - 1$
- $p_2(x) = x^2 + 1$
- $p_3(x) = x^2 - 4x + 4$
- $p_4(x) = x^2 - 4x - 5$
- $p_5(x) = x^2 - 4x + 1$
- $p_6(x) = x^2 - 4x + 13$

(a) For each case, find the roots of the polynomial. Comment on the results in each case.

(b) For each case, find the discriminant of the polynomial and describe any apparent connections between the discriminant of the polynomial and its real and complex roots.

(c) Some of the polynomials have no real roots and their complex roots include the complex number $\boldsymbol{i}$. In these cases, multiply the pair of roots for each polynomial and suggest a definition for the value of $\boldsymbol{i}$.

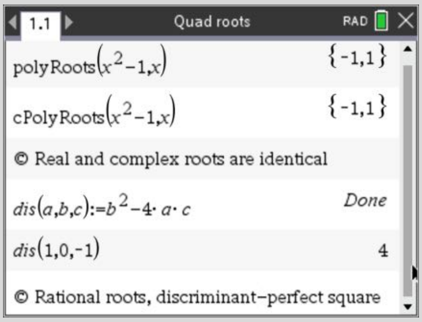
For the case of the polynomial $p_1(x) = x^2 - 1$, on a **Calculator** page, proceed as follows:

(a) To find the roots of $p_1(x) = 0$ over $\mathbb{R}$ and $\mathbb{C}$:

- Press menu > **Algebra** > **Polynomial Tools** > **Real Roots of Polynomial.**
- Enter $\textbf{polyRoots}(\boldsymbol{x^2 - 1, x})$.

To find the roots of $p_1(x) = 0$ over the complex field:

- Press menu > **Algebra** > **Polynomial Tools** > **Complex Roots of Polynomial**.
- Enter $\textbf{cPolyRoots}(\boldsymbol{x^2 - 1, x})$.



(b) To find discriminant, on the **Calculator** page:

- Enter $\textbf{dis}(\boldsymbol{a,b,c}) := \boldsymbol{b^2 - 4a \cdot c}$, then enter $\textbf{dis}(\mathbf{1,0,-1})$.

Note: *To add an optional comment on a* ***Calculator*** *page, press* menu *>* ***Actions*** *>* ***Insert Comment****.*

... continued

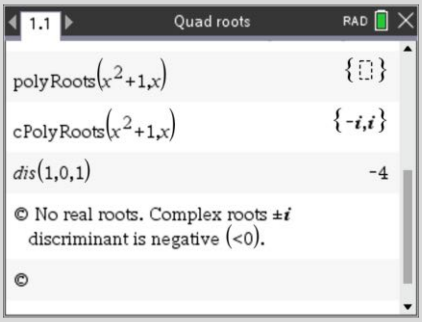
For the case of the polynomial $p_2(x) = x^2 + 1$, on the **Calculator** page, proceed as follows:

(a) To find the roots of $p_2(x) = 0$ over $\mathbb{R}$ and $\mathbb{C}$:

- Press ▲ up to the top entry line of the history and press [enter].
- Edit the pasted input to $\mathbf{polyRoots}(x^2 + 1, x)$ then press [enter].
- Likewise, copy and edit the second entry line of the history to $\mathbf{cPolyRoots}(x^2 + 1, x)$, then press [enter].

(b) To find the discriminant, on a **Calculator** page:

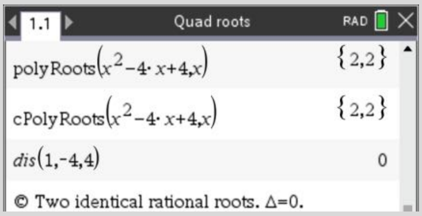
- Press [var], select **dis** and enter $\mathbf{dis}(1,0,1)$.

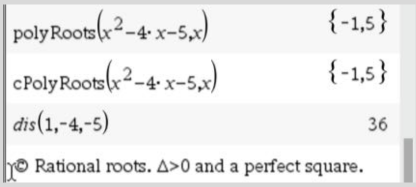


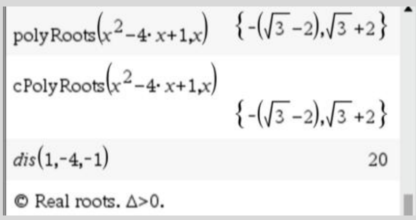
For the cases of the polynomials $p_3(x) = x^2 - 4x + 4$, $p_4(x) = x^2 - 4x - 5$, $p_5(x) = x^2 - 4x + 1$ and $p_6(x) = x^2 - 4x + 13$, on a **Calculator** page, proceed as follows.

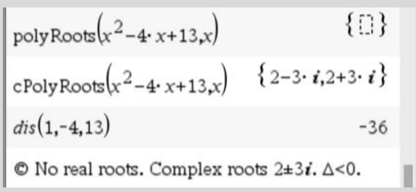
(a) and **(b)** To find the roots over $\mathbb{R}$ and $\mathbb{C}$, and the discriminants:

- Paste the appropriate command from the history and edit the input as required for the polynomials $p_3(x)$ to $p_6(x)$.





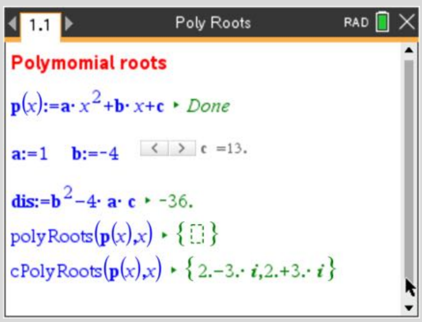




Results summary

$p_k(x)$	$\Delta = b^2 - 4ac$	Roots over $\mathbb{Q}, \mathbb{R}$	Roots over $\mathbb{C}$
$p_3(x)$	$\Delta = 0$	Two identical	Two identical
$p_1(x)$	$\Delta = 4$	Two distinct	Two distinct
$p_4(x)$	$\Delta = 36$	Two distinct	Two distinct
$p_5(x)$	$\Delta = 20$	2 Distinct $\notin \mathbb{Q}$	2 Distinct $\notin \mathbb{Q}$
$p_2(x)$	$\Delta = -4$	No roots $\notin \mathbb{Q}, \mathbb{R}$	Two distinct
$p_6(x)$	$\Delta = -36$	No roots $\notin \mathbb{Q}, \mathbb{R}$	Two distinct

***Note**: The above results could also be found by using a **Notes** page with a slider for k. See sample screen right for the roots of $p_6(x) = 0$.*



... continued

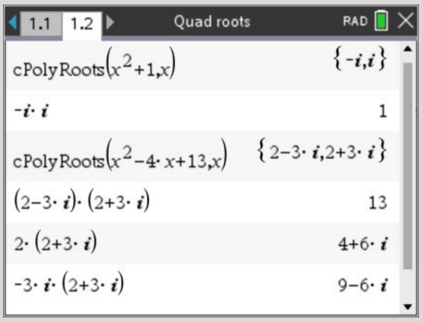
(c) To find the product of the non-real roots of $p_2(x)=0$ and $p_6(x)=0$, on a **Calculator** page:

- Enter $\boldsymbol{-i \cdot i}$ (press [π▸] to select 'imaginary' number, i).
- Enter $(\mathbf{2-3}\boldsymbol{i})\cdot(\mathbf{2+3}\boldsymbol{i})$

The product of the conjugate pairs of roots is a real number, with the 'imaginary' parts cancelling, and the property:

$\boldsymbol{-i\cdot i=-(-1)=1}$, and $\boldsymbol{i^2=-1}$,

$(2-3i)\cdot(2+3i)=13+-6i-6i=13$



***Note*: *Real and Complex mode Document settings*.** *All complex number calculations shown for this subtopic can be performed with **Document Setting** set to the default **Real** mode. The use of **Rectangular** and **Polar Complex** mode is shown in later subtopics.*

Applying Complex Conjugates

Let $a,b\in\mathbb{R}$ and $i^2=-1$. If $z=a+bi$, then the conjugate of z, denoted by $\overline{z}$, is given by $\overline{z}=a-bi$.

(a) Given $z=-3+2i$ find the multiplicative inverse $z^{-1}=\frac{1}{z}$ and verify that $z^{-1}=\frac{\overline{z}}{z\overline{z}}$.

(b) Given $z_1=2-3i$ and $z_2=3+4i$ express the following in Cartesian form, $x+yi$.

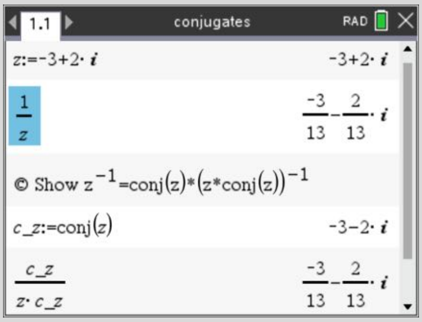
(i) $z_1+\overline{z}_2$ **(ii)** $\overline{z}_1\overline{z}_2$ **(iii)** $\overline{z_1z_2}$ **(iv)** $\frac{z_1}{z_2}$ and verify that $\frac{z_1}{z_2}=\frac{z_1\overline{z}_2}{z_2\overline{z}_2}$

(a) To find the multiplicative inverse of $z=-3+2i$ and verify that $z^{-1}=\frac{\overline{z}}{z\overline{z}}$, on a **Calculator** page:

- Enter $\boldsymbol{z:=-3+2i}$ (using [π▸] to select i), then enter $\boldsymbol{\frac{1}{z}}$.

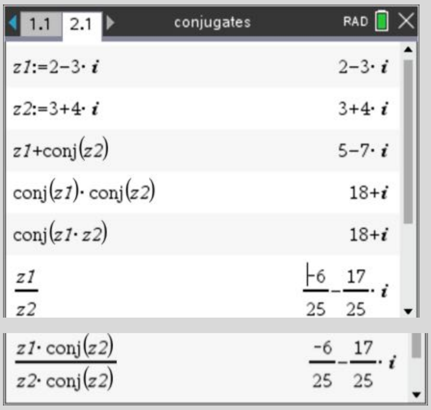
For the conjugate command, press [menu] > **Number** > **Complex Number Tools** > **Complex Conjugate.**

- Using this command, enter $\boldsymbol{c\_z:=}$**conj**$(\boldsymbol{z})$
- Enter $\boldsymbol{\frac{c\_z}{z\cdot c\_z}}$. This displays the same output as $\boldsymbol{\frac{1}{z}}$ and hence the result is verified.



(b) To express these complex numbers in Cartesian form, open a new **Problem** and add a **Calculator** page.

- Enter $\boldsymbol{z1:=2-3i}$, then enter $\boldsymbol{z2:=3+4i}$.
- To evaluate $z_1+\overline{z}_2$, enter $\boldsymbol{z1+}$**conj**$(\boldsymbol{z2})$
- To evaluate $\overline{z}_1\overline{z}_2$, enter **conj**$(\boldsymbol{z1})\cdot$**conj**$(\boldsymbol{z2})$
- To evaluate $\overline{z_1z_2}$, enter **conj**$(\boldsymbol{z1\cdot z2})$
- To verify $\frac{z_1}{z_2}=\frac{z_1\overline{z}_2}{z_2\overline{z}_2}$, enter $\boldsymbol{\frac{z1}{z2}}$, then enter $\frac{\boldsymbol{z1}\cdot\textbf{conj}(\boldsymbol{z2})}{\boldsymbol{z2}\cdot\textbf{conj}(\boldsymbol{z2})}$



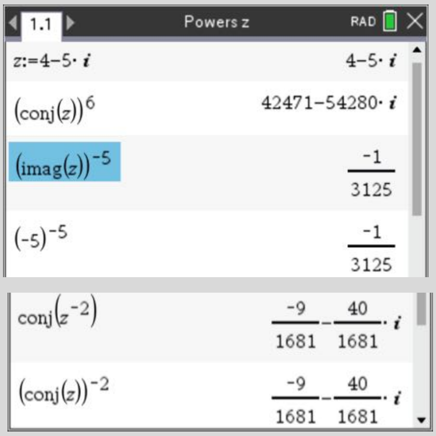
Evaluating powers of complex numbers

Consider $z = 4 - 5i$. Express each of the following in Cartesian form, where $\text{Re}(z)$ and $\text{Im}(z)$ denote the real and imaginary parts of z, respectively.

(i) $(\bar{z})^6$ **(ii)** $(\text{Im}(z))^{-5}$ **(iii)** $\overline{(z^{-2})}$ and verify $\overline{(z^{-2})} = (\bar{z})^{-2}$

To express the powers of the complex numbers in Cartesian form, on a **Calculator** page:

- Enter $\boldsymbol{z := 4 - 5i}$ (using [π▸] to select i).
- To evaluate $(\bar{z})^6$, enter $(\textbf{conj}(z))^6$
- To evaluate $(\text{Im}(z))^{-5}$, press [menu] > **Number** > **Complex Number Tools** > **Imaginary Part** and enter $\textbf{imag}(z)^{-5}$. Confirm that this result is equivalent to $(-5)^{-5}$.
- To verify that $\overline{(z^{-2})} = (\bar{z})^{-2}$, enter $\textbf{conj}(z^{-2})$ then $(\textbf{conj}(z))^{-2}$



2.1.2. The complex plane

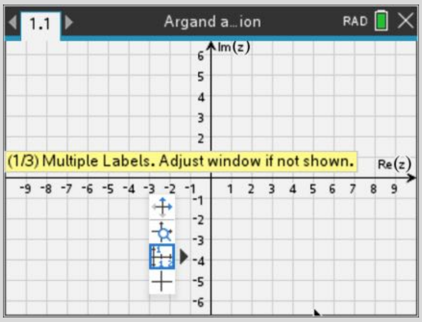
Representing addition of complex numbers as vector addition in the complex plane

Consider $z_1 = 5 + 2i$ and $z_2 = 1 + 2i$. Show the following complex numbers on the complex plane:

(a) $z_1 + z_2$ **(b)** $z_1 + (-2z_2) = z_1 - 2z_2$

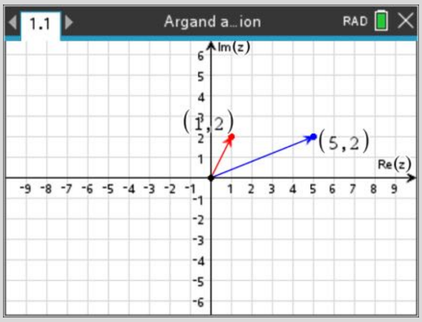
To set up the complex plane (Argand diagram) workspace, on a **Graphs** page:

- Click on the axis label 'y' until it is editable. Edit label to $\textbf{Im}(z)$. Similarly, edit the axis label 'x' to $\textbf{Re}(z)$.
- Press [menu] > **View** > **Grid** > **Lined Grid**.
- Hover over an axis, press [ctrl] [menu] > **Attributes**. Select the axes labels icon and then select **Multiple Labels**.



To illustrate $z_1 = 5 + 2i$ and $z_2 = 1 + 2i$, on a **Graphs** page:

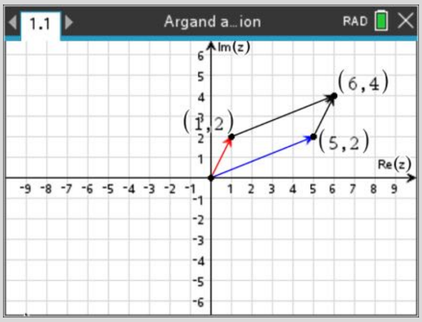
- Press [menu] > **Geometry** > **Points & Lines** > **Vector**.
- For z_1, click on the origin and then on the grid point $(5, 2)$.
- Click on the origin and then on the grid point $(1, 2)$ for z_2, then press [esc].
- Hover over the z_1 vector, press [ctrl] [menu] > **Colour** and select **Blue**. Likewise, for the z_1 point select **Blue**.
- Repeat as above, selecting **Red** for the z_2 vector and point.
- Hover over point at $(5, 2)$, press [ctrl] [menu] > **Coordinates and Equations**. Repeat for point at $(1, 2)$.



... *continued*

(a) To illustrate $z_1 + z_2$, proceed as follows:

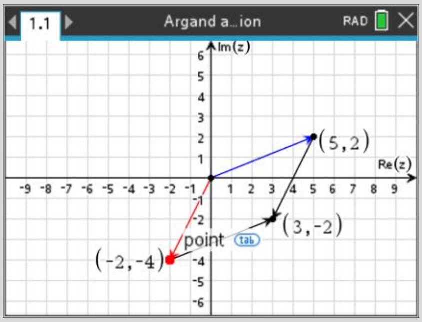
- Press [menu] > **Geometry** > **Transformation** > **Translation**.
- Click the **blue** vector followed by the **red** vector.
  Then click the **red** vector followed by the **blue** vector and press [esc].
- Hover over the new point where the translated vectors intersect. Press [ctrl] [menu] > **Coordinates and Equations**.



The translated vectors intersect at the point with coordinates $(6,4)$, confirming that $z_1 + z_2 = 6 + 4i$.

To illustrate $z_1 + (-2z_2) = z_1 - 2z_2$, proceed as follows:

- Hover over the point at the end of the **red** vector. Press [tab] so that the red point is selected. Grab the point (by pressing [ctrl] [grab]) and move it to coordinates $(-2,-4)$.
- Press [esc].



The translated vectors intersect at the point with coordinates $(3,-2)$, confirming that $z_1 + (-2z_2) = z_1 - 2z_2 = 3 - 2i$.

***Note**: This document could be saved and edited in the future to illustrate other cases of complex number operations.*

In particular, it could be saved into the **MyWidgets** folder.
The page can then be opened in any document as a **Widget**.

- Press [doc▾] > **File** > **Save As** and select the **MyWidget** folder. Save the widget in this folder as 'Argand'.

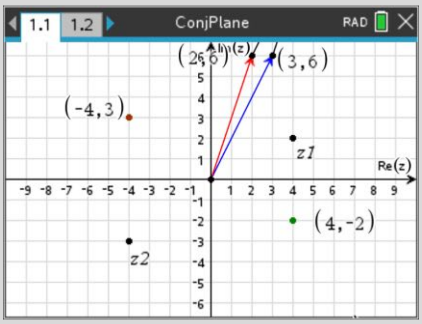
Using a saved Widget to illustrate complex number operations in the complex plane

(a) Using the saved 'Argand' widget from the previous section, represent the complex numbers $z_1 = 5 + 2i$, $\overline{z}_1 = 5 - 2i$, $z_2 = -4 - 3i$ and $\overline{z}_2 = -4 + 3i$ on the complex plane. Hence explain the relationship between the location of a complex number and its complex conjugate.

(b) Illustrate $\overline{z}_1 + \overline{z}_2$ on the complex plane.

To open the saved Widget 'Argand':

- Open a **New** document, or press [ctrl] [+page] in an existing opened document, and select **Add Widget**. Select the previously saved 'Argand' widget.
- Move the red and blue vectors out of the way.
- Press [esc]



(a) To plot $z_1, \overline{z}_1, z_2, \overline{z}_2$ on the complex plane:

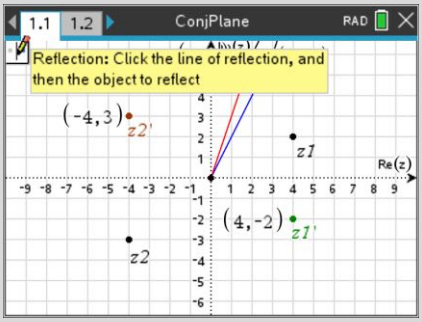
- Press [menu] > **Geometry** > **Points & Lines** > **Point on**.
- Click the grid points with coordinates $(5,2),(5,-2),(-4,-3)$ and $(-4,3)$, then press [esc].
- Hover over point at $(5,2)$, press [ctrl] [menu] > **Label** and enter the label $\mathbf{z1}$. label point at $(-4,-3)$ as $\mathbf{z2}$.

... continued

To confirm the relationship between the location of a complex number and its complex conjugate on the complex plane:

- Press menu > **Geometry** > **Transformation** > **Reflection**.
- Click on the horizontal axis, then click on point ***z*1**. Repeat for point ***z*2**, then press esc to exit this tool.

This illustrates that the complex conjugates, $\overline{z}_1$ and $\overline{z}_2$, are the images, z_1' and z_2', of z_1 and z_2 under a reflection in the horizontal axis.

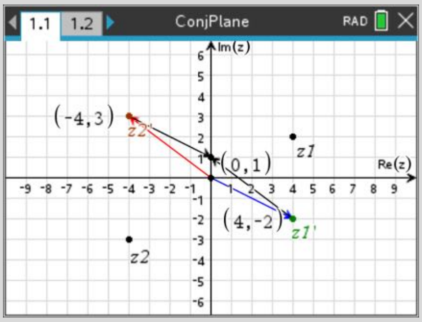


(b) To Illustrate $\overline{z}_1 + \overline{z}_2$ on the complex plane.

- Move the tip of the blue vector to point z_1' and the tip of the red vector to point z_2'.

The translation vectors intersect at $(0,1)$, $\overline{z}_1 + \overline{z}_2 = 0 + 1i = i$.

Note: *To move a vector, hover over the point at the tip and press* tab *until the point is coloured blue/red, then grab it.*

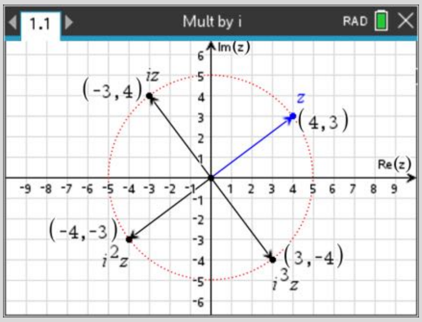


Visualising multiplication by a complex number as a transformation in the complex plane

Consider $z = 4+3i$. Illustrate on the complex plane the complex numbers z, iz, i^2z, i^3z, i^4z . Hence describe a linear transformation in the complex plane that is equivalent to multiplying a complex number by *i*.

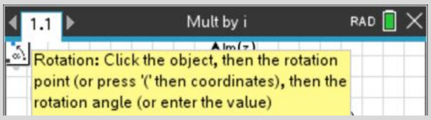
***Note:** This construction is best attempted using the TI-Nspire CX II-T Teacher Software rather than on the handheld device.*

To illustrate $z = i^4z = 4+3i,\ iz = -3+4i,\ i^2z = -4-3i,\ i^3z = 3-4i$ on the complex plane, set up a **Graphs** page as previously described (or add the previously saved 'Argand' Widget, then deleting all vectors except the blue vector). Then proceed as follows:

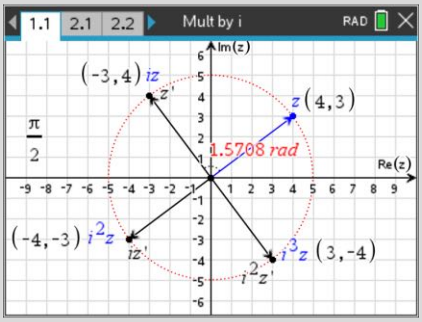


- Press [menu] > **Geometry** > **Points & Lines** > **Vector**.
- To represent z , click on the origin, and then on the grid point $(4,3)$.
- Hover over the point at the tip of the vector, press [ctrl][menu] > **Label** and enter the label z .
- Repeat the above for the other three complex numbers, iz, i^2z, i^3z .
- To show that all these vectors are of equal magnitude, press [menu] > **Geometry** > **Shapes** > **Circle**.
- Click on the origin and then click on the point z . Then press [esc]. All four points should be on the circumference of the circle of radius 5 units.
- Hover over the circle, press [ctrl][menu] > **Attributes** > **Line style is dotted**.

To confirm the transformation that maps z to iz , iz to i^2z etc. proceed as follows:



- Place the cursor in an empty part of the workspace, press [ctrl][menu] > **Text** and enter $\pi / 2$.
- Press [menu] > **Geometry** > **Transformation** > **Rotation**.
- Click on the vector to point z , then the origin (the rotation point), then the text $\frac{\pi}{2}$ (the rotation angle in radians).
- Repeat the previous procedure for the image of vectors to points iz, i^2z, i^3z under a rotation of $\frac{\pi}{2}^c$ or 90° .



***Note:** Under the 90° anticlockwise rotation about O, iz is the image of z , i^2z is the image of iz , etc. This illustrates that multiplying a complex number by i is equivalent to an 90° anticlockwise rotation about the origin. This is consistent with the imaginary axis being set at right angles to the real axis. Also note that if the number i is multiplied by the complex number z , the product iz is obtained by **rotating** i to the direction of iz , and then **dilating** it by a scale factor of 5 (the magnitude of the vector to z).*

2.2. Topic 2: Complex arithmetic and algebra

2.2.1. Complex arithmetic using polar form

Introducing the polar form of a complex number

Let $z_1 = 2 + 2\sqrt{3}i$ and $z_2 = -5 - 5i$.

(a) Illustrate z_1 and z_2 on the complex plane. Draw a line segment from the origin to the points representing z_1 and z_2 .

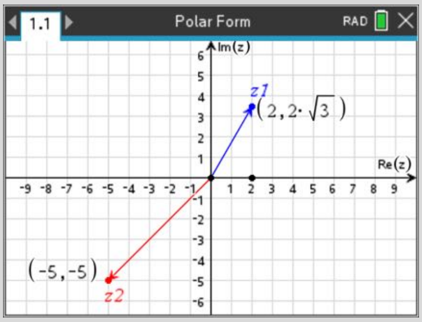
(b) Find $|z_1| = \sqrt{z_1\overline{z_1}}$ and $|z_2| = \sqrt{z_2\overline{z_2}}$, the modulus of z_1 and z_2 . Hence show that $|z_1|$ and $|z_2|$ are equivalent to the magnitudes of the line segments joining z_1 and z_2 to the origin.

(c) Find $\text{Arg}(z_1)$ and $\text{Arg}(z_2)$, the angles, $-\pi < \theta \le \pi$, between the positive real axis and the line segments joining z_1 and z_2 to the origin.

(d) Show more than one method to convert z_1 to polar form, $z_1 = r(\cos(\theta) + i\sin(\theta)) = r\text{cis}(\theta)$, where $r = |z_1|$ and $\theta = \text{Arg}(z_1)$ by converting the Cartesian (rectangular) form to polar.

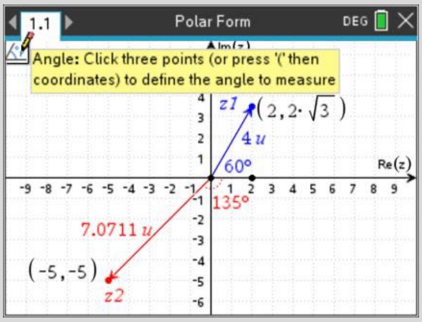
(e) Show more than one method to convert the polar form of $z_2 = 5\sqrt{2}\text{cis}\left(-\frac{3\pi}{4}\right)$ to Cartesian (rectangular) form.

(a) To illustrate z_1, z_2 on the complex plane, set up a **Graphs** page as previously described (or add the previously saved 'Argand' Widget, then delete unwanted vectors).



- Press [menu] > **Geometry** > **Points & Lines** > **Vector**.
- To represent z_1 , click on the origin and then click on the workspace, but *not* on a grid point. Repeat for z_2 .
- Press [esc]
- Hover over the point at the tip of the vector, press [ctrl] [menu] > **Coordinates and Equations**. Edit the coordinates to $(2, 2\sqrt{3})$ for z_1 and $(-3, -3)$ for z_2 .
- Label the points **z1** and **z2** , as described previously.

To measure the magnitudes of the vectors joining the origin to z_1, z_2, and the angles with the positive real axis, in degrees:



- Press [menu] > **Geometry** > **Measurement** > **Length**.
- Click on the vector to z_1 then press [enter]. Repeat for the vector to z_2 . Then press [esc].
- To change geometry angle units, press [menu] > **Settings** and select the preferred unit (degrees/radians).
- Press [menu] > **Geometry** > **Measurement** > **Angle**.
- Click on a point on the positive real axis, then on the origin, then on the point z_1 . Repeat for angle to z_2 .

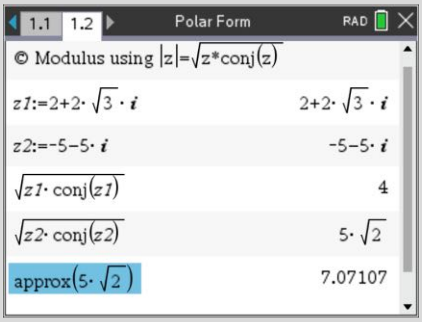
... continued

(b) To find $|z_1| = \sqrt{z_1 \bar{z}_1}$ and $|z_2| = \sqrt{z_2 \bar{z}_2}$, on a **Calculator** page:

- Enter $\boldsymbol{z1 := 2 + 2\sqrt{3}i}$ and $\boldsymbol{z2 := -5 - 5i}$.
- Enter $\sqrt{\boldsymbol{z1 \cdot \text{conj}(z1)}}$ and $\sqrt{\boldsymbol{z2 \cdot \text{conj}(z2)}}$

Note: *Conjugate command:* menu ***> Number > Complex Number Tools > Complex Conjugate****.*

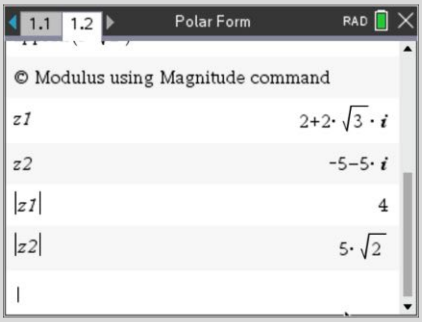
These values accord with the magnitudes in part (a) above.



To find $|z_1|$ and $|z_2|$ using the inbuilt **Magnitude** command, on a **Calculator** page with z_1, z_2 defined:

- Press [template key], and select the **Magnitude** $|\square|$ template.
- Enter $|\boldsymbol{z1}|$, then $|\boldsymbol{z2}|$.

Note: *Alternatively, press* menu ***> Number > Complex Number Tools > Magnitude****.*

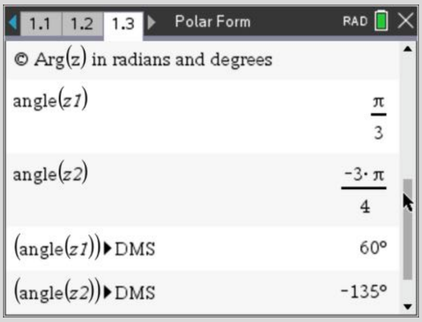


(c) To find $\text{Arg}(z_1)$ and $\text{Arg}(z_2)$ in radians, on a **Calculator** page, with z_1, z_2 defined and default settings:

- Press menu **> Number > Complex Number Tools > Polar Angle**, enter $\boldsymbol{\text{angle}(z1)}$ then enter $\boldsymbol{\text{angle}(z2)}$.

To express $\text{Arg}(z_1)$ and $\text{Arg}(z_2)$ in degrees:

- Press ▲ (up arrow) to select $\boldsymbol{\text{angle}(z1)}$, press enter then [catalog] D to select **▶DMS** and enter **(angle(*z*1))▶DMS**.
- Repeat for $\boldsymbol{\text{angle}(z2)}$.



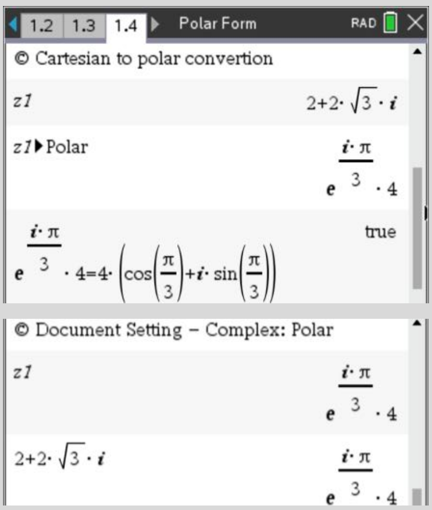
Note: *These values accord with the angles in part (a) above.*

(d) To convert z_1 to polar form, on a **Calculator** page, with z_1 defined and default settings:

- Input $\boldsymbol{z1}$, then press [catalog] 1 P. Select **▶Polar** and enter ***z*1▶Polar**. Verify the equivalence of the output and $4\text{cis}\left(\dfrac{\pi}{3}\right)$.

To convert z_1 to polar form with **Polar** document setting:

- Press [home on] **> Settings > Document Settings > Real or Complex: Polar**. Click **OK** and then select **Current**.
- On the **Calculator** page, enter $\boldsymbol{z1}$ (if previously defined) or $2 + 2\sqrt{3}i$. In this setting, the output will be in polar form.
- Return the **Document Settings** to **Real** mode for part (e).



... continued

(e) To convert $z_2 = 5\sqrt{2}\text{cis}\left(-\dfrac{3\pi}{4}\right)$ to Cartesian (rectangular) form, on a **Calculator** page, with either default **Real** or **Complex: Rectangular** document settings:

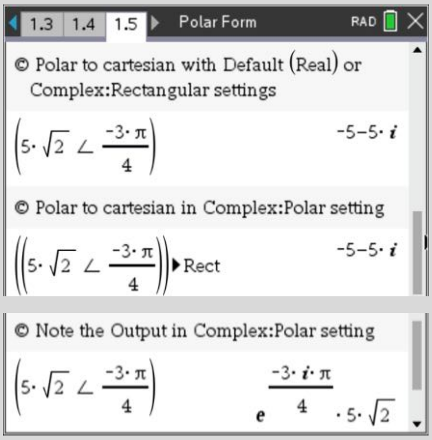
Note: *The shorthand way of entering* $5\sqrt{2}\text{cis}\left(-\dfrac{3\pi}{4}\right)$ *is shown.*

- Enter $\left(5\sqrt{2}\angle-\dfrac{3\pi}{4}\right)$, selecting $\angle$ from ctrl [∞β°] options.

If the **Document Settings** are **Complex: Polar**, then:

- Input $\left(5\sqrt{2}\angle-\dfrac{3\pi}{4}\right)$, then press menu > **Number** > **Complex Number Tools** > **Convert to Rectangular**.

Note: *In* ***Complex: Polar*** *setting, the output of* $\left(5\sqrt{2}\angle-\dfrac{3\pi}{4}\right)$ *is of the form* $5\sqrt{2}\,e^{-\frac{3\pi\cdot i}{4}}$.



Applying multiplication, division and powers of complex numbers in polar form

If $z = 2\left(\cos\left(\dfrac{5\pi}{6}\right) + i\sin\left(\dfrac{5\pi}{6}\right)\right)$ and $w = 4\text{cis}\left(\dfrac{\pi}{6}\right)$, find expressions for the following in both polar and Cartesian forms.

(a) zw **(b)** $\dfrac{z}{w}$ **(c)** z^3

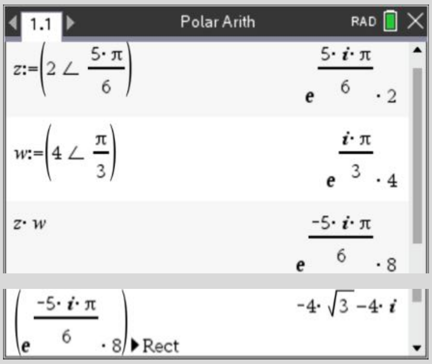
On a **Calculator** page with **Document Settings - Complex: Polar**, define z and w, as follows.

- Enter $z := \left(2\angle\dfrac{5\pi}{6}\right)$, then $w := \left(4\angle\dfrac{\pi}{6}\right)$, selecting $\angle$ from the ctrl [∞β°] options.

(a) To find an expression for zw in polar form:

- Enter $z\cdot w$. Then, to convert to Cartesian form:
- Press menu > **Number** > **Complex Number Tools** > **Convert to Rectangular**, then press enter.

Answer: $2\text{cis}\left(\dfrac{5\pi}{6}\right)\times 4\text{cis}\left(\dfrac{\pi}{3}\right) = 8\text{cis}\left(\dfrac{-5\pi}{6}\right) = -4\sqrt{3}-4i$

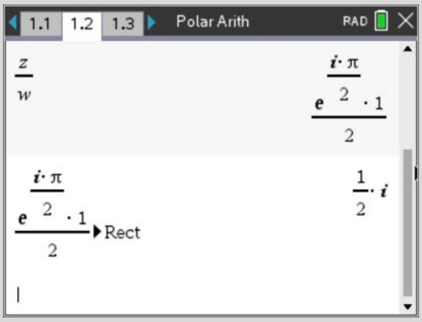


... continued

(b) To find an expression for $\frac{z}{w}$ in polar form:

- Enter $\boldsymbol{z / w}$. Then, to convert to Cartesian form:
- Press menu > **Number** > **Complex Number Tools** > **Convert to Rectangular**, then press enter.

Answer: $\dfrac{2\text{cis}\left(\frac{5\pi}{6}\right)}{4\text{cis}\left(\frac{\pi}{3}\right)}=\frac{1}{2}\text{cis}\left(\frac{\pi}{2}\right)=\frac{1}{2}i$



(c) To find an expression for z^3 in polar form:

- Enter $\boldsymbol{z^3}$. Then, to convert to Cartesian form:
- Press menu > **Number** > **Complex Number Tools** > **Convert to Rectangular**, then press enter.

$\left(2\text{cis}\left(\frac{5\pi}{6}\right)\right)^3=8\text{cis}\left(\frac{\pi}{2}\right)=8i$

2.2.2. Subsets of the complex plane

Representing subsets of the complex plane and interpreting geometrically multiplication in polar form

(a) Illustrate the subsets of the complex plane defined by $\text{Arg}(z_1)=\frac{\pi}{3}$ and $\text{Arg}(z_2)\geq\frac{5\pi}{6}$.

(b) Illustrate the subsets of the complex plane defined by $|z_1|=4$ and $|z_2|=2$.

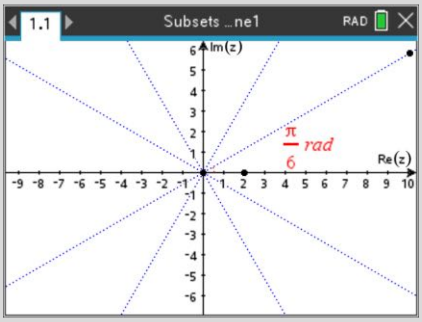
(c) Illustrate $z_1=2+2\sqrt{3}i=4\,\text{cis}\left(\frac{\pi}{3}\right)$ and $z_2=-\sqrt{3}+i=2\,\text{cis}\left(\frac{5\pi}{6}\right)$ as polar coordinates on the complex plane.

(d) Evaluate the product, z_1z_2 in polar form and illustrate z_1z_2 on the complex plane.

(e) Describe the multiplication z_1z_2 in terms of magnitudes and angles of z_1 and z_2.

To set up radial grid lines spaced at $\frac{\pi}{6}^c$, on a **Graphs** page:

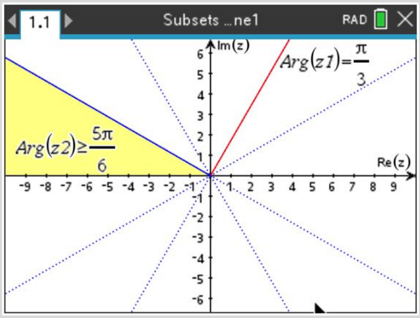
- Enter $\boldsymbol{f1(x)=\tan\left(\frac{n\cdot\pi}{6}\right)\cdot x \mid n=\{1,2,3,4,5\}}$. If prompted to create a slider for $\boldsymbol{n}$, click **cancel** to decline.
- Hover over a radial line, press ctrl menu > **Attributes** > **Line style is dotted**. Repeat for the other radial lines.
- Press menu > **Actions** > **Hide/Show**. Click on unwanted labels to hide them.



... continued

(a) To illustrate $\text{Arg}(z_1) = \dfrac{\pi}{3}$ on the **Graphs** page at right:

- Enter $\boldsymbol{f2(x) = \tan\left(\dfrac{\pi}{3}\right)\cdot x \mid x > 0}$.

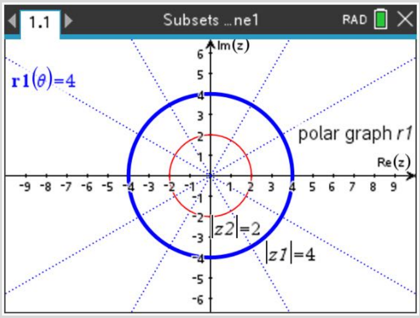


To Illustrate $\text{Arg}(z_2) \geq \dfrac{5\pi}{6}$:

- Press menu > **Graph Entry/Edit > Relation**.
- Enter $\boldsymbol{y \leq \tan\left(\dfrac{5\pi}{6}\right)\cdot x \mid y < 0}$

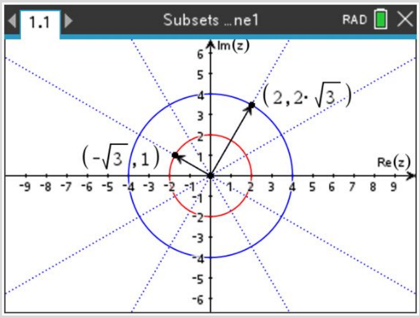
(b) To illustrate $|z_1| = 4$ and $|z_2| = 2$ on the **Graphs** page:

- Press menu > **Graph Entry/Edit > Polar**.
- Enter $\boldsymbol{r1(\theta) = 4}$, then $\boldsymbol{r2(\theta) = 2}$.
- Accept the default settings of $0 \leq \theta \leq 6.28$ $\theta step = 0.13$.



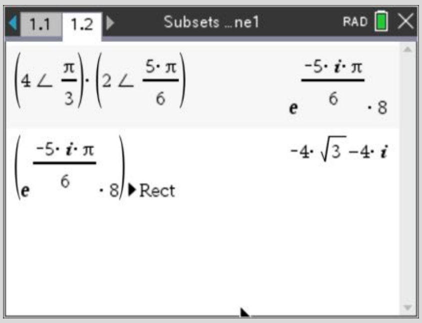
Every point on the circle $r1(\theta) = 4$ is four units from the origin.

(c) To illustrate $z_1 = 2 + 2\sqrt{3}i = 4\,\text{cis}\left(\dfrac{\pi}{3}\right)$ and $z_2 = -\sqrt{3} + i = 2\,\text{cis}\left(\dfrac{5\pi}{6}\right)$ as polar coordinates:



- Press menu > **Geometry > Points & Lines > Vector**.
- Click on the origin and then click on **Point On** the intersection of $\text{Arg}(z_1) = \dfrac{\pi}{3}$, $|z_1| = 4$. Repeat for $\text{Arg}(z_2) = \dfrac{5\pi}{6}$, $|z_1| = 2$.
- Hover over the point at the tip of the first vector, press ctrl menu > **Label** then enter $\boldsymbol{z1}$. Repeat for $\boldsymbol{z2}$.
- Hover over the point at the tip of the $\boldsymbol{z1}$ vector, press ctrl menu > **Coordinates and Equations**. To force exact arithmetic, edit the y-coordinate to be $2\sqrt{3}$.
- Repeat for $\boldsymbol{z2}$, except edit the x-coordinate to be $-\sqrt{3}$.

(d) To evaluate $4\,\text{cis}\left(\dfrac{\pi}{3}\right) \times 2\,\text{cis}\left(\dfrac{5\pi}{6}\right)$, on a **Calculator** page with **Document Settings** > **Complex > Polar**:

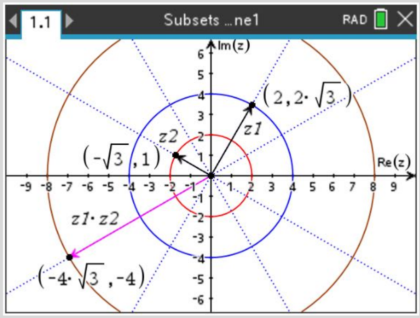


- Enter $\boldsymbol{\left(4\angle\dfrac{\pi}{3}\right)\cdot\left(2\angle\dfrac{5\pi}{6}\right)}$.
- Press menu > **Number > Complex Number Tools > Convert to Rectangular**, then press enter.

Answer: $z_1 z_2 = 8\text{cis}\left(\dfrac{7\pi}{6}\right) = 8\text{cis}\left(-\dfrac{5\pi}{6}\right) = -4\sqrt{3} - 4i$

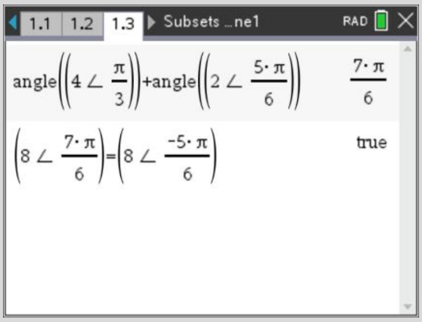
... continued

To plot z_1z_2 as a polar coordinate, on the **Graphs** page:



- Enter $\boldsymbol{r3(\theta)=8}$.
- Press menu > **Geometry > Points & Lines > Vector**.
- Click on the origin and then click on **Point On** the intersection of $r3(\theta)=8$ and the radial line for $\text{Arg}(z_1 \cdot z_2) = -\dfrac{5\pi}{6}$.
- Hover over the point at the tip of the vector, press ctrl menu > **Coordinates and Equations**. To force exact arithmetic, edit the *x*-coordinate to be $-4\sqrt{3}$.

(e) The geometric interpretation of the multiplication z_1z_2 in terms of magnitudes and angles of z_1 and z_2 is evident from the polar plots of z_1, z_2 and z_1z_2.



$|z_1z_2| = |z_1| \times |z_2| = 4 \times 2 = 8$

$$\text{Arg}(z_1z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$$
$$= \frac{\pi}{3} + \frac{5\pi}{6} = \frac{7\pi}{6}(\text{anti-clockwise})$$

$\dfrac{7\pi}{6}(\text{anti-clockwise})$ is an equivalent rotation to $-\dfrac{5\pi}{6}(\text{clockwise})$.

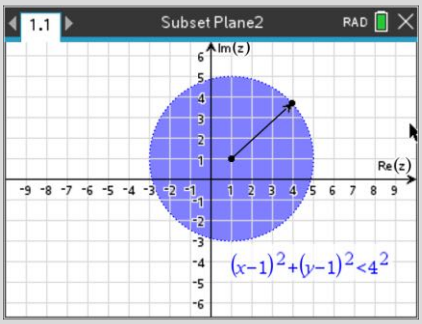
Representing other subsets of the complex plane and their geometric interpretation

Illustrate on the complex plane the following subsets of $\mathbb{C}$.

(a) $|z-(1+i)| < 4$

(b) $|z| = |z-(-3+6i)|$

(a) Geometric interpretation of $|z-(1+i)| < 4$: the set of points less than 4 units from $(1+i)$. The region inside a circle with centre $(1,1)$ and radius of 4.

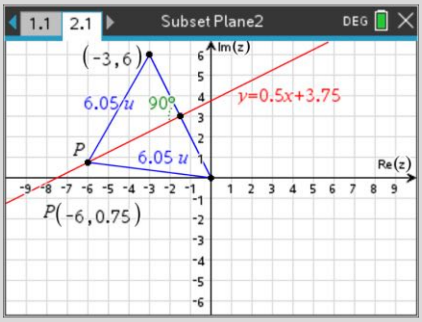


To illustrate the subset of the complex plane geometrically, on a **Graphs** page that is set up as in previous sections:

- Press menu > **Graph Entry/Edit > Relation**.
- Enter $\boldsymbol{(x-1)^2+(y-1)^2<4^2}$

... continued

(b) Geometric interpretation of $|z| = |z - (-3+6i)|$: the set of points equidistant from the origin and $(-3+6i)$. The perpendicular bisector of the segment joining the origin and $(-3,6)$.



To illustrate the subset of the complex plane geometrically, on a **Graphs** page:

- Press [menu] > **Geometry** > **Points & Lines** > **Segment**.
- Click on the origin, then click on the workspace in the second quadrant. Hover over this point, press [ctrl] [menu] > **Coordinates and Equations**. Edit coordinates to $(-3,6)$.
- Press [menu] > **Geometry** > **Construction** > **Perpendicular Bisector**, click the **segment**, then press [esc] to exit.
- Hover over the perpendicular bisector, press [ctrl] [menu] > **Coordinates and Equations**. Equation: $y = 0.5x + 3.75$.
- Use **Geometry** tools for additional embellishments.

2.2.3. Roots of real quadratic equations

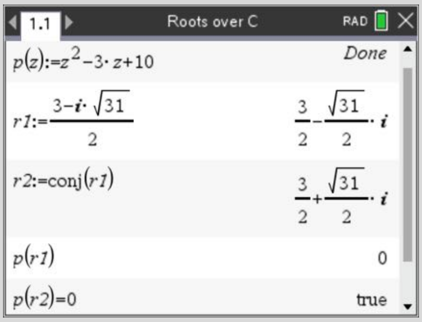
Solving real quadratic equations with complex conjugates

(a) Verify that $\frac{3-i\sqrt{31}}{2}$ is a root of the quadratic polynomial $p(z) = z^2 - 3z + 10$. Hence express $p(z)$ as the product of linear factors over $\mathbb{C}$.
Use a graphical method to confirm the reasonableness of the answer.

(b) Solve the quadratic equation $2z^2 + 1 = \sqrt{2}z$ over $\mathbb{C}$.

(a) To verify that $\frac{3-i\sqrt{31}}{2}$ is a root of $p(z) = 0$, on a **Calculator** page:

- Enter $\boldsymbol{p(z) := z^2 - 3z + 10}$
- Enter $\boldsymbol{r1 := \frac{3 - i\cdot\sqrt{31}}{2}}$, then $\boldsymbol{r2 := \frac{3 + i\cdot\sqrt{31}}{2}}$.
- Enter $\boldsymbol{p(r1)}$, then $\boldsymbol{p(r2) = 0}$.



This confirms that the conjugate pair are the roots of $p(z)$.

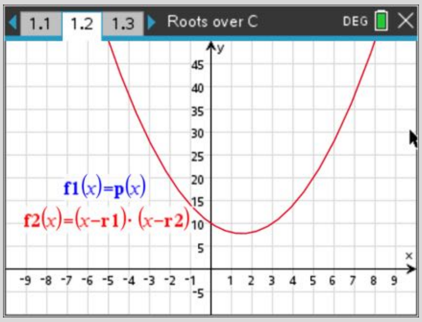
$$p(z) = (z - r1)(z - r2) = \left(z - \left(\frac{3-i\sqrt{31}}{2}\right)\right)\left(z - \left(\frac{3+i\sqrt{31}}{2}\right)\right)$$

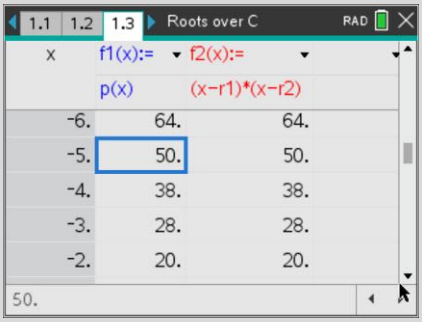
... continued

To graphically confirm the reasonableness of the answer, add a **Graphs** page.

- Enter $\boldsymbol{f1(x)=p(x)}$
- Press [menu] > **Window/Zoom** > **Window Settings**.
  Adjust the window settings as shown.
  XMin = −10   Xmax = 10   XScale = 1
  YMin = −10   YMax = 50   YScale = 5
- Enter $\boldsymbol{f2(x)=(x-r1)\cdot(x-r2)}$
- Press [ctrl] [T] to get a split screen with tables of values.
- Press [doc▾] > **Page Layout** > **Ungroup** to get the table of values on a separate page.

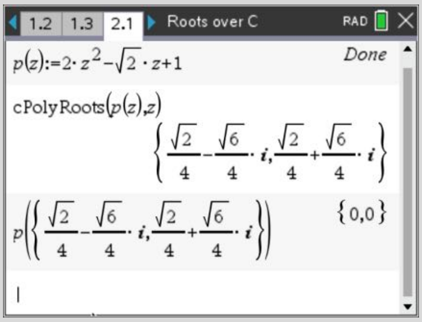
The identical graphs and identical tables of values provide compelling evidence of the reasonableness of the answer.





(b) To solve the quadratic equation $2z^2+1=\sqrt{2}z$, add a **Calculator** page to a new **Problem**, then:

- Enter $\boldsymbol{p(z):=2z^2-\sqrt{2}z+1}$
- Press [menu] > **Algebra** > **Polynomial Roots** > **Complex Roots of Polynomial**.
- Enter $\boldsymbol{p(\text{ans})}$ to confirm that $p(z)=0$ for $z=\dfrac{\sqrt{2}\pm\sqrt{6}\,i}{4}$



Determining the coefficients of a real quadratic equation given one complex root.

The quadratic equation $z^2+bz+c=0$, where $b,c\in\mathbb{R}$, has a solution $z=\dfrac{-5+\sqrt{11}\,i}{6}$.

Determine the values of b and c.

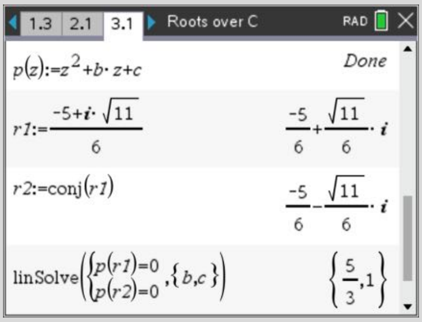
To determine the values of b and c for $z^2+bz+c=0$, where $z=\dfrac{-5+\sqrt{11}\,i}{6}$ is one solution, on a **Calculator** page:

- Enter $\boldsymbol{p(z):=z^2+b\cdot z+c}$.

Note: *Middle term is* $\boldsymbol{b\times z}$.

- Enter $\boldsymbol{r1:=\dfrac{-5+i\cdot\sqrt{11}}{6}}$, then $\boldsymbol{r2:=\text{conj}(r1)}$.
- Press [menu] > **Algebra** > **Solve a System of Linear Equations**. In the dialog box, edit Variables: ***b***, ***c***.
- Enter the equations: $\boldsymbol{p(r1)=0}$ and $\boldsymbol{p(r2)=0}$.

Answer: $b=\dfrac{5}{3},c=1$.



2.3. Topic 3: Circle and geometric proofs

2.3.1. Circle properties and their proofs

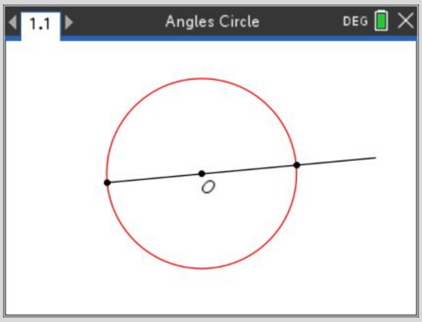
Proving that the angle at the circumference of a semicircle is a right angle

Prove that the angle at the circumference in a semicircle is a right angle.

The **Geometry** application will assist with the proof that an angle in a semicircle is a right angle.

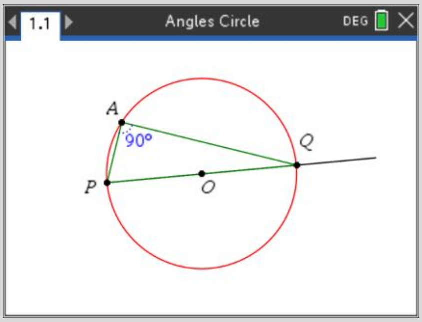
To construct a semicircle, on a **Geometry page:**

- Press [menu] > **Shapes** > **Circle**.
- Click on the centre point of the circle, then drag and click to set the radius. Hover over the centre point, press [ctrl] [menu] > **Label** and enter the label *O*.
- Press [menu] > **Points & Lines** > **Ray**. Construct the diameter by clicking on a circumference point and then the centre. Click on and hold the end of the ray and extend it across the circle.



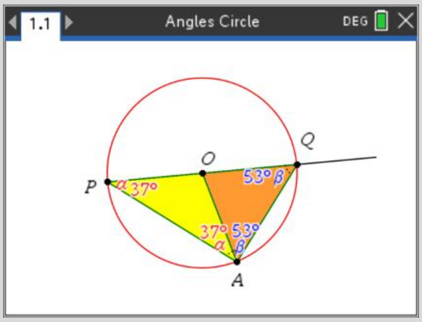
To draw the inscribed triangle and measure the angle:

- Press [menu] > **Shapes** > **Triangle**. Click on each intersection point of the ray and circle, then click on a third point on the circumference. Press [esc] to exit the tool.
- Label the vertices of the triangle *P*, *Q* and *A*, as shown.
- Press [menu] > **Settings**. Set the measurement angle to **Degrees**.
- Press [menu] > **Measurement** > **Angle**. Click on points *P*, *A* and *Q* to measure $\measuredangle PAQ$.
- Click and hold on point *A* and drag it around the circle, visually verifying (but not proving) that $\measuredangle PAQ$ is a right angle.



To prove that $\measuredangle PAQ$ is a right angle:

- Press [menu] > **Points & Lines** > **Segment** and draw a segment *AO*. This forms two isosceles triangles, ΔPOA and ΔQOA, with $d(AO) = d(PO) = d(QO)$.
- Press [menu] > **Measurement** > **Angle**. Click on the required points to measure $\measuredangle OPA, \measuredangle OAP, \measuredangle OQA, \measuredangle OAQ$.



Let $\alpha = \measuredangle OPA = \measuredangle OAP$ and $\beta = \measuredangle OQA = \measuredangle OAQ$.

The internal angles of ΔPQA:

$$\alpha + \alpha + \beta + \beta = 2\alpha + 2\beta = 180^\circ$$

$\alpha + \beta = \measuredangle PAQ = 90^\circ$, as required.

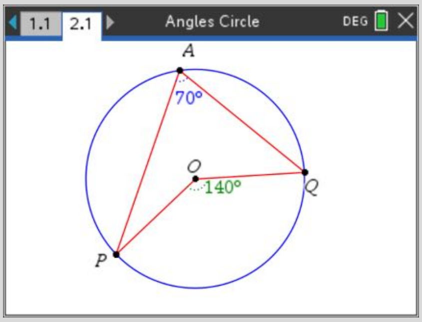
***Note:** This theorem and its converse are proved using vectors in the next subtopic, **2.3.2 Geometric proofs using vectors**.*

Proving inscribed angle theorems

Prove that the angle at the centre subtended by an arc of a circle is twice the angle at the circumference subtended by the same arc. Hence prove that the angles at the circumference of a circle subtended by the same arc are equal.

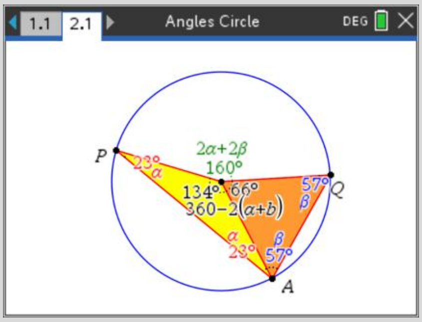
To construct a quadrilateral with vertices on the circumference and centre of a circle, on a **Geometry** page:

- Press menu > **Shapes** > **Circle**. Construct a circle and label the centre *O*, as described in the previous example.
- Press menu > **Shapes** > **Polygon**. Click on point *O*, then on three points on the circumference, and then on *O* again.
- Label the vertices *P*, *Q* and *A*, as shown.
- Press menu > **Measurement** > **Angle**. Click the relevant points to measure $\measuredangle PAQ$ and $\measuredangle POQ$.
- Grab point *A*, *P* or *Q*. Drag the point around the circle, visually verifying that $\measuredangle POQ = 2\measuredangle PAQ$.



To prove that the angle at the centre is twice the angle at the circumference:

- Press menu > **Points & Lines** > **Segment** and draw a segment *AO*. This forms two isosceles triangles, ΔPOA and ΔQOA, with $d(AO) = d(PO) = d(QO)$.
- Press menu > **Measurement** > **Angle**. Click on the required points to measure $\measuredangle OPA, \measuredangle OAP, \measuredangle OQA, \measuredangle OAQ$.



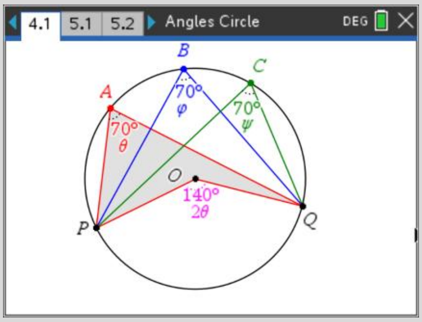
The angle at the circumference is $\alpha + \beta$.

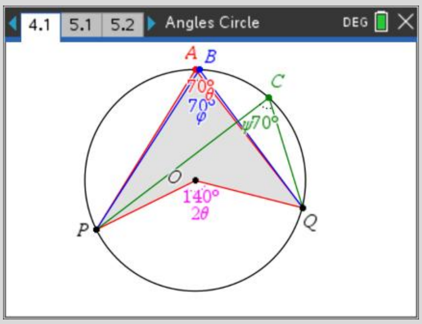
For quadrilateral *POQA*, the interior angle at the centre, $\measuredangle POA + \measuredangle QOA = 360° - 2(\alpha + \beta)$.

Therefore, the exterior angle *POQ* at the centre is $2(\alpha + \beta)$.

To visually verify that the inscribed angles subtended by the same arc are equal:

- Press menu > **Points & Lines** > **Segment**, then draw segments, as shown, to form angles *PBQ* and *PCQ*.
- Press menu > **Measurement** > **Angle** and measure the angles.
- Drag the various points around the circle and observe the relationship between the inscribed and central angles.
- Drag point *A* to points *B* or *C* and observe whether the figures *AQOP*, *BQOP* and *CQOP* appear to be congruent.





To prove that the inscribed angles on the same arc are equal, let the inscribed angles be
$\measuredangle PAQ = \theta, \measuredangle PBQ = \varphi,$ and $\measuredangle PBQ = \psi$.
The corresponding central angle $\measuredangle POQ = 2\theta$.
The inscribed angles at *A*, *B* and *C* are on the same arc and share a common central angle, $\measuredangle POQ$.

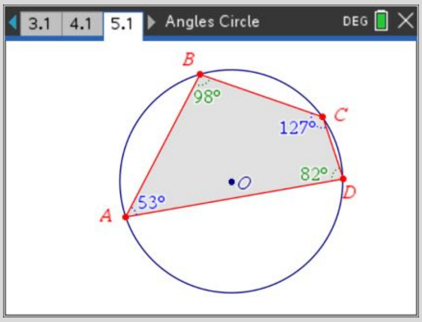
$2\theta = 2\varphi = 2\psi$ therefore $\theta = \varphi = \psi$, as required.

Proving that opposite angles of a cyclic quadrilateral are supplementary

Prove that opposite angles of a cyclic quadrilateral are supplementary.

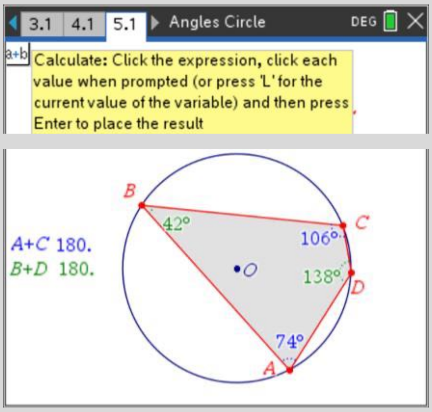
To draw a cyclic quadrilateral, on a **Geometry** page:

- Press [menu] > **Settings** > **Automatically label points** (check this box)
- Press [menu] > **Shapes** > **Circle**. Construct a circle and label the centre ***O***, as described in previous examples.
- Press [menu] > **Shapes** > **Polygon**. Click on four points on the circumference to form quadrilateral *ABCD*.
- Press [menu] > **Measurement** > **Angle**. Click on the relevant points to measure each of the interior angles.



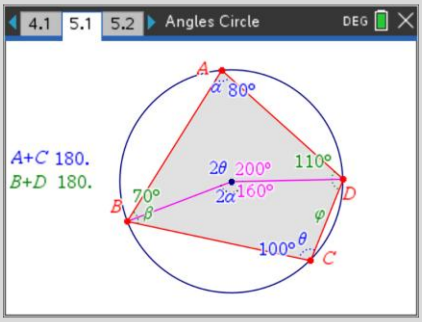
To visually verify that opposite angles are supplementary:

- Press [menu] > **Text**. Enter ***A+C***. Repeat and enter ***B+D***.
- Press [menu] > **Actions** > **Calculate**. Click on the text ***A+C***, then click on the angle measurement at *A* followed by the angle measurement at *C*. Repeat for ***B+D***.
- Grab and drag points *A*, *B*, *C* and *D* in turn and observe the sum of the opposite angles for different quadrilaterals.
- Move the points so that the line segments *AB* and *CD* intersect, and the shape is no longer a cyclic quadrilateral. Observe whether the supplementary angle property still holds, or whether a different theorem is demonstrated.



To prove that opposite angles are supplementary:

- Press [menu] > **Points & Lines** > **Segment** then construct segments *BO* and *OD*.
- Press [menu] > **Measurement** > **Angle** to measure $\measuredangle BOD$.
- Drag point *A* or *C* around the circle and observe the relationship between the inscribed and central angles.

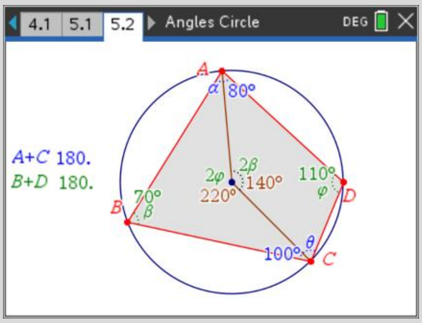


Let the inscribed angles be $\measuredangle BAD = \alpha$ and $\measuredangle BCD = \theta$

The corresponding central angles are therefore 2α and 2θ.

$2\alpha + 2\theta = 360^\circ$ therefore $\alpha + \theta = 180^\circ$.

The angles at *A* and *C* are supplementary, as required.



- Hide segments *BO* and *OD* by pressing [ctrl] [menu] > **Hide**.
- Construct segments *AO* and *OC* and measure angle *AOC*, as described above.

Let the inscribed angles be $\measuredangle ABC = \beta$ and $\measuredangle ADC = \varphi$

The corresponding central angles are therefore 2β and 2φ.

$2\beta + 2\varphi = 360^\circ$ therefore $\beta + \varphi = 180^\circ$.

The angles at *B* and *D* are supplementary, as required.

Note: *The Greek letter labels can be accessed via* [ctrl] [📖]

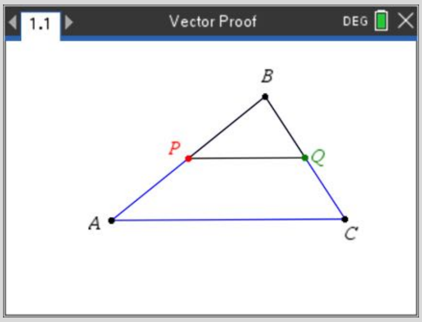
2.3.2. Geometric proofs using vectors

Proving the midsegment theorem using vectors

Use vectors to prove that the line segment connecting the midpoints of two sides of a triangle is parallel to the third side and half its length.

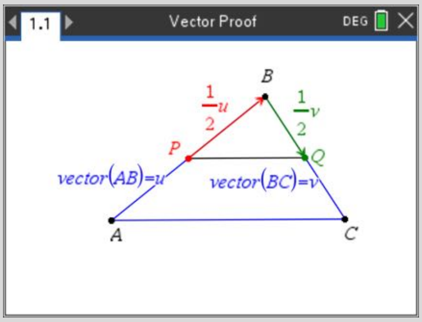
To construct a diagram illustrating the required proof, on a **Geometry** page.

- Press [menu] > **Settings** > **Automatically label points** (check this box).
- Press [menu] > **Shapes** > **Triangle**. Click three points on the workspace to form triangle ABC. Then press [esc] to exit.
- Press [menu] > **Construction** > **Midpoint**. Click on each side AB and BC, then press [esc].
- Label the midpoints as P and Q by hovering over the point, press [ctrl] [menu] > **Label**, enter label $\boldsymbol{P}$ then $\boldsymbol{Q}$.
- Press [menu] > **Points & Lines** > **Segment**. Construct $\overline{PQ}$.



To complete the diagram:

- Press [menu] > **Points & Lines** > **Vector**. Construct $\overrightarrow{PB}$, and $\overrightarrow{BQ}$.
- Hover over $\overrightarrow{PB}$, press [ctrl] [menu] > **Label**. Enter label $\frac{1}{2}\boldsymbol{u}$.
- Similarly, label $\overrightarrow{BQ}$ as $\frac{1}{2}\boldsymbol{v}$. Label $\overrightarrow{AB}$ and $\overrightarrow{BC}$ as shown.



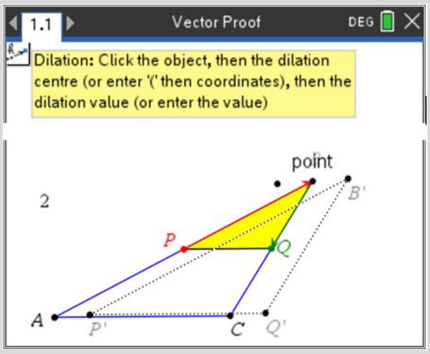
To prove that $\overrightarrow{PQ}=\frac{1}{2}\overrightarrow{AC}$:

$$\overrightarrow{PQ}=\overrightarrow{PB}+\overrightarrow{BQ}=\frac{1}{2}(\underset{\sim}{u}+\underset{\sim}{v}) \text{ and } \overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{BC}=(\underset{\sim}{u}+\underset{\sim}{v})$$

$\overrightarrow{PQ}=\frac{1}{2}\overrightarrow{AC}$ as required.

The **Dilation** transformation tool can be used to visually verify that triangle ABC is a dilation of triangle PBQ by a factor of 2.

- Press [menu] > **Geometry** > **Shapes** > **Triangle**. Construct ΔPBQ, colour as required and hide unwanted labels.
- Press [menu] > **Transformation** > **Dilation**.
- Click on triangle PBQ, then press [2] [enter] (dilation factor of 2), and then move the cursor to point B.
- Press [enter] to superimpose the dilation of triangle PBQ over triangle ABC. The image of PBQ is shown as $P'B'Q'$.
- Click and hold any vertex, and then drag it to show that the property holds for any type and shape of triangle.



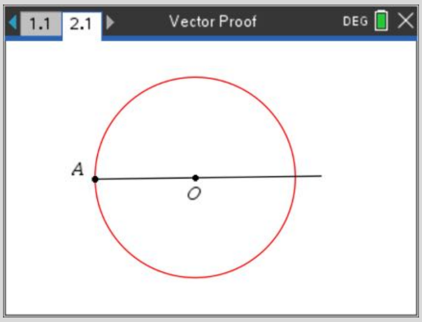
Proving, using vectors, that the angle inscribed in a semicircle is a right angle

Use vectors to prove that an angle inscribed in a semicircle is a right angle.

Note: *It is important to understand and use the vector notation* $\overrightarrow{AB}$, $\underset{\sim}{b}$, $\boldsymbol{d}$ *and* $\hat{\boldsymbol{n}}$. *In this section, the vector notation* $\overrightarrow{AB}$ *and* $\underset{\sim}{b}$ *is used.*

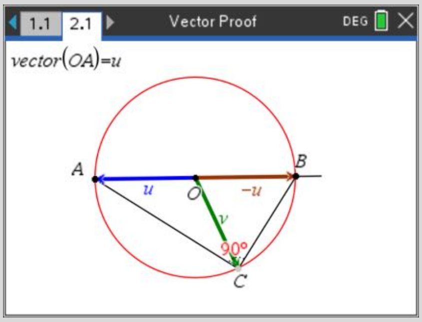
To construct a semicircle, on a **Geometry page:**

- Press menu > **Geometry** > **Shapes** > **Circle**.
- Click on the page to position the centre point of the circle, then drag and click on the page to set the radius. Hover over the centre point, press ctrl menu > **Label** and enter the label *O*.
- Press menu > **Geometry** > **Points & Lines** > **Ray**. Construct the diameter by clicking on a circumference point and then on the centre. Click and hold the end of the ray and then extend it across the circle.



To draw the inscribed triangle and vectors:

- Press menu > **Geometry** > **Shapes** > **Triangle**. Click on each intersection point of the ray and the circle, then click on a third point on the circumference. Press esc to exit the tool.
- Press menu > **Geometry** > **Points and Lines** > **Vector**. Construct vectors $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$.
- Label the objects as shown. Hover over each object, press ctrl menu > **Label**, and enter the label as required.
- Press menu > **Geometry** > **Measurement** > **Angle**. Click the points A, C, B. Drag point *C* and observe $\measuredangle ACB$. Observe that $\measuredangle ACB$ is 90 degrees.



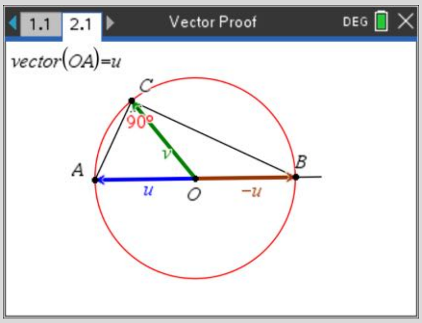
To prove $\overrightarrow{AC} \perp \overrightarrow{BC}$:

$$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = \left(-\underset{\sim}{u} + \underset{\sim}{v}\right) \text{ and } \overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC} = \left(\underset{\sim}{u} + \underset{\sim}{v}\right)$$

$$\overrightarrow{AC} \cdot \overrightarrow{BC} = \left(-\underset{\sim}{u} + \underset{\sim}{v}\right) \cdot \left(\underset{\sim}{u} + \underset{\sim}{v}\right) = -\underset{\sim}{u} \cdot \underset{\sim}{u} - \underset{\sim}{u} \cdot \underset{\sim}{v} + \underset{\sim}{v} \cdot \underset{\sim}{u} + \underset{\sim}{v} \cdot \underset{\sim}{v}$$

$$\overrightarrow{AC} \cdot \overrightarrow{BC} = \left|\underset{\sim}{v}\right|^2 - \left|\underset{\sim}{u}\right|^2 = 0 \text{ because } \left|\underset{\sim}{v}\right| = \left|\underset{\sim}{u}\right| = \text{radius.}$$

Therefore $\overrightarrow{AC} \perp \overrightarrow{BC}$. Angle *ACB* is a right angle, as required.

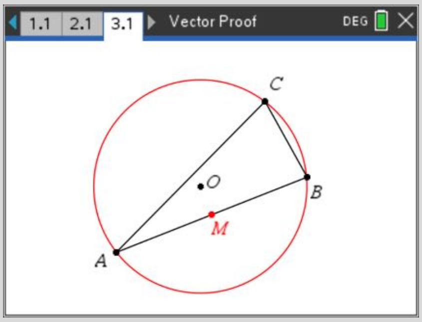


Proving, with the use of vectors, the converse of the inscribed angle semicircle theorem

Use vectors to prove the converse of the inscribed angle semicircle theorem, which was proved above, is also true. That is, prove that if an inscribed angle is a right angle, then the midpoint of the opposite side is the centre of the circle.

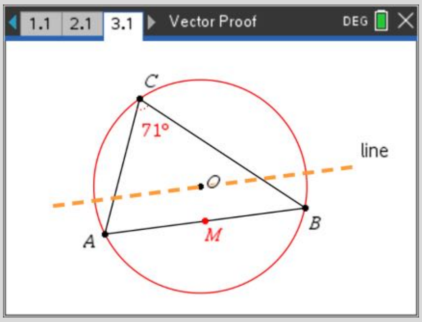
To construct a diagram of this situation, on a **Geometry** page:

- Press menu > **Geometry** > **Shapes** > **Circle**.
- Click on the page to position the centre point of the circle, then drag and click to set the radius. Hover over the centre point and label it *O*.
- Press menu > **Geometry** > **Shapes** > **Triangle**. Click on three arbitrary points on the circumference to form triangle *ABC*.
- Press menu > **Geometry** > **Construction** > **Midpoint** then click on side *AB*. Label the midpoint ***M***, as described above.



To visually verify that the converse theorem is true:

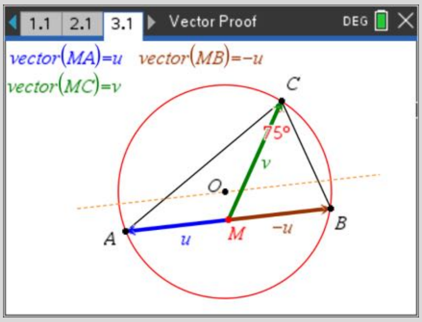
- Press menu > **Geometry** > **Construction** > **Parallel**, click on side *AB* and then on point *O*. Grab and extend the parallel line.
- Press menu > **Geometry** > **Measurement** > **Angle**. Click on the points A, C, B to measure $\measuredangle ACB$.
- In turn, drag the points *A*, *B* and *C* around the circle. Observe the measurements of $\measuredangle ACB$. Observe the location of point *M* and the size of side *AB* when the angle measurement is 90°.



To set up the diagram for the vector proof:

- Press menu > **Geometry** > **Points and Lines** > **Vector**. Construct vectors $\overrightarrow{MA}, \overrightarrow{MB}, \overrightarrow{MC}$.
- Label the objects as shown. Hover over each object, press ctrl menu > **Label**, and enter the label as required.

Note: *To hide unwanted objects or labels, hover over the object, press* ctrl menu *>* ***Hide****. To 'unhide' hidden objects, press* menu *>* ***Actions*** *>* ***Hide/Show****, then click on the object.*



To prove that if $\overrightarrow{AC} \perp \overrightarrow{BC}$ then *M* is the centre of the circle:

$$\overrightarrow{AC} = \overrightarrow{AM} + \overrightarrow{MC} = \left(-\underset{\sim}{u} + \underset{\sim}{v}\right) \text{ and } \overrightarrow{BC} = \overrightarrow{BM} + \overrightarrow{MC} = \left(\underset{\sim}{u} + \underset{\sim}{v}\right)$$

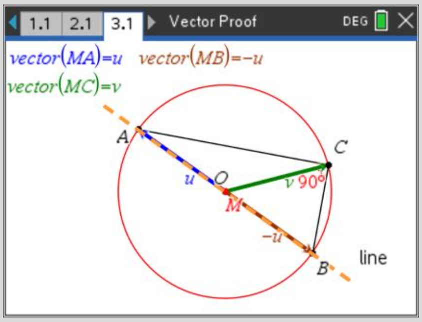
$\overrightarrow{AC} \cdot \overrightarrow{BC} = 0$ because it is given that $\overrightarrow{AC} \perp \overrightarrow{BC}$

$$\overrightarrow{AC} \cdot \overrightarrow{BC} = \left(-\underset{\sim}{u} + \underset{\sim}{v}\right) \cdot \left(\underset{\sim}{u} + \underset{\sim}{v}\right) = -\underset{\sim}{u} \cdot \underset{\sim}{u} - \underset{\sim}{u} \cdot \underset{\sim}{v} + \underset{\sim}{u} \cdot \underset{\sim}{v} + \underset{\sim}{v} \cdot \underset{\sim}{v}$$

Therefore $-\underset{\sim}{u} \cdot \underset{\sim}{u} + \underset{\sim}{v} \cdot \underset{\sim}{v} = 0 \Leftrightarrow \underset{\sim}{v} \cdot \underset{\sim}{v} = \underset{\sim}{u} \cdot \underset{\sim}{u}$

$$\left|\underset{\sim}{v}\right|^2 = \left|\underset{\sim}{u}\right|^2 \Rightarrow \left|\underset{\sim}{v}\right| = \left|\underset{\sim}{u}\right|$$

Since $d(MA) = d(MC) =$ radius, *M* must be the circle centre.



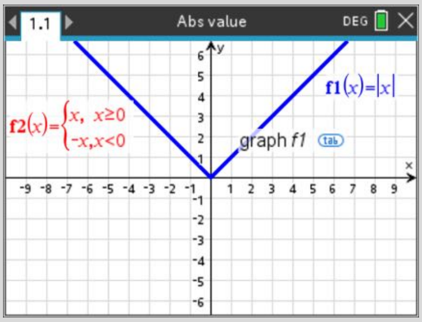
2.4. Topic 4: Trigonometry and functions

2.4.1. Sketching graphs

Introducing the absolute value function

Sketch the graphs of the following functions and determine the range of each function.

(a) $f_1(x)=|x|$ **(b)** $f_2(x)=\begin{cases} x & x\ge 0 \\ -x & x<0 \end{cases}$ **(c)** $f_3(x)=|x+1|-3$ **(d)** $f_4(x)=3-|x+1|$

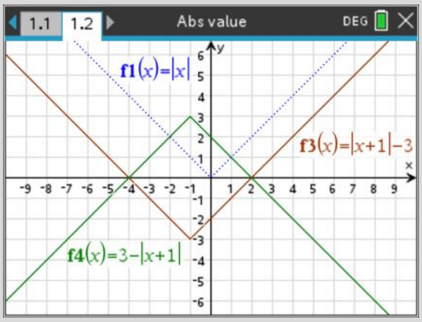


To plot the graphs for parts **(a)** and **(b)** and determine the range, on a **Graphs** page:

- Enter $\boldsymbol{f1(x)=|x|}$ by pressing [template key] to select the absolute value template, $|\square|$.

Note: *Entering* **abs(*x*)** *has the same effect as entering* $|\boldsymbol{x}|$.

- Press [tab] to open the graph entry line, then enter $\boldsymbol{f2(x)=\begin{cases} x & x\ge 0 \\ -x & x<0 \end{cases}}$ by pressing [template key] to select the **Piecewise function** template
- Hover over the graph and press [tab] to toggle between the graphs.



To plot the graphs for parts **(c)** and **(d)** and determine the range, on a **Graphs** page:

- Enter $\boldsymbol{f3(x)=|x+1|-3}$ then $\boldsymbol{f4(x)=3-|x+1|}$.

Answer: $f_1(x)=|x|=f_2(x)$. Range f_1, f_2 is $[0, \infty)$.

$f_3(x)=f_1(x+1)-3$. Range f_3 is $[-3, \infty)$.

$f_4(x)=-f_3(x)=-(f_1(x+1)-3)$. Range f_4 is $(-\infty, 3]$.

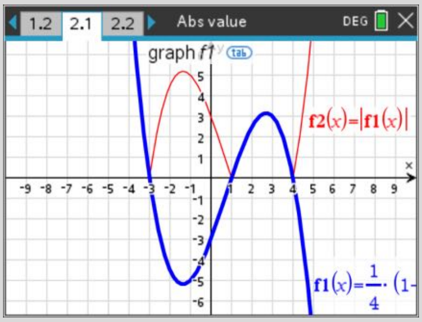
Exploring the relationship between the graphs of $y = f(x)$, $y = |f(x)|$ and $y = f(|x|)$

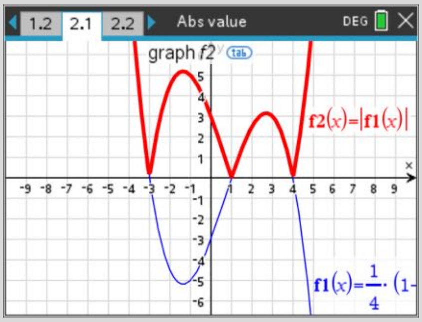
Sketch the graphs of the following functions and determine if any relationships exist between the functions.

(a) $f_1(x) = \frac{1}{4}(1-x)(x+3)(x-4)$ **(b)** $f_2(x) = |f_1(x)|$ **(c)** $f_3(x) = f_1(|x|)$

To plot the graphs of $y = f_1(x)$ and $y = |f_1(x)|$, and determine their relationship, on a **Graphs** page in a **New Problem**:

- Enter $\boldsymbol{f1(x) = \frac{1}{4}(1-x)(x+3)(x-4)}$, observe key features of the graph, then enter $\boldsymbol{f2(x) = |f1(x)|}$.
- Hover over the upper-left branch of the graphs and press tab to toggle between graphs and to observe the relationship.



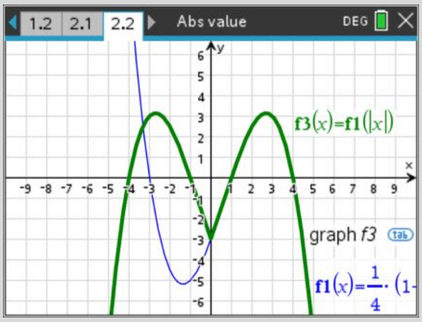


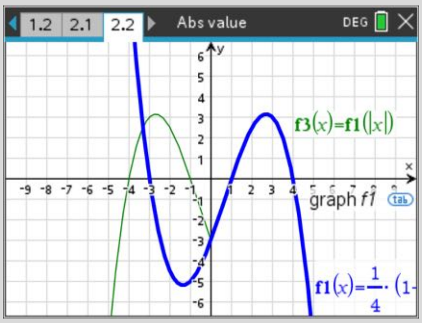
The graph of $y = |f_1(x)|$ has the negative parts of the graph of $y = f_1(x)$ (i.e. where $f_1(x) < 0$) reflected in the *x*-axis.

Therefore, $|f_1(x)| = \begin{cases} f_1(x), & f_1(x) \geq 0 \\ -f_1(x), & f_1(x) < 0 \end{cases}$.

To draw the graphs of $y = f_1(x)$ and $y = f_1(|x|)$, and determine their relationship, add a **Graphs** page to the existing problem, then:

- In the graph entry line, select $\boldsymbol{f1(x)}$, and press enter to plot the graph, and then enter $\boldsymbol{f3(x) = f1(|x|)}$.
- Hover over the lower-right branch of the graphs and press tab to toggle between graphs and to observe the relationship.





The graph of $y = f_1(|x|)$ is symmetrical about the *y*-axis because $f_1(|x|) = \begin{cases} f_1(x), & x \geq 0 \\ f_1(-x), & x < 0 \end{cases}$.

The graph of $y = f_1(|x|)$ for $x \in (-\infty, \infty)$ is a reflection in the *y*-axis of the graph of $y = f_1(x)$ for $x \in [0, \infty)$, the non-negative values of *x*.

2.4.2. The reciprocal trigonometric functions, secant, cosecant and cotangent

Graphing reciprocal trigonometric functions

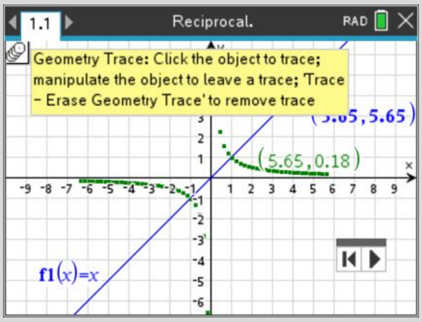
Use a 'pointwise first principles' approach to establish the key features of the graphs of

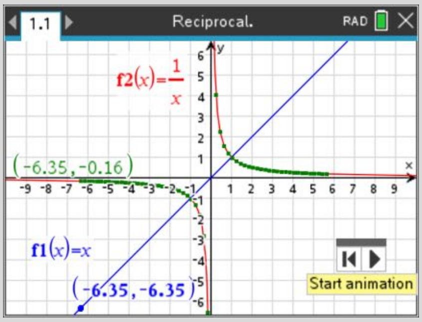
(a) $y=\dfrac{1}{x}$ **(b)** $y=\dfrac{1}{\sin(x)}$ **(c)** $y=\dfrac{1}{\tan(x)}$.

(a) To obtain a plot of $y=\dfrac{1}{x}$ using a 'pointwise first principles' approach, on a **Graphs** page:

- Enter $\boldsymbol{f1(x)=x}$.
- Press [P] > **Point**. Click on the graph $\boldsymbol{f1}$ to place a point on the graph and edit the *x*-coordinate to obtain $(\mathbf{6,6})$.
- Hover over the *x*-coordinate value, press [var] and assign the variable $\boldsymbol{x1}$. Repeat for the *y*-coordinate with $\boldsymbol{y1}$.
- Hover over the point, press [ctrl] [menu] > **Attributes**. Press ▼ to **Unidirection speed**, press [1] followed by [enter] [enter].
- Press [P] > **Point by Coordinates**. Edit the coordinates to be $(\boldsymbol{x1, 1/y1})$.
- Hover over the point, press [ctrl] [menu] > **Geometry Trace**.
- Use the control buttons to start, pause or reset the animation of point $(\boldsymbol{x1, y1})$.
- Enter $(\boldsymbol{f2(x)=1/x})$. Compare this graph and the plot – what do you notice?

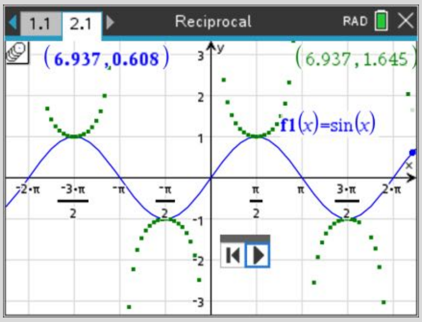
Note: Press [esc] is an important instruction when exiting a tool.

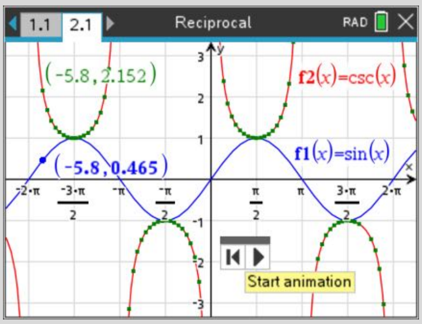




(b) To obtain a pointwise plot of $y=\dfrac{1}{\sin(x)}$:

- On the document from part **(a)** above, press [ctrl] ▲. In thumbnail view, press ▲ to select **Problem 1**.
- Press [ctrl] [C] then [ctrl] [V] to obtain a 'clone' of **Problem 1**.
- On **page 2.1**, hide $\boldsymbol{f2}$ and edit $\boldsymbol{f1}$ to $\boldsymbol{f1(x)=\sin(x)}$.
- Press [menu] > **Window/Zoom** > **Window Settings**. In the dialog box that follows, enter the following values:
  Xmin: $-9\pi/4$ Xmax: $9\pi/4$ XScale: $\pi/2$
  Ymin: -3.33 Ymax: 3.33 YScale: 1
- Edit *x*-coordinate of $(\boldsymbol{x1, y1})$ to -0.58, then hover over point $(\boldsymbol{x1, 1/y1})$ and press [ctrl] [menu] > **Geometry Trace**.
- Using the control buttons, animate the point $(\boldsymbol{x1, y1})$.
- Unhide $\boldsymbol{f2}$ and edit to $\boldsymbol{f2(x)=\csc(x)}$. Observe the relationship of $\boldsymbol{f2}$ to the pointwise plot. Then edit $\boldsymbol{f2}$ to $\boldsymbol{f2(x)=1/\sin(x)}$ and observe the relationship.

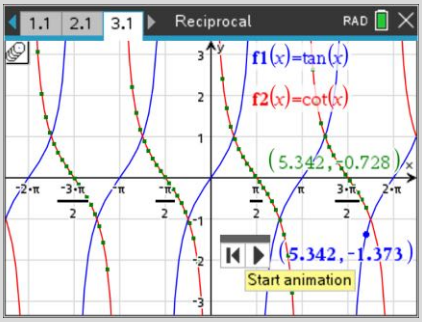




... continued

***Note**: Press* ctrl menu 3 *to hide/show a graph.*

(c) To obtain a pointwise plot of $y=\dfrac{1}{\tan(x)}$:



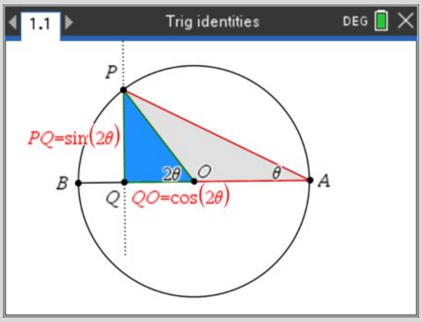
- On the document from part **(b)** above, press ctrl ▲. In thumbnail view, press ▲ to select **Problem 2**.
- Press ctrl C then ctrl V to obtain a 'copy' of **Problem 2**.
- On page 3.1, hide $\boldsymbol{f2}$ and edit $\boldsymbol{f1}$ to $\boldsymbol{f1(x)=\tan(x)}$.
- Edit x-coordinate of $(\boldsymbol{x1,y1})$ to -0.58, then hover over point $(\boldsymbol{x1,1/y1})$ and press ctrl menu > **Geometry Trace**.
- Using the control buttons, animate the point $(\boldsymbol{x1,y1})$.
- Unhide $\boldsymbol{f2}$ and edit to $\boldsymbol{f2(x)=\cot(x)}$. Observe the relationship of $\boldsymbol{f2}$ to the pointwise plot. Then edit $\boldsymbol{f2}$ to $\boldsymbol{f2(x)=1/\tan(x)}$ and observe the relationship.

Proving the double angle identity

With the aid of the **Geometry** application, prove that:

(a) $\cos(2\theta)=2\cos(\theta)-1$ **(b)** $\sin(2\theta)=2\cos(\theta)\sin(\theta)$

To set up the first part of the proof, on a **Geometry** page:



- Draw the unit circle: press menu > **Shapes** > **Circle**. Click to set the centre then drag and click again and move the cursor to set the radius.
- To draw diameter *AB*, press menu > **Points & Lines** > **Ray**, click on the circle at *A* then click on the centre and extend the ray through centre *O* to *B*, as shown.

***Note**: Press* esc *is an important instruction when exiting a tool.*
***Note:** To label a point, hover over the point then press* ctrl menu > ***Label** and enter the desired label.*

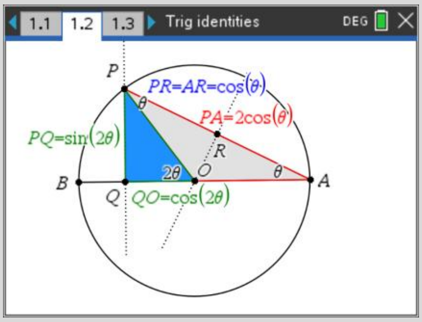
- To construct *PQ*, press menu > **Construction** > **Perpendicular**. Click on the line *AB* then point *P*.
- Press menu > **Points & Lines** > **Segment** and draw segments *AP, OA* and *OP*.
- Label all points as shown.

***Note**: Part 1 of the proof.*
For the angles on arc PB, if $\measuredangle PAB=\theta$, *then* $\measuredangle POB=2\theta$.
Hence for ΔPOQ, $PO=1$, $QO=1\cdot\cos(2\theta)=\cos(2\theta)$,
$PQ=1\cdot\sin(2\theta)=\sin(2\theta)$.

... continued

For the second part of the proof, on **page 1.1** above:

- Press ctrl ▲. In thumbnail view, press ctrl C then ctrl V to obtain a 'copy' of **page 1.1** on **page 1.2**.
- On **page 1.2**, construct *OR*. Press menu > **Construction** > **Perpendicular**. Click on segment *PA* and then click on point *O*.
- Construct and label the point R, by pressing menu > **Points and Lines** > **Intersection**.

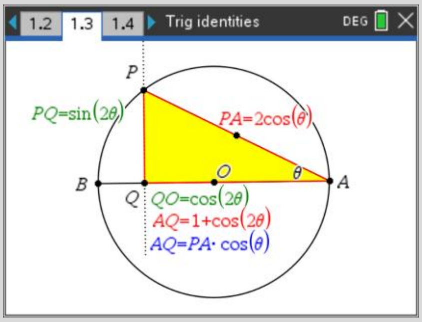


Part 2 of the proof. $PO = AO = 1$, hence $\Delta PRO \cong \Delta ARO$ and $PR = AR = 1 \times \cos(\theta)$. Therefore, $PA = 2 \times \cos(\theta)$.

***Note:** In this chapter, the symbol ≅ is used to denote **congruence** of shapes.*

For the third part of the proof, on **page 1.2** above:

- Press ctrl ▲. In thumbnail view, press ctrl C then ctrl V to obtain a 'copy' of **page 1.2** on **page 1.3**.
- On **page 1.3**, press menu > **Actions** > **Hide/Show**. Click to hide objects, except those for ΔPAQ, as shown.



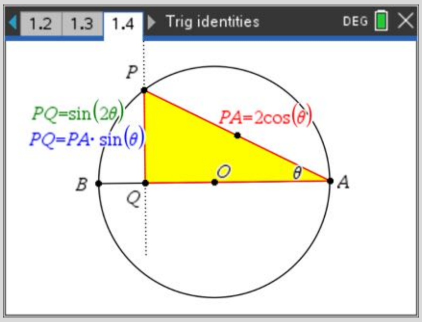
Part 3 of proof. $AQ = 1 + \cos(2\theta) = PA\cos(\theta)$ (equation 1)

(a) Substituting $PA = 2\cos(\theta)$ in equation 1,

$1 + \cos(2\theta) = 2\cos(\theta)\cos(\theta) \Leftrightarrow \cos(2\theta) = 2\cos^2(\theta) - 1$.

For the fourth part of the proof, on **page 1.3** above:

- Press ctrl ▲. In thumbnail view, press ctrl C then ctrl V to obtain a 'copy' of **page 1.3** on **page 1.4**.
- On **page 1.4**, press menu > **Actions** > **Hide/Show**. Click to hide objects, except for those shown.



Part 4 of proof. $PQ = \sin(2\theta) = PA\sin(\theta)$ (equation 2)

(b) Substituting $PA = 2\cos(\theta)$ in equation 2,

$\sin(2\theta) = 2\cos(\theta)\sin(\theta)$, as required.

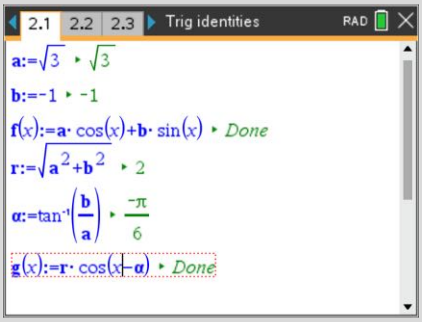
Exploring equivalent forms of $a\cos(x)+b\sin(x)$

Show that the graph of $f(x)=\sqrt{3}\cos(x)-\sin(x)$ is equivalent to the graph of a function of the form $g(x)=r\cos(x-\alpha)$, with appropriate amplitude and phase shift.

Hence solve the equation $f(x)=-1, x\in[-2\pi,2\pi]$.

To determine the values of r and α, on a **Notes** page:

- Insert **Maths Boxes** by pressing ctrl M and enter values for the coefficients ***a*** and ***b***, and the formulas to determine ***r*** and $\boldsymbol{\alpha}$, as shown.

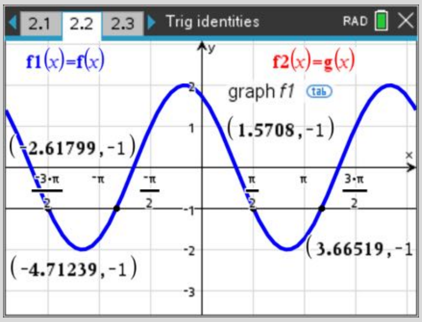


Answer: $\sqrt{3}\cos(x)-\sin(x)=2\cos\left(x+\dfrac{\pi}{6}\right)$

Note: *The alpha character 'α' can be found via* ctrl 📖.

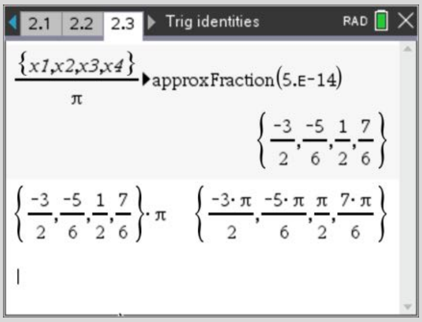
To compare the graphs of $f(x)=\sqrt{3}\cos(x)-\sin(x)$ and $g(x)=2\cos\left(x+\dfrac{\pi}{6}\right)$, on a **Graphs** page:

- Enter $\boldsymbol{f1(x)=f(x)}$, $\boldsymbol{f2(x)=g(x)}$ and $\boldsymbol{f3(x)=-1}$
  Press menu > **Window Zoom > Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin: -2π XMax: 2π XScale: $\pi/2$
  YMin: –3.33 YMax: 3.33 YScale: 1



To solve $f(x)=-1, x\in[-2\pi,2\pi]$, on the **Graphs** page:

- Press menu > **Geometry > Points & Lines > Intersection Point(s)**. Click on graph $\boldsymbol{f1}$ then on graph $\boldsymbol{f3}$.
  Hover over the x-coordinate of the leftmost intersection point, press var and store as ***x*1**. Similarly, store the other x-coordinates: ***x*2**, ***x*3**, ***x*4**.
- For solutions in terms of π, on a **Calculator** page enter $\dfrac{\{\mathbf{x1,x2,x3,x4}\}}{\boldsymbol{\pi}}$**►approxFraction**, then enter $\mathbf{ans}\cdot\boldsymbol{\pi}$.



Note:* ►approxFraction** *is found in* ***Number *menu.*

2.5. Topic 5: Matrices and transformations

2.5.1. Transformations in the plane

Representing translations as column vectors

The lines $y=-\frac{7}{3}$ and $x=7$ intersect at point R. These lines intersect the line with equation $3x+y=7$ at points P and Q, respectively.

(a) Show on the Cartesian plane the image of triangle PQR under this transformation $\begin{bmatrix}-6\\4\end{bmatrix}$.

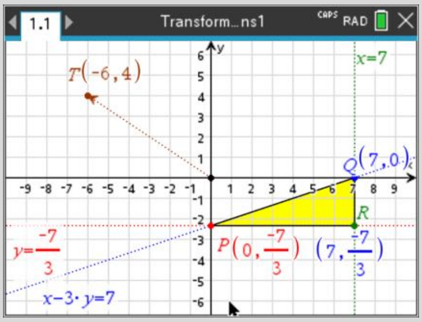
Hence find the coordinates of the vertices of the image of triangle PQR under the transformation.

(b) Determine the equation of the line that passes through the image of points P and Q.

(c) Explore the image of triangle PQR under the transformation $\begin{bmatrix}a\\b\end{bmatrix}$ for various values of a and b.

To construct ΔPQR and the translation vector, on a **Graphs** page:

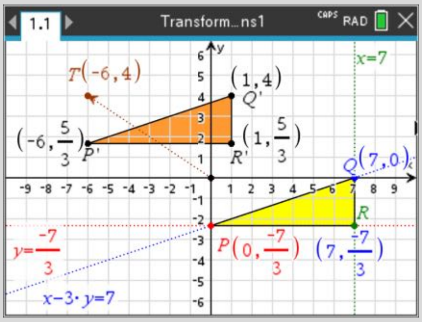
- Press menu > **Graph Entry/Edit** > **Relation** and enter the relations $\boldsymbol{x-3y=7}$, $\boldsymbol{y=-\frac{7}{3}}$ and $\boldsymbol{x=7}$.
- Press menu > **Geometry** > **Points & Lines** > **Intersection Point(s)**. Click on the lines in pairs and then label the vertices.
- Edit intersection point coordinates to exact values.
- Press P > **Point by Coordinates**. Enter $(\boldsymbol{-6,4})$ and label as shown.
- Press menu > **Geometry** > **Points & Lines** > **Vector**. Click on the origin and then on the point at $T(-6,4)$.



Note: *To label a point, hover over the point, press* ctrl menu > ***Label*** *and enter the label.*

(a) To obtain the image of ΔPQR under the transformation:

- Press menu > **Geometry** > **Transformation** > **Translation**. Click on ΔPQR and then click on the vector to $T(-6,4)$.
- Hover over point P', press ctrl menu > **Coordinates and Equations**. Repeat for Q' and R'. Edit these coordinates to exact form.



Coordinates of the image are $P'\left(-6,\frac{5}{3}\right), Q'(1,4), R'\left(1,\frac{5}{3}\right)$

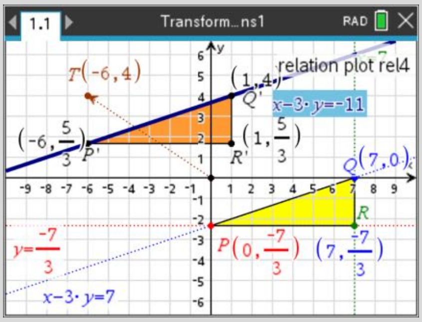
... *continued*

(b) To verify the equation of the line through $P'Q'$:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -6 \\ 4 \end{bmatrix} \Rightarrow \begin{matrix} x = x' + 6 \\ y = y' - 4 \end{matrix}.$$

$x - 3y = 7 \to (x'+6) - 3(y'-4) = 7$, or $x' - 3y' = -11$

- Enter the relation with equation $\boldsymbol{x - 3y = -11}$.



(c) To explore the image of ΔPQR under the transformation $\begin{bmatrix} a \\ b \end{bmatrix}$, edit/drag the vector endpoint coordinates to $(\boldsymbol{a}, \boldsymbol{b})$.

Representing dilations of the form $(x, y) \to (ax, by)$ as matrices

The triangle PQR from the previous problem is transformed according to $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1.5 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

(a) Determine the coordinates of the image of points P, Q and R under this transformation.

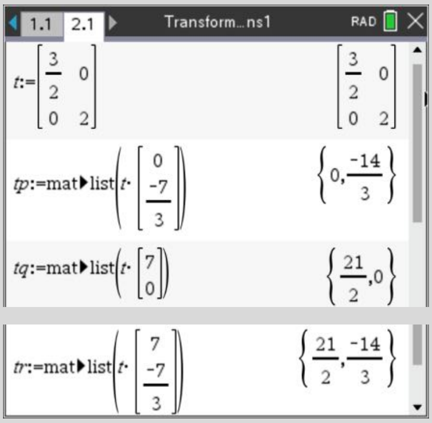
(b) Plot the image of triangle PQR and compare the area of PQR and the area of its image.

(c) Calculate the determinant of the dilation matrix and interpret its geometric significance.

(d) Use the **Dilation** tool to explore transformations of the form $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

(a) To determine the coordinates of the image of $P(0,-7/3), Q(7,0)$ and $R(7,-7/3)$, in the document from the previous problem, add a **Calculator** page to a **New Problem**.

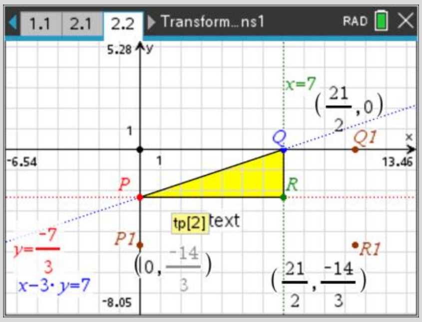
- Enter $\boldsymbol{t} := \begin{bmatrix} \mathbf{3/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{2} \end{bmatrix}$, then $\boldsymbol{tp} :=$ **mat▶list** $\left(\boldsymbol{t} \cdot \begin{bmatrix} \mathbf{0} \\ \mathbf{-7/3} \end{bmatrix}\right)$.
- Copy and paste the previous entry and edit it to $\boldsymbol{tq} :=$ **mat▶list** $\left(\boldsymbol{t} \cdot \begin{bmatrix} \mathbf{7} \\ \mathbf{0} \end{bmatrix}\right)$ and $\boldsymbol{tr} :=$ **mat▶list** $\left(\boldsymbol{t} \cdot \begin{bmatrix} \mathbf{7} \\ \mathbf{-7/3} \end{bmatrix}\right)$



Note: *To select the '**mat▶list**' command, press* [book] [M].

(b) To plot the image of the vertices of ΔPQR:

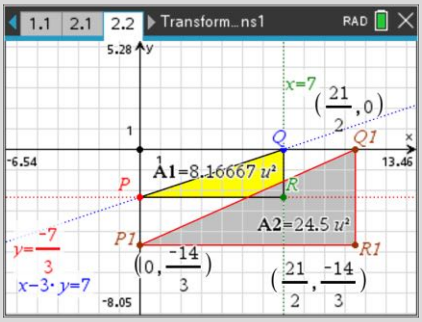
- Press [ctrl] ▲. In thumbnail view, select **page 1.1**, press [ctrl] [C], select **Problem 2** then press [ctrl] [V] to obtain a 'copy' of **page 1.1** on **page 2.2**.
- On **page 2.2**, delete all objects except those for ΔPQR.
- Press [P] > **Point by coordinates**. Enter coordinates $(\boldsymbol{tp}[1], \boldsymbol{tp}[2])$. Repeat for $(\boldsymbol{tq}[1], \boldsymbol{tq}[2])$ and $(\boldsymbol{tr}[1], \boldsymbol{tr}[2])$.
- Press [ctrl] [hand]. Drag the workspace so reveal all points.



... continued

To find the area of ΔPQR and its image, on **page 2.2**:

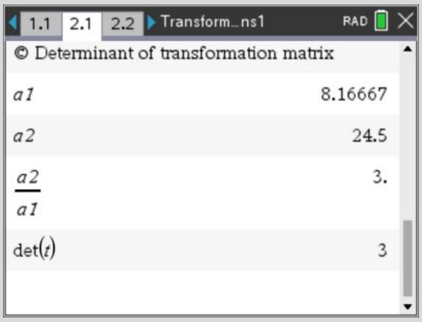
- Label the image of the vertices: $P1$, $Q1$ and $R1$, as shown.
- Press [menu] > **Shapes** > **Triangle**. Construct $\Delta P1Q1R1$.
- Press [menu] > **Geometry** > **Measurement** > **Area**. Click on ΔPQR then $\Delta P1Q1R1$ to find their areas.
- Hover over the area measure for ΔPQR, press [var] > **Store Var** and enter **A1**. Repeat for $\Delta P1Q1R1$, and enter **A2** instead.



(c) To explore the geometric meaning of the determinant:

- On **page 2.1**, enter the ratio of areas: $\dfrac{A2}{A1}$.
- To calculate the determinant of matrix T, enter $\mathbf{det}(\boldsymbol{t})$.

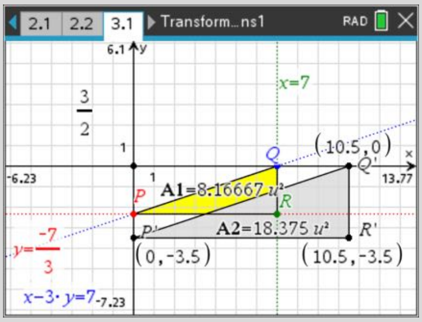
In this case, the determinant is equal to the ratio of the area of the object, triangle PQR, and its image. This can be explored further with other transformation matrices.



(d) To explore the **Dilation** tool, on a 'copy' of **page 1.1**:

- Delete all objects except those for ΔPQR.
- For a dilation of $a = 1.5$ parallel to both x and y axes, press [menu] > **Geometry** > **Transformation** > **Dilation**.
- Click on ΔPQR, then click on the origin and then enter **3/2**.
- Hover over point P', press [ctrl] [menu] > **Coordinates and Equations**. Repeat for points Q' and R'.

Note that, as in in previous case, $A2 / A1 = \det(T)$.

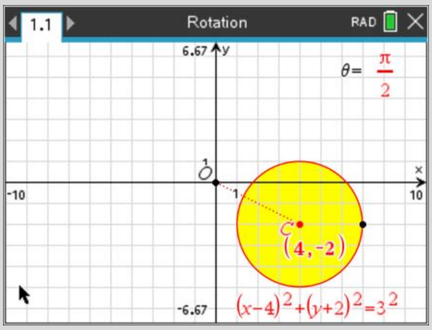


Exploring rotation of angle θ anticlockwise about the origin: $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

Use the **Rotation** transformation tool, together with rotation matrices, to explore the coordinates of the image of the centre of circles with equation $(x-h)^2 + (y-k)^2 = r^2$ under different rotation angles, including $\theta = \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}$ and $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}$.

To set up the **Rotation** tool exploration, on a **Graphs** page:

- Press [menu] > **Geometry** > **Shapes** > **Circle**. To draw the circle, click on the point at $(\mathbf{4}, \mathbf{-2})$ and then on the point at $(\mathbf{7}, \mathbf{-2})$.
- Label centre C. Press [menu] > **Actions** > **Coordinates and Equations**. Click the circle circumference and point C.
- To input the rotation angle, with the cursor near the top right corner, press [ctrl] [menu] > **Text**. In the textbox, enter $\boldsymbol{\pi} / \mathbf{2}$.

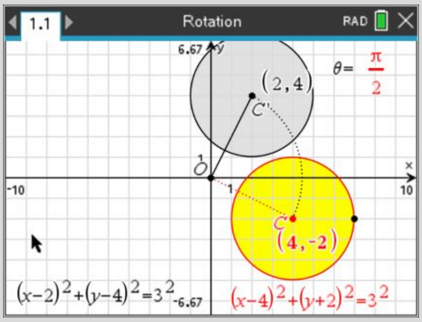


... continued

- Label the origin, O. Draw a line segment OC.

Note: *To label a point, hover over the point, press* ctrl menu *>* ***Label*** *and enter the label.*

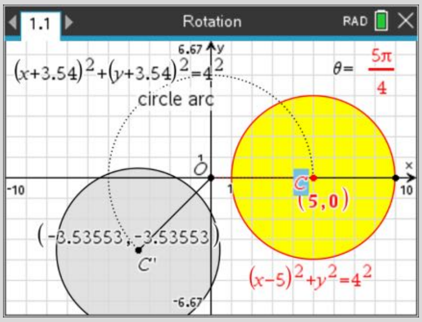
To apply the rotation to the circle and segment:



- Press menu > **Geometry** > **Transformation** > **Rotation**. Click on the circle, then the origin, then the angle textbox.
- Click in turn on: the segment OC, the origin, the angle textbox.
- Press menu > **Actions** > **Coordinates and Equations**. Click on the images, C' and on the circle centred at C'.

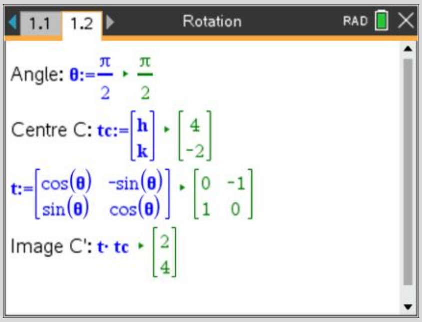
Note: *The image of the circle and segment will now interactively update if a change is made to the circle centre, C, or to the circle radius, or to the rotation angle textbox.*

To explore changing the circle centre and rotation angle:



- Drag point C or the point on the circle to vary the circle centre or radius. Edit the textbox to vary the rotation angle.
- Click in turn on: segment OC, the origin, the angle textbox.
- Press menu > **Actions** > **Coordinates and Equations**. Click on the images: C' and the circle centred at C'. The coordinates of C' and the equation of the circle centred at C' will be interactively displayed following any changes.

To confirm geometric results using the transformation matrix:



- Store the coordinates of C as variables h and k. (Hover over the coordinate, press var and enter variable h or k).
- Add a **Notes** page, and in Maths Boxes, enter the variables, matrices and calculations as shown.

Note: *To insert a* ***Maths Box****, press* ctrl M.

If the coordinates of point C change on **page 1.1**, this will automatically update in vector tc on **page 1.2**. However, changes to θ need to be manually changed on both pages.

Exploring reflection in the x and y axes, geometrically and with matrices

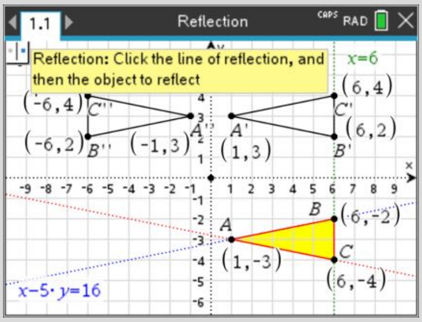
Let R be the region bounded by the lines $x-5y=16$, $x+5y=-14$ and $x=6$. Region R is reflected in the x-axis followed by a reflection in the y-axis.

(a) Use the **Reflection** transformation tool to explore the image of R.

(b) Use the **Rotation** transformation tool and matrix multiplication to show that the combined effect of the two reflections is a rotation.

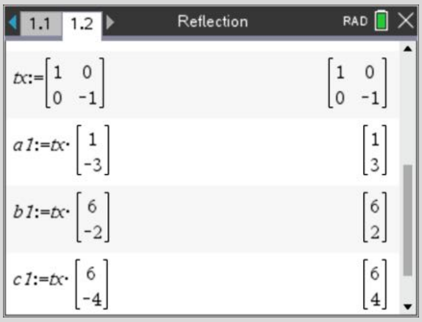
To construct region R and its image, on a **Graphs** page:

- Press menu > **Graph Entry/Edit** > **Relation** and enter the relations $\boldsymbol{x-5y=16}$, $\boldsymbol{x+5y=-14}$ and $\boldsymbol{x=6}$.
- Press menu > **Geometry** > **Shape** > **Triangle**, then click the vertices of the region. Label the vertices A, B, C by hovering over a vertex, pressing ctrl menu > **Label** and entering the label.
- Press menu > **Geometry** > **Transformation** > **Reflection**. Click on the x-axis, then ΔABC.
  Click on the y-axis, then $\Delta A'B'C'$.
- Press menu > **Actions** > **Coordinates and Equations**. To show coordinates, click on the points $A, B, C, A', B', C', A'', B'', C''$.



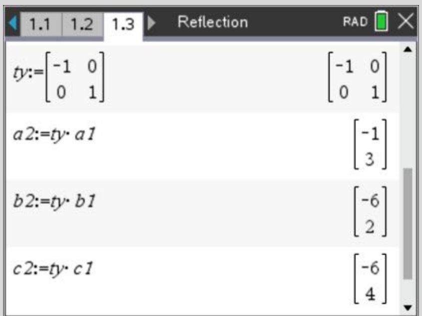
To find the coordinates of the image of the vertices of the triangle ABC, using matrices, on a **Calculator** page:

- Enter $\boldsymbol{tx}:=\begin{bmatrix}\mathbf{1} & \mathbf{0}\\ \mathbf{0} & \mathbf{-1}\end{bmatrix}$, where $\boldsymbol{tx}$ is the matrix for reflection mapping in the x-axis, such that $(x_1,y_1)\to(x_1,-y_1)$.
- Enter the matrix products as shown, where the numbers in the column vectors are the coordinates of A, B and C.



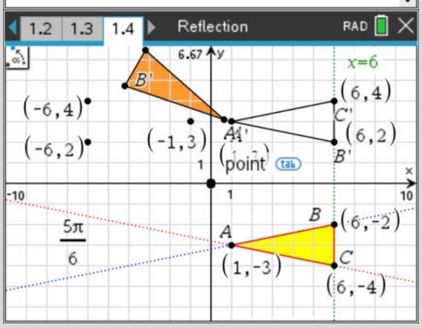
To find the coordinates of the vertices for the combined reflections in the x and y axes, on a **Calculator** page:

- Enter $\boldsymbol{ty}:=\begin{bmatrix}\mathbf{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{1}\end{bmatrix}$, where $\boldsymbol{ty}$ is the matrix for reflection mapping in the y-axis, such that $(x_1,y_1)\to(-x_1,y_1)$.
- Enter the matrix products, as shown, where $a1$, $b1$ and $c1$ are column vectors of the coordinates of A', B' and C'.



To explore geometrically the equivalent rotation:

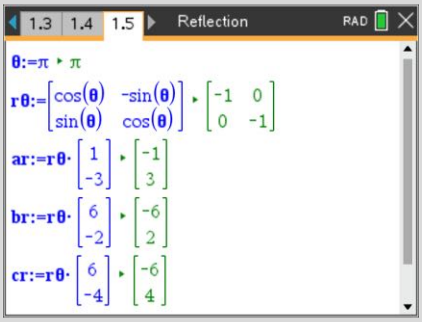
- Copy **page 1.1** (press ctrl ▲, then ctrl C, and ctrl V).
- On copy of **page 1.1**, hide $\Delta A''B''C''$ by pressing menu > **Hide/Show** and clicking the triangle (show only vertices).
- Input the rotation angle: press ctrl menu > **Text**. In the textbox, enter, say, $\mathbf{3\pi/2}$.
- Press menu > **Geometry** > **Transformation** > **Rotation**. Click on ΔABC, then the origin, then the rotation angle text box.
- Edit rotation angle until the rotation image fits the vertices of the double reflection. The rotation $\boldsymbol{\theta=5\pi/6}$ is shown.



... continued

To explore the problem using matrices, and confirm the geometric result that the combined reflections in the x and y axes is equivalent to a rotation of $\theta = \pi$, on a **Notes** page:

- Press [ctrl] [menu] to insert a **Maths Box**, then enter a rotation angle e.g. $\theta := \pi$.
- Insert additional **Maths Boxes** and enter the matrix operations, as shown. Explore the value of θ for which the coordinates of the image of A, B and C correspond to those of the combined reflections in the x and y axes.

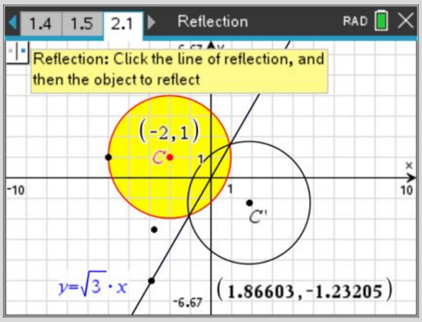


Representing reflection in the line $y = m \cdot x = \tan(\theta) \cdot x$ as matrix $\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$

Let the point C with coordinates $(-2,1)$ be the centre of a circle of radius $r = 3$. Using the **Reflection** transformation tool, together with matrices calculations, show that a reflection in the line $y = \sqrt{3}x$, corresponds to the transformation $\begin{bmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$.

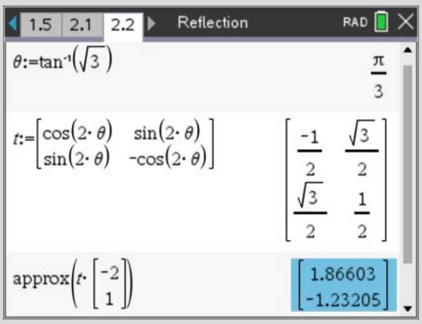
To set up the geometric transformation, on a **Graphs** page:

- Press [menu] > **Graph Entry/Edit** > **Relation** and enter the relation $\boldsymbol{y = \sqrt{3}x}$. Select **Line** from the **Geometry** menu and click on the graph of $\boldsymbol{y = \sqrt{3}x}$ at two distinct points.
- Press [menu] > **Geometry** > **Shapes** > **Circle**. Click on the grid point $C(-2,1)$ then click on $(-5,1)$ for circle with $r = 3$.
- Press [menu] > **Geometry** > **Transformation** > **Reflection**. Click on the line along $\boldsymbol{y = \sqrt{3}x}$, then click on the circle.
- Press [menu] > **Actions** > **Coordinates and Equations**. Click on the point C' to obtain its coordinates.



To confirm the transformation matrix, on a **Calculator** page:

- Enter $\boldsymbol{\theta := \tan^{-1}(\sqrt{3})}$.
- Enter the transformation matrix, as shown.
- To obtain a decimal approximation of the coordinates of the image of C, enter $\mathbf{approx}\left(\boldsymbol{t} \cdot \begin{bmatrix} \mathbf{-2} \\ \mathbf{1} \end{bmatrix}\right)$



Unit 3: Further complex numbers, proof, vectors and matrices

3.1. Topic 1: Further complex numbers

3.1.1. Complex arithmetic using polar form

Verifying some complex number identities

Required complex number identities involving modulus and argument include, for example:

- $z\bar{z} = |z|^2$ where $|z| = \sqrt{a^2 + b^2}$ and $z = a + bi$
- $|z_1||z_2| = |z_1 z_2|$
- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

Note that $\arg(z) = \text{Arg}(z) + 2\pi n$ where $n \in \mathbb{Z}$.

Consider $z = 1 - \sqrt{3}i$ and $w = 1 + i$.

Verify the following results:

(a) $z\bar{z} = |z|^2$ **(b)** $|z||w| = |zw|$ **(c)** $\arg(zw) = \arg(z) + \arg(w)$

(a) On a **Calculator** page, assign z and w as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [ctrl] [x²] to access [√].
- Press [π▸] ▶ to select $\boldsymbol{i}$.
- Enter as shown.

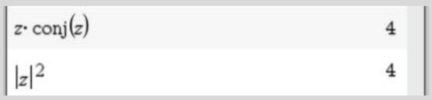
Note: On the keypad, do not confuse complex number $\boldsymbol{i}$ *with the letter i.*

To enter $z\bar{z}$:

- Press [menu] > **Number** > **Complex Number Tools** > **Complex Conjugate.**

To enter $|z|^2$:

- Press [menu] > **Number** > **Complex Number Tools** > **Magnitude.**



Answer: $z\bar{z} = 4$ and $|z|^2 = 4$.

Note: *Alternatively, to find the modulus of a complex number, press* [📖] [5] *and select the* ***Absolute Value*** *template.*

Note: *In general, for* $z = a + bi$, $z\bar{z} = a^2 + b^2$ and $|z|^2 = a^2 + b^2$.

... continued

(b) Enter $|z||w|$ and $|zw|$:

Answer: $|z||w| = 2\sqrt{2}$ and $|zw| = 2\sqrt{2}$.

(c) Enter $\arg(zw)$:

- Press menu > **Number** > **Complex Number Tools** > **Polar Angle.**

Enter $\arg(z)+\arg(w)$:

$\arg(zw) = -\dfrac{\pi}{12}$ and $\arg(z)+\arg(w) = -\dfrac{\pi}{12}$.

Note: *Alternatively, the steps required to calculate* $\arg(z)+\arg(w)$ *could be performed one at a time as shown at right.*

Note: *These argument calculations were performed in* ***Radian*** *mode. To express an argument in degrees while in* ***Radian*** *mode, press ▲ to select one of the arguments and press* enter. *Press* 📖 D *to select* ▶***DMS*** *and press* enter.

Using De Moivre's theorem for integral powers

De Moivre's theorem allows us to simplify expressions of the form z^n when z is expressed in polar form.

- $z^n = (r\text{cis}\theta)^n = r^n\text{cis}(n\theta)$ where $n \in \mathbb{Z}$

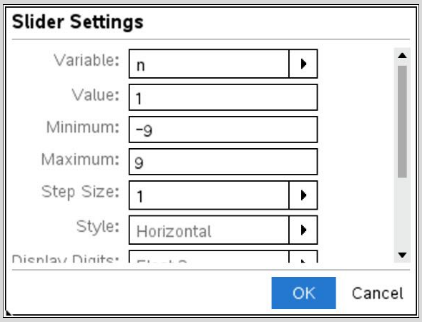
Find the values of n such that $\left(1+\sqrt{3}i\right)^n$ is a real number.

Start with an exploratory verification by using a slider to change the value of n.

On a **Notes** page:

Insert a **Slider** to control the value of n as follows:

- Press menu > **Insert** > **Slider**.
- Set the **Slider Settings** as shown.
- Ensure to check the **Minimised** box.



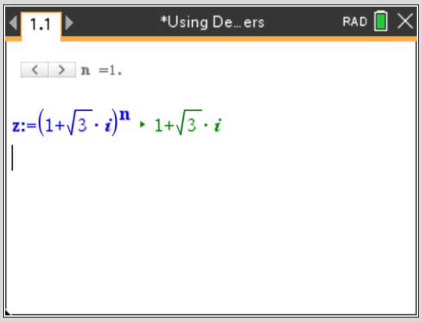
Insert a **Maths Box** as follows:

- Press menu > **Insert** > **Maths Box**.

Note: *Alternatively, to insert a* ***Maths Box****, press* ctrl M.

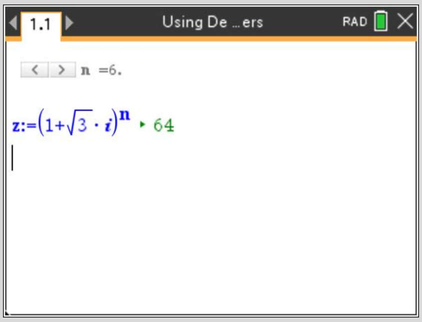
Assign z as follows:

- Press ctrl [var] to access the **Assign** := command.
- Press ctrl x² to access √.
- Press π▸ ▶ to select i.



... continued

Click on the slider to change the value of n.



It appears that $\left(1+\sqrt{3}i\right)^n$ is a real number when $n=0,\pm3,\pm6,\ldots$.

Expressing $1+\sqrt{3}i$ in modulus-argument form:

$$1+\sqrt{3}i=2\operatorname{cis}\left(\frac{\pi}{3}\right)$$

Using De Moivre's theorem:

$$\left(1+\sqrt{3}i\right)^n=2^n\operatorname{cis}\left(\frac{n\pi}{3}\right)=2^n\left(\cos\left(\frac{n\pi}{3}\right)+i\sin\left(\frac{n\pi}{3}\right)\right)$$

$$\sin\left(\frac{n\pi}{3}\right)=0\Rightarrow n=0,\pm3,\pm6,\ldots$$

Answer: So $n=3k$ where $k\in\mathbb{Z}$.

3.1.2. Roots of complex numbers

Determining and examining the nth roots of unity

The solutions of the equation $z^n=1$ are called the nth roots of unity.

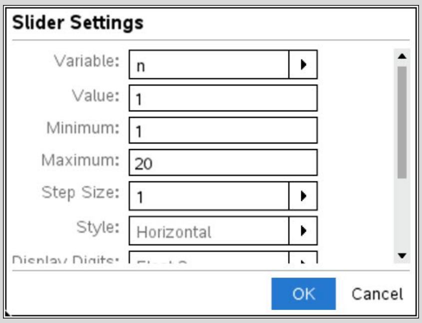
- The solutions of $z^n=1$ lie on the unit circle.
- There are n solutions equally spaced around the circle at intervals of $\frac{2\pi}{n}$.
- $z=1$ is always a solution.

The following instructions describe how to construct a dynamic demonstration of the nth roots of unity.

On a **Notes** page:

Insert a **Slider** to control the value of n as follows:

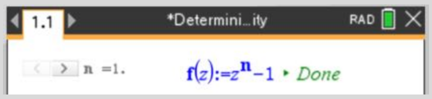
- Press [menu] > **Insert** > **Slider**.
- Set the **Slider Settings** as shown.
- Ensure to check the **Minimised** box.



Insert a **Maths Box** as follows:

- Press [menu] > **Insert** > **Maths Box**.

***Note:** Alternatively, to insert a **Maths Box**, press [ctrl] [M].*

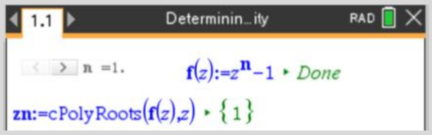


Enter $f(z):=z^n-1$:

- Press [ctrl] [templates] to access the **Assign** [:=] command.

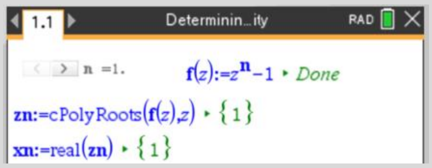
... continued

Enter $zn := \textbf{cPolyRoots}(f(z), z)$:



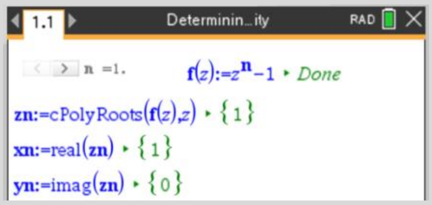
- Press ctrl [:=] to access the **Assign** [:=] command.
- Press menu > **Calculations** > **Algebra** > **Polynomial Tools** > **Complex Roots of Polynomial**.

Enter $xn := \textbf{real}(zn)$:



- Press ctrl [:=] to access the **Assign** [:=] command.
- Either type the command **real** or press menu > **Calculations** > **Number** > **Complex Number Tools** > **Real Part**.

Enter $yn := \textbf{imag}(zn)$:

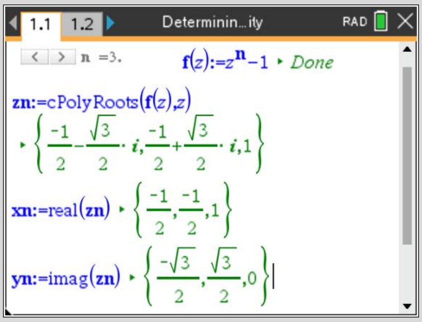


- Press ctrl [:=] to access the **Assign** [:=] command.
- Either type the command **imag** or press menu > **Calculations** > **Number** > **Complex Number Tools** > **Imaginary Part**.

Note: *If desired, to hide the input, press* menu *>* ***Maths Box Options*** *>* ***Maths Box Attributes****. Change* ***Show Input & Output*** *to* ***Hide Input****, press* tab *to highlight* ***OK*** *and press* enter*. Alternatively, to access* ***Maths Box Attributes****, press* ctrl menu *and select* ***Maths Box Attributes****.*

Click on the slider to change the value of n.

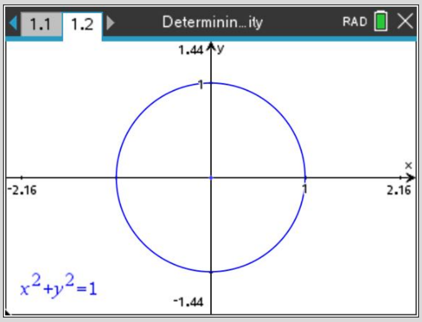
The *n*th roots of unity are displayed.



For example, the screenshot at right shows the roots of the equation $z^3 = 1$, namely, $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}, 1$.

To set up a graphical display of the roots of unity, on a **Graphs** page:

- Press menu > **Window/Zoom** > **Window Settings**.



In the dialog box that follows, enter the following values:

XMin = –2.16	Xmax = 2.16	XScale = 1
YMin = –1.44	YMax = 1.44	YScale = 1

To graph the unit circle $x^2 + y^2 = 1$:

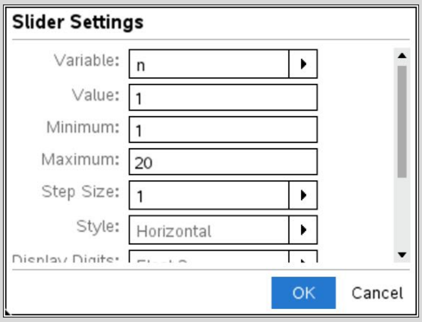
- Press menu > **Graph Entry/Edit** > **Relation**.
- Enter $\boldsymbol{x^2 + y^2 = 1}$.

Note: *Alternatively, to plot the unit circle, press* menu *>* ***Equation Templates*** *>* ***Circle*** *>* ***Centre form*** $\boldsymbol{(x-h)^2 + (y-k)^2 = r^2}$.

... continued

Insert a **Slider** to control the value of n as follows:

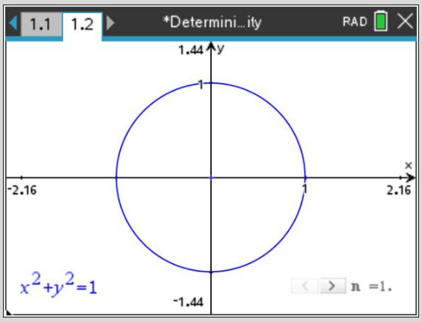
- Press menu > **Actions** > **Insert Slider**.
- Set the **Slider Settings** as shown.
- Ensure to check the **Minimised** box.



To move the **Slider**:

- Click the slider, then press ctrl menu > **Move** and move it to the bottom right-hand corner as shown.

Note*: The slider is moveable when it is framed by a blue border. If the blue border is not showing, click the slider, press* ctrl menu *>* ***Move*** *and move it.*



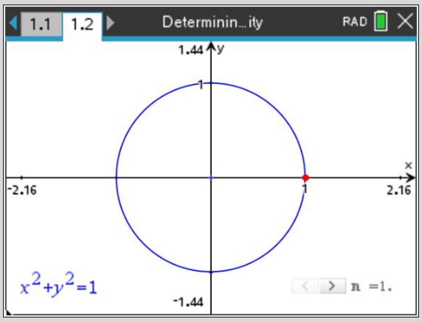
To plot the nth roots of unity:

- Press menu > **Graph Entry/Edit** > **Scatter Plot**.
- Next to $\boldsymbol{x} \leftarrow$ enter $\boldsymbol{xn}$ and next to $\boldsymbol{y} \leftarrow$ enter $\boldsymbol{yn}$.

To hide the coordinates $(\boldsymbol{xn}, \boldsymbol{yn})$:

- Move the cursor over the coordinates and press ctrl menu > **Hide**.

Note: *To see hidden objects, press* menu *>* ***Actions*** *>* ***Hide/Show****. To bring an object back onto a page, move the cursor over the object and press* enter.

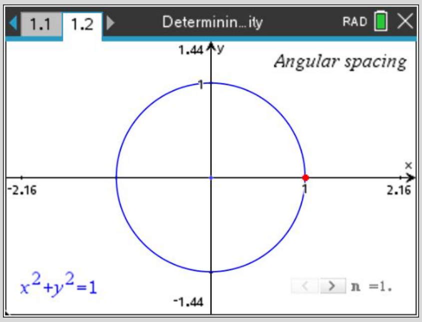


To change the colour of the point at $(1,0)$:

- Move the cursor over the point, press ctrl menu > **Colour** > **Line Colour**.

To add the text '*Angular spacing*' to the top right-hand corner of the page:

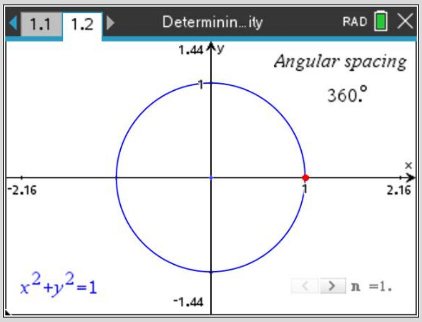
- Press menu > **Actions** > **Text**.
- Move the cursor up towards the top right-hand corner of the page and press enter to open a text box.
- Enter as shown and press enter esc.



... continued

To add a text box displaying the angular spacing in degrees:

- Press [menu] > **Actions** > **Text**.
- Move the cursor up towards the top right-hand corner of the page (underneath the text '*Angular spacing*') and press [enter] to open a text box.
- Enter **360 / *n*** and press [enter] [esc].



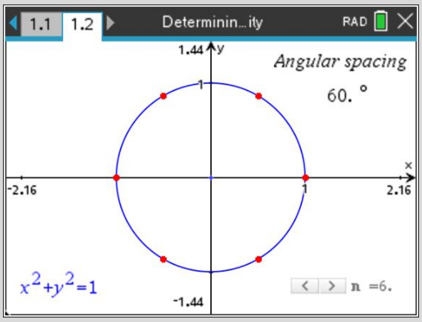
To display the size of the angular spacing for particular values of n :

- Press [menu] > **Actions** > **Calculate**.
- Move the cursor over $\mathbf{\frac{360}{n}}$ and press [enter] [L] [enter] [esc].
- Move the cursor over $\mathbf{\frac{360}{n}}$ and press [ctrl] [menu] > **Hide**.

To add a text box displaying the degrees symbol:

- Press [menu] > **Actions** > **Text**.
- Move the cursor next to the number, press [enter] [π▸] and select the degrees symbol.
- Press [enter] [esc].

Note: *The degrees symbol may need to be moved closer to the number.*



Click on the slider to change the value of n .

The *n*th roots of unity appear with the angular spacing between each root displayed.

For example, the screenshot at right shows the six roots of the equation $z^6 = 1$ with these roots equally spaced around the circle at intervals of $\frac{2\pi}{6} = \frac{\pi}{3} = 60°$.

Determining the nth roots of complex numbers

Equations of the form $z^n = a$ where $a \in \mathbb{C}$ are often solved using De Moivre's theorem.

Given that $z = r\text{cis}(\theta)$ and $a = s\text{cis}(\alpha)$.

By substituting into $z^n = a$, using De Moivre's theorem and comparing both modulus and argument, it can be shown that $\theta = \frac{1}{n}(\alpha + 2k\pi)$ where $k \in \mathbb{Z}$.

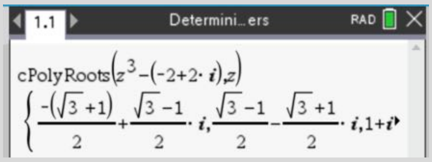
- The solutions of $z^n = a$ lie on a circle with centre the origin and radius $|a|^{\frac{1}{n}}$.
- There are n solutions equally spaced around the circle at intervals of $\frac{2\pi}{n}$.

Solve the equation $z^3 = -2 + 2i$, giving your answers in

(a) Cartesian form **(b)** modulus-argument form.

(a) On a **Calculator** page:

- Press [menu] > **Algebra** > **Polynomial Tools** > **Complex Roots of Polynomial**.
- Enter as shown.



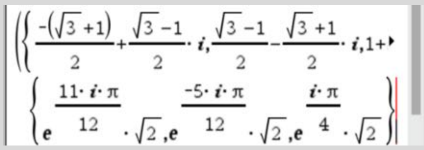
***Note:** The **Complex Roots of Polynomial** command assumes an equation set to zero.*

Answer: In Cartesian form, the roots are

$$\frac{-(\sqrt{3}+1)}{2} + \frac{\sqrt{3}-1}{2}i, \frac{\sqrt{3}-1}{2} - \frac{\sqrt{3}+1}{2}i, 1+i.$$

(b) To express these roots in modulus-argument form:

- Press [ctrl] [(−)] to access [ans].
- Press [menu] > **Number** > **Complex Number Tools** > **Convert to Polar**.
- Enter as shown.



***Note:** Alternatively, press [📖] [P], scroll down and select **▶Polar**.*

Answer: In modulus-argument form, the roots are

$$\sqrt{2}\text{cis}\left(\frac{11\pi}{12}\right), \sqrt{2}\text{cis}\left(-\frac{5\pi}{12}\right), \sqrt{2}\text{cis}\left(\frac{\pi}{4}\right).$$

Expressing $-2+2i$ in modulus-argument form:

$$-2+2i = 2\sqrt{2}\text{cis}\left(\frac{3\pi}{4}\right) \text{ and so } z^3 = 2\sqrt{2}\text{cis}\left(\frac{3\pi}{4}\right).$$

$$z_1 = \sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$$

Adding $\frac{2\pi}{3}$ gives $z_2 = \sqrt{2}\text{cis}\left(\frac{\pi}{4} + \frac{2\pi}{3}\right) = \sqrt{2}\text{cis}\left(\frac{11\pi}{12}\right)$.

Subtracting $\frac{2\pi}{3}$ gives $z_3 = \sqrt{2}\text{cis}\left(\frac{\pi}{4} - \frac{2\pi}{3}\right) = \sqrt{2}\text{cis}\left(-\frac{5\pi}{12}\right)$.

3.1.3. Factorisation of polynomials

Applying the factor theorem and remainder theorem for polynomials

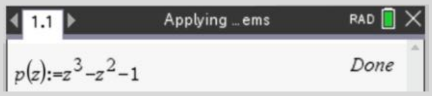
Let $\alpha \in \mathbb{C}$. The remainder theorem states that when a polynomial $p(z)$ is divided by $z-\alpha$, the remainder is $p(\alpha)$.

The factor theorem states that $z-\alpha$ is a factor of a polynomial $p(z)$ if and only if $p(\alpha)=0$.

Consider $p(z)=z^3-z^2-1$.

Find the remainder in modulus-argument form when $p(z)$ is divided by $z-i$.

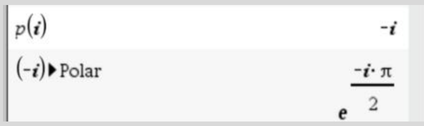
On a **Calculator** page, assign $p(z)$ as follows:



- Press ctrl [templates] to access the **Assign** [:=] command.
- Enter as shown.

The remainder is given by $p(i)$.

- Press [π▸] ▶ to select ***i*** and enter as shown.



To express in modulus-argument form:

- Press ctrl [(−)] to access [ans].
- Press [menu] > **Number** > **Complex Number Tools** > **Convert to Polar**.
- Enter as shown.

Note: *Alternatively, press* [book] [P]*, scroll down and select* ***▶Polar***.

In modulus-argument form when $p(z)$ is divided by $z-i$, the remainder is $\text{cis}\left(-\frac{\pi}{2}\right)$.

Understanding and using the complex conjugate root theorem

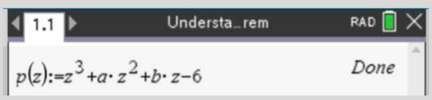
Let $p(z)$ be a polynomial with real coefficients.

If $a+bi$, where $a,b\in\mathbb{R}$, is a solution of the equation $p(z)=0$, then the complex conjugate $a-bi$ is also a solution.

Consider $p(z)=z^3+az^2+bz-6$ where $a,b\in\mathbb{R}$.

Given that $p(-1+i)=0$, find the solutions to the equation $p(z)=0$.

On a **Calculator** page, assign $p(z)$ as follows:



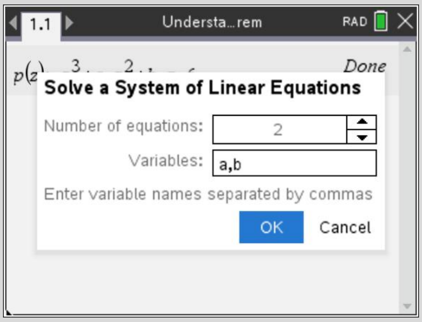
- Press ctrl [templates] to access the **Assign** [:=] command.
- Enter as shown.

From the conjugate root theorem, $p(-1-i)=0$.

... continued

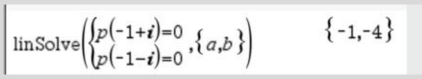
To find the values of a and b, solve the system of linear equations $p(-1+i)=0$ and $p(-1-i)=0$ for a and b.

- Press menu > **Algebra** > **Solve System of Linear Equations**.
- Complete the required fields as shown.



Complete the template as shown:

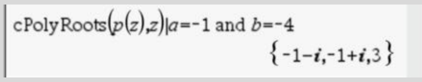
- Press π▸ ▶ to select $\boldsymbol{i}$.



So $a=-1, b=-4$ and the equation is $z^3-z^2-4z-6=0$.

To find the other solution:

- Press menu > **Algebra** > **Polynomial Tools** > **Complex Roots of Polynomial**.
- Press ctrl = to access the 'with' or 'given' symbol |.
- Enter as shown.



The other solution is $z=3$.

Alternatively:

$$(z-(-1+i))(z-(-1-i))=z^2+2z+2$$

Equating the coefficients of $(z^2+2z+2)(z-\alpha)$ and $p(z)$ and solving gives $\alpha=3$.

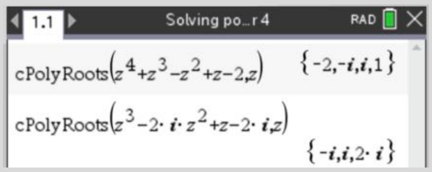
Solving polynomial equations over C to order 4

Solve the following polynomial equations over $\mathbb{C}$.

(a) $z^4+z^3-z^2+z-2=0$ **(b)** $z^3-2iz^2+z-2i=0$

To solve the equations on a **Calculator** page:

- Press menu > **Algebra** > **Polynomial Tools** > **Complex Roots of Polynomial**.
- Press ctrl = to access the 'with' or 'given' symbol |.
- Press π▸ ▶ to select $\boldsymbol{i}$.
- Enter as shown.



Answer:

(a) $z=-2, \pm i, 1$

(b) $z=\pm i, 2i$.

3.2. Topic 2: Mathematical induction and trigonometric proofs

3.2.1. Mathematical induction

Inductive proof involves an initial statement, assumption statement, inductive step and conclusion.

In practice, there are three main parts to an induction proof:

- Verify the statement for any initial terms.
- Prove the implication that if the statement is true for some integer k then it is true for the next integer $(k+1)$.
- Provide a concluding statement of truth that appeals to the principle of mathematical induction.

Mathematical induction is used to prove divisibility results for any positive integer n.

Proving by mathematical induction

Use mathematical induction to prove that 5^n+9^n+2 is divisible by 4, for $n \in \mathbb{Z}^+$.

Start with an exploratory verification establishing that 5^n+9^n+2 is divisible by 4 for positive integers between 1 and 10.

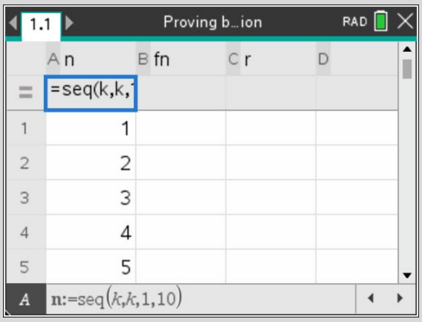
On a **Lists & Spreadsheet** page:

- In the column A heading cell, enter the variable $\boldsymbol{n}$.
- In the column B heading cell, enter the variable $\boldsymbol{fn}$.
- In the column C heading cell, enter the variable $\boldsymbol{r}$.

Generate the required sequences of values as follows:

To enter $\boldsymbol{n := \mathrm{seq}(k,k,1,10)}$ in the column A formula cell:

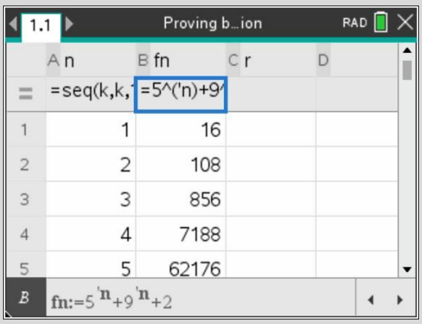
- Press 📖 S, scroll down and select **seq(**.
- Enter as shown.



__Note:__ The syntax for expressing a sequence as a list is __seq(Expression, Variable, Low, High[,Step])__. The default value for __Step__ is 1.

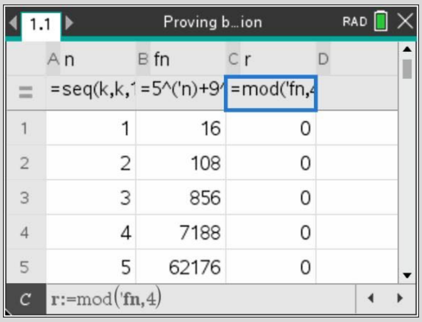
To enter $\boldsymbol{fn := 5^{'n}+9^{'n}+2}$ in the column B formula cell:

- Press =.
- Press ?!▸ to access the ' symbol.
- Enter as shown.



__Note:__ The symbol ' in __'n__ specifies n as a variable reference. Otherwise, TI-Nspire CX II-T will consider n as a column reference. If the ' symbol is omitted, a prompt may appear asking whether the variable being used refers to a variable or to a spreadsheet column.

... continued



To enter $r := \textbf{mod}(\text{'}fn, 4)$ in the column C formula cell:

- Press [📖] [M], scroll down and select **mod(**.
- Press [var] to access ***fn***.
- Enter as shown.

Note: *The* ***remain(*** *command, accessed from the* ***Catalog****, can be used instead of* ***mod(****.*
Note: *Press* [ctrl] [1] *to go to the last entry in a column. Press* [ctrl] [7] *to go to the first entry in a column. Press* [ctrl] [3] *to go down a page and* [ctrl] [9] *to go up a page. To go to a specific cell, press* [ctrl] [G] *and type in the cell reference.*

Column C of the spreadsheet shows that $5^n + 9^n + 2$ is divisible by 4 for positive integers between 1 and 10.

In each case, the remainder is 0.

Note: *A number is exactly divisible by 4 if the number formed by its last two digits is divisible by 4.*

Proof:

Let $f(n) = 5^n + 9^n + 2$ and let P_n be the proposition that $f(n)$ is divisible by 4.

$f(1) = 16$ and so P_1 is true.

Assume P_k is true for $n = k$, i.e. $f(k)$ is divisible by 4.

Consider $f(k+1)$.

$$\begin{aligned} f(k+1) &= 5^{k+1} + 9^{k+1} + 2 \\ &= 5^k(4+1) + 9^k(8+1) + 2 \\ &= f(k) + 4\left(5^k + 2 \times 9^k\right) \end{aligned}$$

Both terms are divisible by 4, so $f(k+1)$ is divisible by 4.

Since P_1 is true and P_k true $\Rightarrow P_{k+1}$ true, P_n is proved true by mathematical induction for $n \in \mathbb{Z}^+$.

3.2.2. Trigonometric proofs using De Moivre's theorem

Proving trigonometric identities

Multi-angle trigonometric identities established by equating parts using the binomial expansion and De Moivre's theorem include, for example:

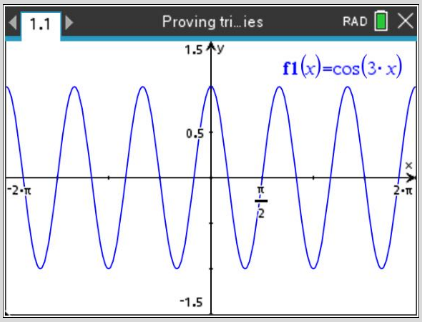
- $\cos(3x) = 4\cos^3(x) - 3\cos(x)$
- $\sin(3x) = 3\sin(x) - 4\sin^3(x)$

Verify graphically that $\cos(3x) = 4\cos^3(x) - 3\cos(x)$ for $-2\pi \le x \le 2\pi$.

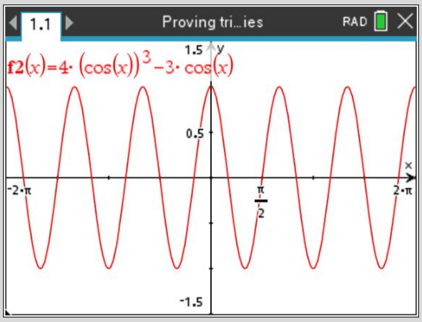
On a **Graphs** page:

- Enter $\boldsymbol{f1(x) = \cos(3x)}$.
- Enter $\boldsymbol{f2(x) = 4\cos^3(x) - 3\cos(x)}$.
- Press menu > **Window/Zoom** > **Window Settings**.

In the dialog box that follows, enter the following values:
XMin = -2π Xmax = 2π XScale = $\pi/2$
YMin = –1.5 YMax = 1.5 YScale = 0.5



Note: *To hide one of the graphs, either press* tab *to show the entry line and press* [click] *to remove the tick mark or hover the cursor over one of the graphs and press* ctrl menu ***> Hide***.



The two graphs appear identical for $-2\pi \le x \le 2\pi$.

Note: *The same approach can be used to verify graphically that* $\sin(3x) = 3\sin(x) - 4\sin^3(x)$.

Proof:

$$\cos(3x) + i\sin(3x)$$
$$= (\cos(x) + i\sin(x))^3$$
$$= \cos^3(x) + 3i\cos^2(x)\sin(x) + 3i^2\cos(x)\sin^2(x) + i^3\sin^3(x)$$
$$= \cos^3(x) + 3i\cos^2(x)\sin(x) - 3\cos(x)(1 - \cos^2(x)) - i\sin^3(x)$$
$$= \cos^3(x) + 3i\cos^2(x)\sin(x) - 3\cos(x) + 3\cos^3(x) - i\sin^3(x)$$
$$= 4\cos^3(x) - 3\cos(x) + 3i\cos^2(x)\sin(x) - i\sin^3(x)$$

Equating the real parts gives $\cos(3x) = 4\cos^3(x) - 3\cos(x)$.

3.3. Topic 3: Vectors in two and three dimensions

3.3.1. Vectors in three dimensions

It is important to understand and use the vector notation $\overrightarrow{AB}$, $\underset{\sim}{c}$, $\boldsymbol{d}$ and $\hat{\boldsymbol{n}}$.

In Section 3.3, the vector notation $\overrightarrow{AB}$, $\boldsymbol{d}$ and $\hat{\boldsymbol{n}}$ is used.

Calculating the magnitude of a vector

A position vector in three dimensions can be represented using ordered pair notation (x, y, z) and column vector notation $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. The magnitude of a vector is defined as: $|\boldsymbol{a}| = \left|\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right| = \sqrt{a_1{}^2 + a_2{}^2 + a_3{}^2}$

Consider the vectors $\boldsymbol{a} = \begin{pmatrix} -4 \\ -3 \\ 2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix}$. Find the exact value of $|\boldsymbol{a} + 2\boldsymbol{b}|$.

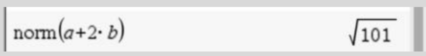
On a **Calculator** page, assign $\boldsymbol{a}$ and $\boldsymbol{b}$ as follows:

- Press [ctrl] [⁝⁼] to access the **Assign** [:=] command.
- Press [📖] [5], select the **m-by-n Matrix** template, fix the dimensions as 3-by-1 and enter as shown.

***Note:** Here, column vectors are used. This is because there was enough space on the screen to display all the calculations. In subsequent examples where this is not possible, row vectors will be used.*

To find $|\boldsymbol{a} + 2\boldsymbol{b}|$:

- Press [menu] > **Matrix & Vector** > **Norms** > **Norm**.
- Enter as shown.



***Note:** If required, press [ctrl] [enter] to obtain a decimal magnitude.*

$$|\boldsymbol{a} + 2\boldsymbol{b}| = \left|\begin{pmatrix} -4 \\ -3 \\ 2 \end{pmatrix} + 2\begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix}\right|$$

$$= \left|\begin{pmatrix} 4 \\ -7 \\ 6 \end{pmatrix}\right|$$

$$= \sqrt{4^2 + (-7)^2 + 6^2}$$

$$= \sqrt{101}$$

***Note:** The vector $\boldsymbol{a} + 2\boldsymbol{b}$ can be entered and calculated before finding its magnitude.*

Using vectors in Cartesian form

A unit vector, $\hat{\boldsymbol{n}}$, in three-dimensional space is given by $\hat{\boldsymbol{n}} = \dfrac{\boldsymbol{n}}{|\boldsymbol{n}|}$.

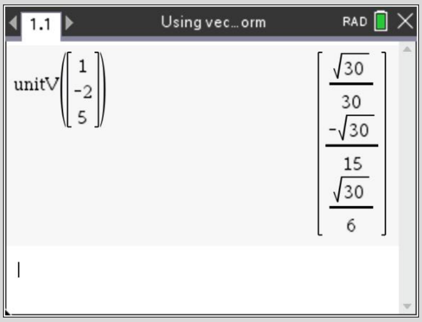
Vectors in Cartesian (component) form are expressed using unit perpendicular vectors $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$. A vector parallel to $\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$ and with magnitude 6 is

(A) $6\left(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}\right)$ **(B)** $\dfrac{\sqrt{30}}{5}\left(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}\right)$

(C) $-\dfrac{6}{\sqrt{30}}\left(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}\right)$ **(D)** $\dfrac{1}{5}\left(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}\right)$

On a **Calculator** page:

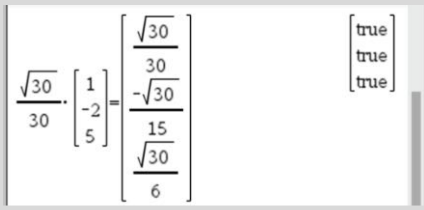
- Press [menu] > **Matrix & Vector** > **Vector** > **Unit Vector**.
- Press [templates] [5], select the **m-by-n Matrix** template, fix the dimensions as 3-by-1 and enter as shown.



A unit vector in the direction of $\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$ is

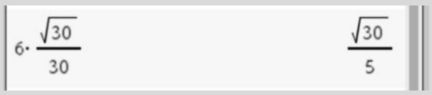
$$\left(\frac{\sqrt{30}}{30}\hat{\boldsymbol{i}} - \frac{\sqrt{30}}{15}\hat{\boldsymbol{j}} + \frac{\sqrt{30}}{6}\hat{\boldsymbol{k}}\right) = \frac{\sqrt{30}}{30}\left(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}\right).$$

The screenshot at right confirms the two equivalent forms.



Hence a vector with magnitude 6 parallel to $\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$ is

$$\frac{6\sqrt{30}}{30}\left(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}\right) = \frac{\sqrt{30}}{5}\left(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}\right)$$



Answer is **B**.

Option **C**, $-\dfrac{6}{\sqrt{30}}\left(\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}\right)$ has a magnitude of 6 but is not parallel to $\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$.

3.3.2. Algebra of vectors in three dimensions

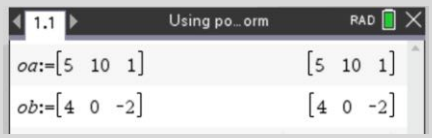
Using position vectors in Cartesian form

The position vectors of points A and B are given by $\overrightarrow{OA} = 5\hat{\boldsymbol{i}} + 10\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ and $\overrightarrow{OB} = 4\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{k}}$.

Find the exact distance between points A and B.

On a **Calculator** page, assign $\overrightarrow{OA}$ and $\overrightarrow{OB}$ as row vectors as follows:

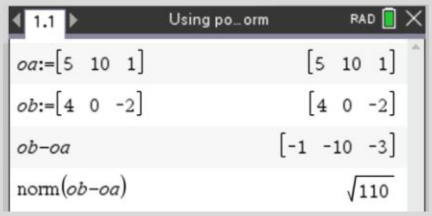
- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [templates] [5], select the **m-by-n Matrix** template, fix the dimensions as 1-by-3 and enter as shown.



... *continued*

To find the exact distance between points A and B:

- Press [menu] > **Matrix & Vector** > **Norms** > **Norm**.
- Enter as shown.



$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \left(4\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{k}}\right) - \left(5\hat{\boldsymbol{i}} + 10\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}\right)$

$= -\hat{\boldsymbol{i}} - 10\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$

$\left|\overrightarrow{AB}\right| = \sqrt{(-1)^2 + (-10)^2 + (-3)^2}$

$= \sqrt{110}$

The exact distance between A and B is $\sqrt{110}$.

***Note:** Press [var] to access assigned/stored variables.*

Using the scalar (dot) product to find the angle between two vectors

The scalar (dot) product is defined as:

- $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos(\theta)$
- $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$

Points P and Q are defined by the position vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ respectively, where

$\boldsymbol{p} = 2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - \hat{\boldsymbol{k}}$ and $\boldsymbol{q} = -4\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{k}}$.

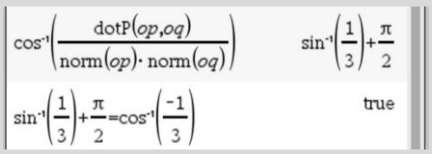
Find the angle between $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, giving your answer correct to the nearest tenth of a degree.

On a **Calculator** page, assign $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ as row vectors as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [catalog] [5], select the m-by-n **Matrix** template, fix the dimensions as 1-by-3 and enter as shown.

To determine the angle, θ, between $\overrightarrow{OP}$ and $\overrightarrow{OQ}$:

- Press [trig] and select **cos**$^{-1}$.
- Press [ctrl] [÷] to access the **Fraction** template.
- Press [menu] > **Matrix & Vector** > **Vector** > **Dot Product**.
- Enter the numerator as shown.
- Press [menu] > **Matrix & Vector** > **Norms** > **Norm**.
- Enter the denominator as shown.

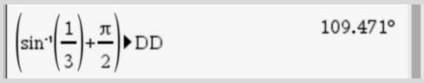


***Note:** Press [var] to access assigned/stored variables.*

... continued

Note: *The exact output* $\sin^{-1}\left(\frac{1}{3}\right)+\frac{\pi}{2}$ *is equivalent to the more familiar intermediate answer* $\cos^{-1}\left(-\frac{1}{3}\right)$ *as shown at right.*

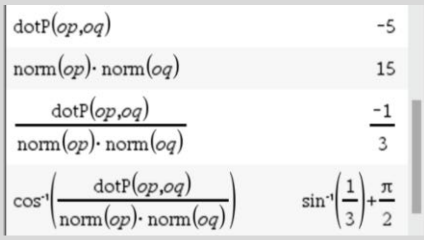
To express θ correct to the nearest tenth of a degree:



- Press ▲ [enter] to paste the exact angle (in radians) to a new entry line.
- Press [📖] [D], scroll down and select ▶DD.
- Press [ctrl] [enter] to obtain a decimal angle in degrees.

$$\overrightarrow{OP}\cdot\overrightarrow{OQ}=\left|\overrightarrow{OP}\right|\left|\overrightarrow{OQ}\right|\cos(\theta)$$

$$\begin{aligned}\theta&=\cos^{-1}\left(\frac{\overrightarrow{OP}\cdot\overrightarrow{OQ}}{\left|\overrightarrow{OP}\right|\left|\overrightarrow{OQ}\right|}\right)\\&=\cos^{-1}\left(\frac{\left(2\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}-\hat{\boldsymbol{k}}\right)\cdot\left(-4\hat{\boldsymbol{i}}-3\hat{\boldsymbol{k}}\right)}{\left|2\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}-\hat{\boldsymbol{k}}\right|\left|-4\hat{\boldsymbol{i}}-3\hat{\boldsymbol{k}}\right|}\right)\\&=\cos^{-1}\left(\frac{-8+0+3}{3\times5}\right)\\&=\cos^{-1}\left(-\frac{1}{3}\right)\\&=109.471...^\circ\\&=109.5^\circ\end{aligned}$$



Note: *Instead of performing all the steps at once on TI-Nspire CX II-T, it is a good idea from a teaching viewpoint to show the required steps one at a time as shown at right.*

Using the scalar (dot) product to determine when two vectors are perpendicular

Using the scalar product, $\boldsymbol{a}\cdot\boldsymbol{b}=0$ if and only if $\boldsymbol{a}=\boldsymbol{0}$ or $\boldsymbol{b}=\boldsymbol{0}$ or $\boldsymbol{a}\perp\boldsymbol{b}$.

Find the value(s) of p for which the vectors $\boldsymbol{u}=p\hat{\boldsymbol{i}}+\hat{\boldsymbol{j}}+2\hat{\boldsymbol{k}}$ and $\boldsymbol{v}=(p-1)\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}-4\hat{\boldsymbol{k}}$ are perpendicular.

Find $\boldsymbol{u}\cdot\boldsymbol{v}$:

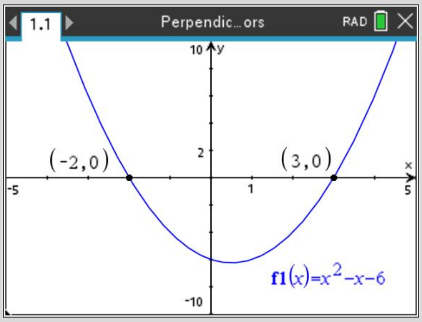
$$\begin{aligned}\boldsymbol{u}\cdot\boldsymbol{v}&=\left(p\hat{\boldsymbol{i}}+\hat{\boldsymbol{j}}+2\hat{\boldsymbol{k}}\right)\cdot\left((p-1)\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}-4\hat{\boldsymbol{k}}\right)\\&=p(p-1)+2-8\\&=p^2-p-6\end{aligned}$$

Find the values of p for which $\boldsymbol{u}\cdot\boldsymbol{v}=0$.

On a **Graphs** page:

- Enter $\boldsymbol{f1(x)=x^2-x-6}$.
- Press menu > **Window/Zoom** > **Window Settings**.

  In the dialog box that follows, enter the following values:
  XMin = –5 Xmax = 5 XScale = 1
  YMin = –10 YMax = 10 YScale = 2



To determine the x-intercepts:

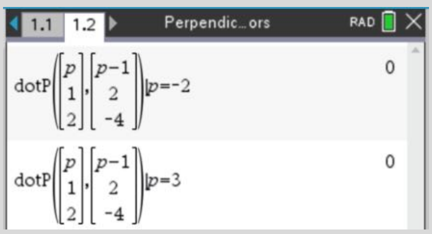
- Press menu > **Geometry** > **Points & Lines** > **Intersection Point(s)**.
- Click (press) on the graph and click (press) on the x-axis.
- On each point of intersection, press ctrl menu > **Coordinates and Equations**.
- The coordinates $(-2,0)$ and $(3,0)$ are now pasted on the screen.

So $p=-2$ or 3.

This can be verified algebraically as follows:

$$\begin{aligned}p^2-p-6&=0\\(p-3)(p+2)&=0\\p&=-2,3\end{aligned}$$

On a **Calculator** page, the solutions $p=-2,3$ can be checked as follows:



- Press menu > **Matrix & Vector** > **Vector** > **Dot Product**.
- Press 5, select the **m-by-n Matrix** template, fix the dimensions as 3-by-1 and enter the vectors as shown.
- Press ctrl = to access the 'with' or 'given' symbol | and enter the condition on p as shown.

When $p=-2$, $\left(-2\hat{\boldsymbol{i}}+\hat{\boldsymbol{j}}+2\hat{\boldsymbol{k}}\right)\cdot\left(-3\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}-4\hat{\boldsymbol{k}}\right)=0$.

When $p=3$, $\left(3\hat{\boldsymbol{i}}+\hat{\boldsymbol{j}}+2\hat{\boldsymbol{k}}\right)\cdot\left(2\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}-4\hat{\boldsymbol{k}}\right)=0$.

Finding the vector projection of one vector onto another

The scalar projection of $\boldsymbol{a}$ on $\boldsymbol{b}$ is defined as:

- $|\boldsymbol{a}|\cos(\theta) = \boldsymbol{a} \cdot \hat{\boldsymbol{b}}$

The vector projection of $\boldsymbol{a}$ on $\boldsymbol{b}$ is defined as:

- $|\boldsymbol{a}|\cos(\theta)\hat{\boldsymbol{b}} = (\boldsymbol{a} \cdot \hat{\boldsymbol{b}})\hat{\boldsymbol{b}} = \left(\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\right)\boldsymbol{b}$

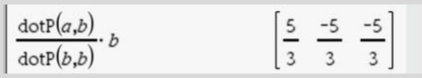
Find the vector projection of $\boldsymbol{a} = 4\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$ onto $\boldsymbol{b} = -\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$.

On a **Calculator** page, assign $\boldsymbol{a}$ and $\boldsymbol{b}$ as row vectors as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [catalog] [5], select the **m-by-n Matrix** template, fix the dimensions as 1-by-3 and enter the vectors as shown.

To determine the vector projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$:

- Press [(].
- Press [ctrl] [÷] to access the **Fraction** template.
- Press [menu] > **Matrix & Vector** > **Vector** > **Dot Product**.
- Enter the numerator as shown.
- Press [menu] > **Matrix & Vector** > **Vector** > **Dot Product**.
- Enter the denominator as shown.
- Press ▶ until the cursor is outside the bracket. Press [×] and enter as shown.



The vector projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ is
$\boldsymbol{a} = \frac{5}{3}\hat{\boldsymbol{i}} - \frac{5}{3}\hat{\boldsymbol{j}} - \frac{5}{3}\hat{\boldsymbol{k}} = -\frac{5}{3}(-\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$.

This can be verified as follows:

Let $\boldsymbol{u}$ be the vector projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$.

$$\begin{aligned}\boldsymbol{u} &= \left(\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\right)\boldsymbol{b} \\ &= \left(\frac{-4+2-3}{3}\right)(-\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}) \\ &= -\frac{5}{3}(-\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})\end{aligned}$$

The scalar projection of $\boldsymbol{a} = 4\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$ onto $\boldsymbol{b} = -\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$ is $-\frac{5}{3}$.

Modelling and solving problems with vectors

Let $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$ be unit vectors in the east, north and vertically up directions respectively.

Po leaves her base camp at point O and walks on flat terrain for 6 km in a SE direction to point A.

She then walks 4.5 km east to point B.

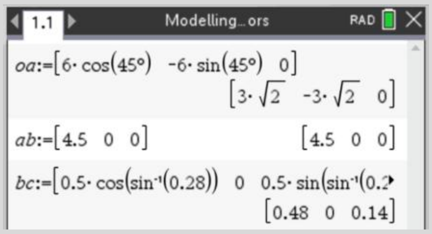
From B, Po walks 0.5 km east up a steep slope inclined at an angle of $\sin^{-1}(0.28)$ to the horizontal to point C.

Find $\left|\overrightarrow{OC}\right|$, giving your answer correct to one decimal place.

***Note:** In **Document Settings > Real or Complex** (accessed by pressing [home on]), there is a choice to set the TI-Nspire CX II-T to either **Real** or **Rectangular** or **Polar** mode. In this example, TI-Nspire CX II-T was set to **Rectangular** mode and **Radian** mode.*

On a **Calculator** page, assign $\overrightarrow{OA}$ and $\overrightarrow{AB}$ as row vectors as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [catalog] [5], select the **m-by-n Matrix** template, fix the dimensions as 1-by-3.
- Press [trig] to access **cos**, **sin** and **sin⁻¹**.
- Press [π▸] to access the degree symbol.
- Enter as shown.



$$\overrightarrow{OA} = \begin{pmatrix} 3\sqrt{2} \\ -3\sqrt{2} \\ 0 \end{pmatrix},\ \overrightarrow{AB} = \begin{pmatrix} 4.5 \\ 0 \\ 0 \end{pmatrix} \text{ and } \overrightarrow{BC} = \begin{pmatrix} 0.5\cos\left(\sin^{-1}(0.28)\right) \\ 0 \\ 0.5\sin\left(\sin^{-1}(0.28)\right) \end{pmatrix}$$

Resolving into $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$ components:

$$\overrightarrow{OA} = 6\cos(45°)\hat{\boldsymbol{i}} - 6\sin(45°)\hat{\boldsymbol{j}}$$

$$= 6\left(\frac{\sqrt{2}}{2}\hat{\boldsymbol{i}} - \frac{\sqrt{2}}{2}\hat{\boldsymbol{j}}\right)$$

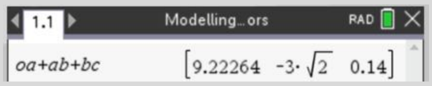
$$= 3\sqrt{2}\left(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}\right)$$

$$\overrightarrow{AB} = 4.5\hat{\boldsymbol{i}}$$

$$\overrightarrow{BC} = 0.5\cos\left(\sin^{-1}(0.28)\right)\hat{\boldsymbol{i}} + 0.5\sin\left(\sin^{-1}(0.28)\right)\hat{\boldsymbol{k}}$$

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC}$$

Enter $\overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC}$ as shown.



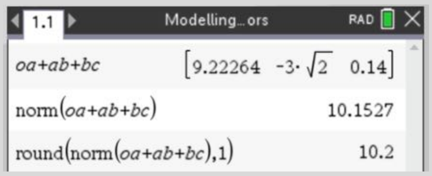
***Note:** Press [var] to access assigned/stored variables.*

... continued

To find $\left|\overrightarrow{OC}\right|$:

- Press [menu] > **Matrix & Vector** > **Norms** > **Norm**.
- Press ▲ to select the row vector and press [enter].
- Press [menu] > **Number** > **Number Tools** > **Round** to give $\left|\overrightarrow{OC}\right|$ correct to one decimal place.

Note: *The syntax for the* ***Round*** *command is* ***round(Value [,Digits])***.



$\overrightarrow{OC} = 9.22...\hat{\boldsymbol{i}} - 3\sqrt{2}\hat{\boldsymbol{j}} + 0.14...\hat{\boldsymbol{k}}$

$$\left|\overrightarrow{OC}\right| = \sqrt{(9.22...)^2 + \left(-3\sqrt{2}\right)^2 + (0.14...)^2}$$

$$= 10.15...$$

$$= 10.2 \text{ (km)}$$

Using vectors to prove geometric results in two dimensions

The following properties of the scalar product will be useful in the following proof:

- Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.
- $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$

Prove that the diagonals of a rhombus are perpendicular.

A rhombus is a quadrilateral that has two pairs of parallel, congruent sides.

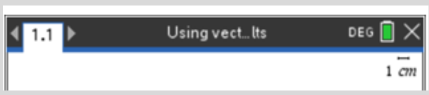
To construct a rhombus, use the centres and points on three intersecting circles to determine the vertices of the rhombus and then connect these vertices to form its sides.

Start with a geometric verification that the diagonals of a rhombus are perpendicular.

On a **Geometry** page construct the rhombus as follows:

Note: *After using each geometric tool, press* [esc] *to ensure completion of its use.*

It is a good idea to set the **Geometry Angle** on the page to **Degree** mode. To set this:



- Activate the cursor and click (press [click]) on **RAD**.

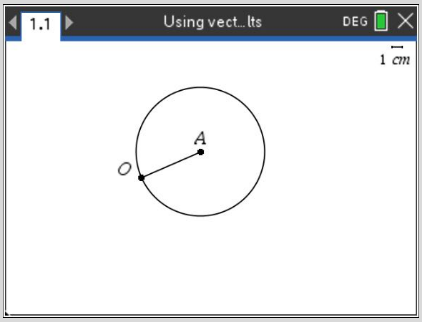
Construct a line segment OA:

- Press [menu] > **Points & Lines** > **Segment**.
- Click (press [click]) to create the initial point of the segment and click (press [click]) to create the end point of the segment.
- Hover the cursor over the initial point.
- Press [ctrl] [menu] > **Label** and label as O.
- Hover the cursor over the end point.
- Press [ctrl] [menu] > **Label** and label as A.

... continued

Construct a circle centred at A with radius OA:

- Press menu > **Construction** > **Compass**.
- Click (press [click]) on point O, click (press [click]) on point A and click (press [click]) on point A again.
- Press esc to exit the circle construction tool

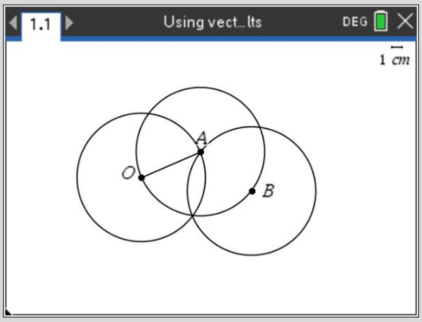


Construct a point B as the third vertex of the rhombus on the circle centred at A.

- Press menu > **Points & Lines** > **Point On**.
- Hover the cursor over the circle and click twice (press [click] [click]).
- Press ctrl menu > **Label** and label as B.

Use the **Compass** tool to construct a circle centred at B with radius OA.

Use the **Compass** tool to construct another circle centred at O with radius OA.



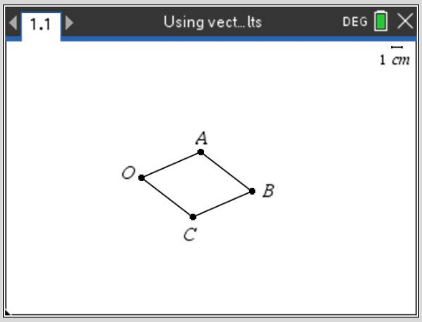
Generate the intersection point of the circles to locate the fourth vertex, point C, of the rhombus:

- Press menu > **Points & Lines** > **Intersection Point(s)**.
- Click (press [click]) on two of the circles.
- Press ctrl menu > **Label** and label as C.

Construct line segments OC, AB and BC.

To hide the three circles:

- Hover the cursor over each circle and press ctrl menu > **Hide**.



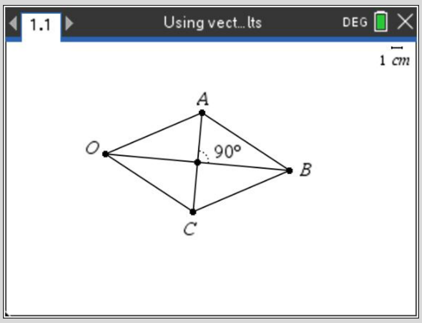
Note: *Alternatively, press* menu *>* ***Actions*** *>* ***Hide/Show****. Use the cursor to hide or show objects as appropriate.*

Grab and drag points O and B around the page.

Construct line segments OB and AC.

To measure the angle between the diagonals:

- Press menu > **Measurement** > **Angle**.
- Click (press [click]) on point A, click (press [click]) where the diagonals meet and click (press [click]) on point B.



The diagonals OB and AC are $\perp$.

... continued

Proof:

$OABC$ is a rhombus.

Let $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{c} = \overrightarrow{OC}$.

The diagonals of the rhombus are OB and AC.

$$\begin{aligned}\overrightarrow{OB} &= \overrightarrow{OC} + \overrightarrow{CB}\\ &= \overrightarrow{OC} + \overrightarrow{OA}\\ &= \boldsymbol{c} + \boldsymbol{a}\end{aligned}$$

$$\begin{aligned}\overrightarrow{AC} &= \overrightarrow{AO} + \overrightarrow{OC}\\ &= -\boldsymbol{a} + \boldsymbol{c}\end{aligned}$$

$$\begin{aligned}\overrightarrow{OB} \cdot \overrightarrow{AC} &= (\boldsymbol{c} + \boldsymbol{a}) \cdot (\boldsymbol{c} - \boldsymbol{a})\\ &= \boldsymbol{c} \cdot \boldsymbol{c} - \boldsymbol{c} \cdot \boldsymbol{a} + \boldsymbol{a} \cdot \boldsymbol{c} - \boldsymbol{a} \cdot \boldsymbol{a}\\ &= \boldsymbol{c} \cdot \boldsymbol{c} - \boldsymbol{a} \cdot \boldsymbol{a}\\ &= |\boldsymbol{c}|^2 - |\boldsymbol{a}|^2\end{aligned}$$

A rhombus has four sides of equal length and hence $|\boldsymbol{c}| = |\boldsymbol{a}|$.

$$\overrightarrow{OB} \cdot \overrightarrow{AC} = |\boldsymbol{c}|^2 - |\boldsymbol{a}|^2 = 0 \Rightarrow \overrightarrow{AC} \perp \overrightarrow{OB}$$

Hence the diagonals of a rhombus are perpendicular.

3.3.3. Vector and Cartesian equations

The form of the vector equation of a sphere is identical to that of a circle in two dimensions.

The sphere with centre $\boldsymbol{c}$ has vector equation:

- $|\boldsymbol{v} - \boldsymbol{c}| = r$ where $\boldsymbol{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} h \\ k \\ l \end{pmatrix}$.

The Cartesian equation of a sphere is given by:

- $(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$

Understanding and using equations of spheres

Two spheres, S_1 and S_2, have equations

$$\left|\boldsymbol{v} - \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}\right| = 3 \text{ and } \left|\boldsymbol{v} - \begin{pmatrix} -5 \\ 0 \\ -1 \end{pmatrix}\right| = 4 \text{ respectively.}$$

(a) Show that S_1 and S_2 touch each other at a single point.

(b) Find the coordinates of this point, P.

The Cartesian equations of S_1 and S_2 are:

$$S_1 : (x-1)^2 + (y+2)^2 + (z-2)^2 = 3^2$$

$$S_1 : (x+5)^2 + y^2 + (z+1)^2 = 4^2$$

Spheres can be plotted using 3D graphing.

On a **Graphs** page:

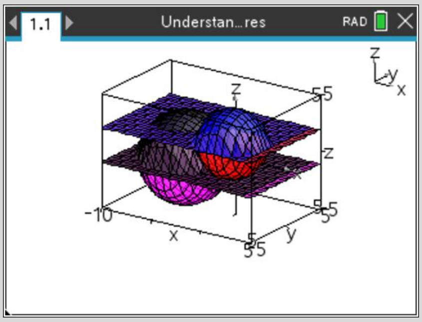
- Press [menu] > **View** > **3D Graphing**.

Note: *In* ***Document Settings*** *>* ***Real or Complex*** *(accessed by pressing [on]), there is a choice to set the TI-Nspire CX II-T to either* ***Real*** *or* ***Rectangular*** *or* ***Polar*** *mode. To obtain a clearer plot of the spheres, TI-Nspire CX II-T was set to* ***Rectangular*** *mode.*

- Enter $z1(x,y) = \text{real}\left(2 + \sqrt{3^2 - (x-1)^2 - (y+2)^2}\right)$.
- Enter $z2(x,y) = \text{real}\left(2 - \sqrt{3^2 - (x-1)^2 - (y+2)^2}\right)$.
- Enter $z3(x,y) = \text{real}\left(-1 + \sqrt{4^2 - (x+5)^2 - y^2}\right)$.
- Enter $z4(x,y) = \text{real}\left(-1 - \sqrt{4^2 - (x+5)^2 - y^2}\right)$.

To set the viewing window:

- Press [menu] > **Range/Zoom** > **Range Settings**.



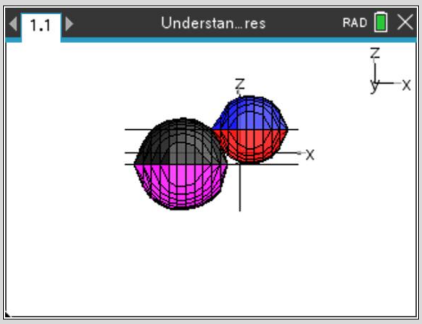
In the dialog box that follows, enter the following values:

XMin = –10	Xmax = 5	XScale = 5
YMin = –5	YMax = 5	YScale = 5
ZMin = –5	ZMax = 5	ZScale = 5

Note: *Press [menu] >* ***Actions*** *or* ***View*** *or* ***Range/Zoom*** *to explore a suite of viewing alternatives.*

For example, for the screenshot shown at right:

- Press [menu] > **Range/Zoom** > **x-z Orientation**.
- Press [menu] > **View** > **Hide Box**.



Note: *Press [Y] to obtain an* x-z *orientation.*

The two spheres appear to touch at a point.

(a) S_1 has centre $C_1(1,-2,2)$ and radius $R_1 = 3$.

S_2 has centre $C_2(-5,0,-1)$ and radius $R_2 = 4$.

... continued

The distance between C_1 and C_2 is:

$$\left|\begin{pmatrix}-5\\0\\-1\end{pmatrix}-\begin{pmatrix}1\\-2\\2\end{pmatrix}\right|=\left|\begin{pmatrix}-6\\2\\-3\end{pmatrix}\right|$$

$$=\sqrt{(-6)^2+2^2+(-3)^2}$$

$$=7$$

The sum of the radii, $R_1+R_2=3+4=7$.

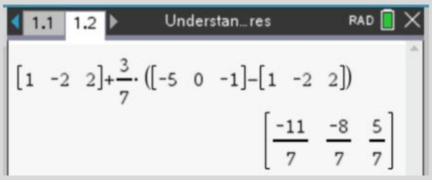
As these two distances are equal, S_1 and S_2 touch each other at a single point.

(b) The point, P, where S_1 and S_2 touch lies on the line segment joining the centres of the two spheres.

Hence the coordinates of P are given by:

$$(x,y,z)=(1,-2,2)+\frac{3}{7}\big((-5,0,-1)-(1,-2,2)\big)$$

$$=\left(-\frac{11}{7},-\frac{8}{7},\frac{5}{7}\right)$$



***Note:** On a **Calculator** page, row vectors can be used to execute this calculation as shown at right. To form a row vector, press 5, select the **m-by-n Matrix** template, fix the dimensions as 1-by-3 and enter as shown.*

Determining Cartesian equations of curves from vector equations

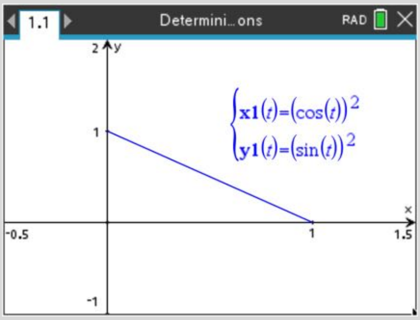
Find the Cartesian equation for the curve represented by the vector equation $\boldsymbol{r}(t)=\cos^2(t)\hat{\boldsymbol{i}}+\sin^2(t)\hat{\boldsymbol{j}}$, where $t\in\mathbb{R}$.

State the domain and range of the Cartesian relation.

Use parametric graphing mode to plot the curve.

On a **Graphs** page:

- Press menu > **Graph Entry/Edit** > **Parametric**.
- Enter $\boldsymbol{x1(t)=\cos(t)^2}$
- Enter $\boldsymbol{y1(t)=\sin(t)^2}$
- Press menu > **Window/Zoom** > **Window Settings**.



In the dialog box that follows, enter the following values:

XMin = –0.5 Xmax = 1.5 XScale = 1
YMin = –1 YMax = 2 YScale = 1

***Note:** To plot this curve, there is no need to change the settings for t in the parametric graphing entry line.*

The curve appears to have Cartesian equation $y=1-x$ for $0\le x\le 1$.

***Note:** To confirm this Cartesian equation, return to function graphing mode and plot* $\boldsymbol{f1(x)=1-x\,|\,0\le x\le 1}$.

Let (x,y) be any point on the curve defined by $\boldsymbol{r}(t)$.

The parametric equations are:

$x=\cos^2(t)$ (1)

$y=\sin^2(t)$ (2)

$$\begin{aligned}y&=\sin^2(t)\\&=1-\cos^2(t)\\&=1-x\end{aligned}$$

As $0\le\cos^2(t)\le 1$ for $t\in\mathbb{R}$, the domain is $[0,1]$.

As $0\le\sin^2(t)\le 1$ for $t\in\mathbb{R}$, the range is $[0,1]$.

Finding where two lines intersect

The vector equation of a line, l, is given by:

- $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{d}$ where $t \in \mathbb{R}$

Given position vectors, $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{b} = \overrightarrow{OB}$, of two points on a line, l, then:

- $\boldsymbol{r} = \boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a})$ where $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$ and $t \in \mathbb{R}$

Alternatively:

- $\boldsymbol{r} = (1-t)\boldsymbol{a} + t\boldsymbol{b}$ where $t \in \mathbb{R}$

A line, l, in three-dimensional space can also be described by $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{d}$ where $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ is the position vector of a point A on l, $\boldsymbol{d} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$ is a vector parallel to l and $t \in \mathbb{R}$.

The parametric equations of a line, l, are given by:

- $x = a_1 + td_1$, $y = a_2 + td_2$, $z = a_3 + td_3$ where $t \in \mathbb{R}$

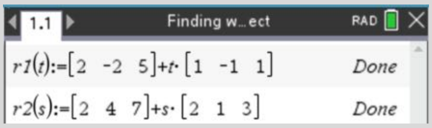
The Cartesian equation of a line, l, is given by:

- $\dfrac{x - a_1}{d_1} = \dfrac{y - a_2}{d_2} = \dfrac{z - a_3}{d_3}$

Find the position vector of the point of intersection of the lines
$\boldsymbol{r}_1(t) = 2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}} + t(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}})$ and $\boldsymbol{r}_2(t) = 2\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} + 7\hat{\boldsymbol{k}} + s(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + 3\hat{\boldsymbol{k}})$ where $t, s \in \mathbb{R}$.

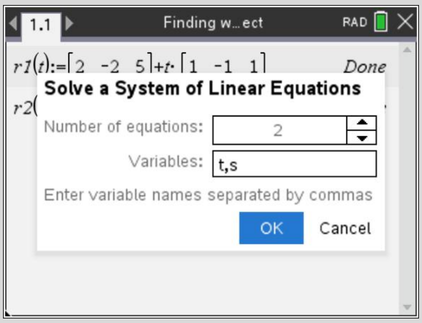
On a **Calculator** page, assign $\boldsymbol{r}_1(t)$ and $\boldsymbol{r}_2(t)$ as row vectors as follows:

- Press [ctrl] [=] to access the **Assign** [:=] command.
- Press [templates] [5], select the **m-by-n Matrix** template, fix the dimensions as 1-by-3 and enter as shown.



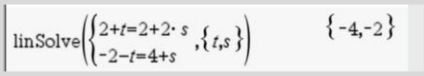
Equate the $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ components, for example, and solve:

- Press [menu] > **Algebra** > **Solve System of Linear Equations**.
- Complete the required fields as shown



... continued

Complete the template as shown and press enter.



$2+t=2+2s$

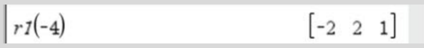
$-2-t=4+s$

So $t=-4, s=-2$.

Checking:

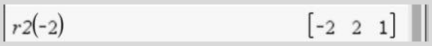
$r_1(-4)=2\hat{\boldsymbol{i}}-2\hat{\boldsymbol{j}}+5\hat{\boldsymbol{k}}-4\left(\hat{\boldsymbol{i}}-\hat{\boldsymbol{j}}+\hat{\boldsymbol{k}}\right)$

$\quad =-2\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}+\hat{\boldsymbol{k}}$



$r_2(-2)=2\hat{\boldsymbol{i}}+4\hat{\boldsymbol{j}}+7\hat{\boldsymbol{k}}-2\left(2\hat{\boldsymbol{i}}+\hat{\boldsymbol{j}}+3\hat{\boldsymbol{k}}\right)$

$\quad =-2\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}+\hat{\boldsymbol{k}}$

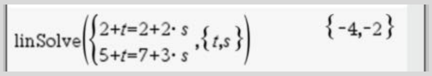


As an extra check, set the $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{k}}$ components equal, for example, and solve:

$2+t=2+2s$

$5+t=7+3s$

So $t=-4, s=-2$.



Defining and using the vector (cross) product

The vector (cross) product can be used to determine a vector normal to a given plane.

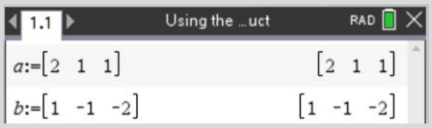
The vector (cross) product is defined as:

- $\boldsymbol{a}\times\boldsymbol{b}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=\begin{pmatrix}a_2b_3-a_3b_2\\a_3b_1-a_1b_3\\a_1b_2-a_2b_1\end{pmatrix}$
- $|\boldsymbol{a}\times\boldsymbol{b}|=|\boldsymbol{a}||\boldsymbol{b}|\sin(\theta)$ where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$

Find a vector perpendicular to the vectors $\boldsymbol{a}=2\hat{\boldsymbol{i}}+\hat{\boldsymbol{j}}+\hat{\boldsymbol{k}}$ and $\boldsymbol{b}=\hat{\boldsymbol{i}}-\hat{\boldsymbol{j}}-2\hat{\boldsymbol{k}}$.

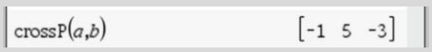
On a **Calculator** page, assign $\boldsymbol{a}$ and $\boldsymbol{b}$ as row vectors as follows:

- Press ctrl [templates] to access the **Assign** := command.
- Press [catalog] 5, select the **m-by-n Matrix** template, fix the dimensions as 1-by-3 and enter as shown.



To calculate $\boldsymbol{a}\times\boldsymbol{b}$:

- Press menu > **Matrix & Vector** > **Vector** > **Cross Product**.
- Enter as shown.



... continued

$$\boldsymbol{a}\times\boldsymbol{b}=\begin{pmatrix}2\\1\\1\end{pmatrix}\times\begin{pmatrix}1\\-1\\-2\end{pmatrix}$$

$$=\begin{pmatrix}(1)(-2)-(1)(-1)\\(1)(1)-(2)(-2)\\(2)(-1)-(1)(1)\end{pmatrix}$$

$$=\begin{pmatrix}-1\\5\\-3\end{pmatrix}$$

Note: $\boldsymbol{b}\times\boldsymbol{a}=\begin{pmatrix}1\\-5\\3\end{pmatrix}$. *There is not a unique normal vector for a given plane. If the vector* $\boldsymbol{n}$ *is normal to a plane, then so are the vectors* $s\boldsymbol{n}$ *and* $-s\boldsymbol{n}$ *for* $s\in\mathbb{R}^+$.

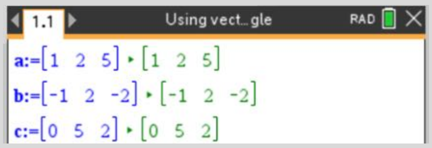
Using vectors to determine the area of a triangle

Vectors can be used to determine the area of shapes such as triangles.

Find the area of the triangle ABC with vertices $A(1,2,5)$, $B(-1,2,-2)$ and $C(0,5,2)$.

On a **Notes** page, assign $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ as row vectors as follows:

- Press [menu] > **Insert** > **Maths Box**.
- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [book] [5], select the **m-by-n Matrix** template, fix the dimensions as 1-by-3 and enter as shown.



Note: *Alternatively, to insert a* ***Maths Box****, press* [ctrl] [M].

- Press [ctrl] [M] and enter $\boldsymbol{b}-\boldsymbol{a}$.
- Press [ctrl] [M] and enter $\boldsymbol{c}-\boldsymbol{a}$.

To display an equals sign in a **Maths Box**:

- Click on the **Maths Box**.
- Press [menu] > **Maths Box Options** > **Maths Box Attributes**.
- Press [tab] to highlight the **Insert Symbol** field.
- Press ▸ and select $=$.

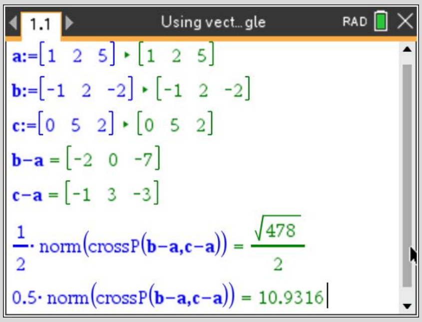
Note: Maths Box Attributes *can also be accessed within a* ***Maths Box*** *by pressing* [ctrl] [menu].

Let A be the area of the triangle where $A=\frac{1}{2}\left|(\boldsymbol{b}-\boldsymbol{a})\times(\boldsymbol{c}-\boldsymbol{a})\right|$.

... continued

Enter this formula as follows:

- Press ctrl ÷ to access the **Fraction** template.
- Press menu > **Calculations** > **Matrix & Vector** > **Norms** > **Norm**.
- Press menu > **Calculations** > **Matrix & Vector** > **Vector** > **Cross Product**.
- Press menu > **Maths Box Options** > **Maths Box Attributes**, press tab to highlight the **Insert Symbol** field, press ▸ and select $=$.



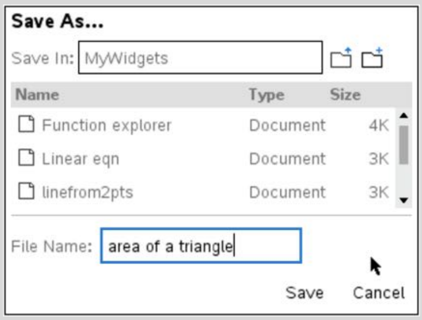
***Note:** There are many ways to express this area in decimal form. The easiest way is to change $\frac{1}{2}$ to 0.5 as shown.*

$$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a} = \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -7 \end{pmatrix}$$

$$\overrightarrow{AC} = \boldsymbol{c} - \boldsymbol{a} = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -3 \end{pmatrix}$$

$$\begin{aligned} A &= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| \\ &= \frac{1}{2}\left|\begin{pmatrix} -2 \\ 0 \\ -7 \end{pmatrix} \times \begin{pmatrix} -1 \\ 3 \\ -3 \end{pmatrix}\right| \\ &= \frac{1}{2}\left|\begin{pmatrix} 21 \\ 1 \\ -6 \end{pmatrix}\right| \\ &= \frac{\sqrt{478}}{2} \end{aligned}$$

***Note:** A **Notes** page such as this one that calculates the area of a triangle can be saved as a widget. As a widget, this page can be accessed at any time and used to solve a similar area of a triangle problem.*
*To save the page as a widget, press doc▾ > **File** > **Save As** > **MyWidgets**. Press tab to highlight **Save** and press enter. Ensure that you give the Widget a name such as 'area of a triangle'.*
*To access the widget, press doc▾ > **Insert** > **Widget**, select the widget, press tab to highlight **Add** and press enter.*



Finding and plotting the Cartesian equations of planes

The vector equation of a plane is given by:

- $\boldsymbol{r}\cdot\boldsymbol{n}=\boldsymbol{a}\cdot\boldsymbol{n}$

The Cartesian equation of a plane is given by:

- $ax+by+cz+d=0$

The plane Π contains the points $A(0,1,1)$, $B(-2,0,3)$ and $C(2,1,0)$.

(a) Find a Cartesian equation for the plane Π.

(b) Use the **3D Graphing** feature to plot the plane Π.

(c) State the axis intercepts of Π.

On a **Notes** page, assign $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ as row vectors as follows:

- Press [menu] > **Insert** > **Maths Box**.
- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [catalog] [5], select the **m-by-n Matrix** template, fix the dimensions as 1-by-3 and enter as shown.

***Note:** Alternatively, to insert a **Maths Box**, press* [ctrl] [M].

- Press [ctrl] [M] and enter $\boldsymbol{b}-\boldsymbol{a}$.
- Press [ctrl] [M] and enter $\boldsymbol{c}-\boldsymbol{a}$.

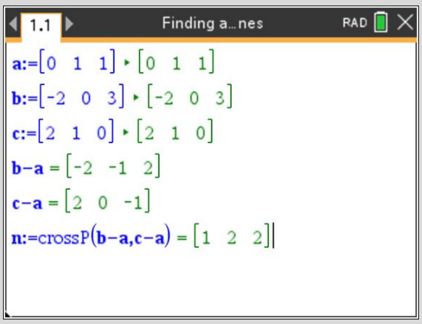
To display an equals sign in a **Maths Box**:

- Click on the **Maths Box**.
- Press [menu] > **Maths Box Options** > **Maths Box Attributes**.
- Press [tab] to highlight the **Insert Symbol** field.
- Press ▸ and select $=$.

***Note:** **Maths Box Attributes** can also be accessed within a **Maths Box** by pressing* [ctrl] [menu].

Assign $\boldsymbol{n}$, a vector normal to Π as follows:

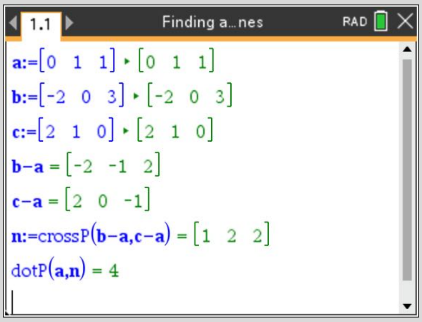
- Press [ctrl] [M] and enter $\boldsymbol{n}$.
- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [menu] > **Calculations** > **Matrix & Vector** > **Vector** > **Cross Product**.
- Enter as shown.
- Press [menu] > **Maths Box Options** > **Maths Box Attributes**, press [tab] to highlight the **Insert Symbol** field, press ▸ and select $=$.



... continued

To calculate $\boldsymbol{a}\cdot\boldsymbol{n}$ enter as follows:

- Press ctrl M.
- Press menu > **Calculations** > **Matrix & Vector** > **Vector** > **Dot Product**.
- Press menu > **Maths Box Options** > **Maths Box Attributes**, press tab to highlight the **Insert Symbol** field, press ▶ and select $=$.



A Cartesian equation for the plane of Π is $x+2y+2z=4$.

(a) $\boldsymbol{b}-\boldsymbol{a}=-2\hat{\boldsymbol{i}}-\hat{\boldsymbol{j}}+2\hat{\boldsymbol{k}}$ and $\boldsymbol{c}-\boldsymbol{a}=2\hat{\boldsymbol{i}}-\hat{\boldsymbol{k}}$.

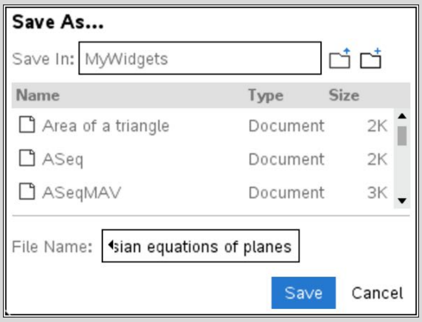
$$(\boldsymbol{b}-\boldsymbol{a})\times(\boldsymbol{c}-\boldsymbol{a})=\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}+2\hat{\boldsymbol{k}}$$

So $\boldsymbol{n}=\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}+2\hat{\boldsymbol{k}}$ is a vector normal to Π.

$$\boldsymbol{a}\cdot\boldsymbol{n}=\left(\hat{\boldsymbol{j}}+\hat{\boldsymbol{k}}\right)\cdot\left(\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}+2\hat{\boldsymbol{k}}\right)=4$$

Using $\boldsymbol{r}\cdot\boldsymbol{n}=\boldsymbol{a}\cdot\boldsymbol{n}$, a Cartesian equation plane of Π is $x+2y+2z=4$.

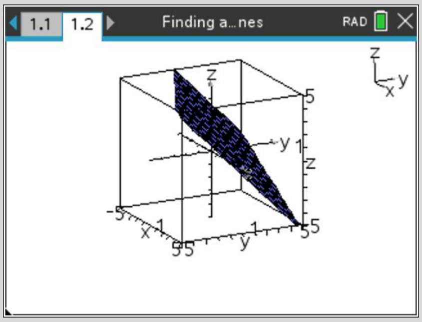
Note: *A* ***Notes*** *page such as this one that calculates the Cartesian equation of a plane can be saved as a widget. As a widget, this page can be accessed at any time and used to solve a similar problem.*
To save the page as a widget, press doc▾ *>* ***File*** *>* ***Save As*** *>* ***MyWidgets****. Press* tab *to highlight* ***Save*** *and press* enter*.*
Ensure that you give the Widget a name such as 'Cartesian equations of plane'.
To access the widget, press doc▾ *>* ***Insert*** *>* ***Widget****, select the widget, press* tab *to highlight* ***Add*** *and press* enter*.*



(b) Planes can be plotted using 3D graphing.

On a **Graphs** page:

- Press menu > **View** > **3D Graphing**.
- Enter $z1(x,y)=\dfrac{4-x-2y}{2}$.



Note: *Press* menu *>* ***Actions*** *or* ***View*** *or* ***Range/Zoom*** *to explore a suite of viewing alternatives including the range settings. To rotate the view of the plane, for example, press* menu *>* ***Actions*** *>* ***Rotate*** *(or press* R*) and then use the arrow keys.*

(c) The axis intercepts of Π are $(4,0,0)$, $(0,2,0)$ and $(0,0,2)$.

3.4. Topic 4: Vector calculus

3.4.1. Vector calculus

Finding when and where two particles meet

Two particles meet when they share the same position at the same time.

The motion of two particles are given by the position vectors

$$r_1(t) = (2t-3)\hat{i} + (t^2+10)\hat{j} \text{ and } r_2(t) = (t+2)\hat{i} + 7t\hat{j}\text{, where } t \geq 0.$$

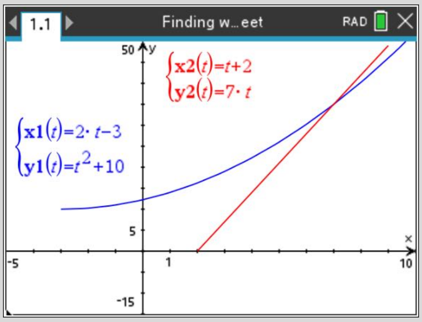
(a) Find when the two particles meet.

(b) Determine the coordinates of their meeting point.

Use parametric graphing mode to plot the paths of the two particles.

On a **Graphs** page:

- Press menu > **Graph Entry/Edit** > **Parametric**.
- Enter $\boldsymbol{x1(t) = 2t - 3}$
- Enter $\boldsymbol{y1(t) = t^2 + 10}$.
- Enter $\boldsymbol{0 \leq t \leq 7\ tstep = 0.1}$
- Enter $\boldsymbol{x2(t) = t + 2}$
- Enter $\boldsymbol{y2(t) = 7t}$
- Enter $\boldsymbol{0 \leq t \leq 7\ tstep = 0.1}$
- Press menu > **Window/Zoom** > **Window Settings**.

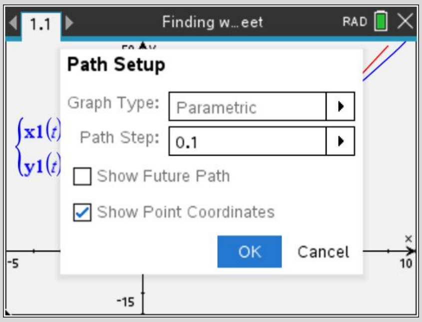


In the dialog box that follows, enter the following values:
XMin = –5 Xmax = 10 XScale = 1
YMin = –15 YMax = 50 YScale = 5

To set the path setup:

- Press menu > **Trace** > **Path Plot** > **Path Setup**.
- Complete the required fields as shown.

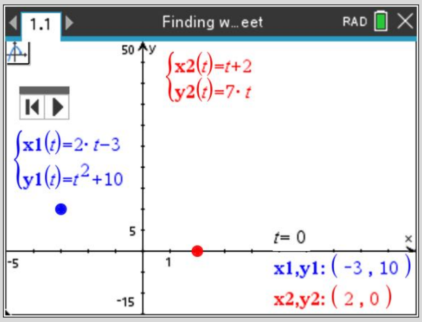
*Note: If desired, the future path(s) can be shown on the page by checking the **Show Future Path** box.*



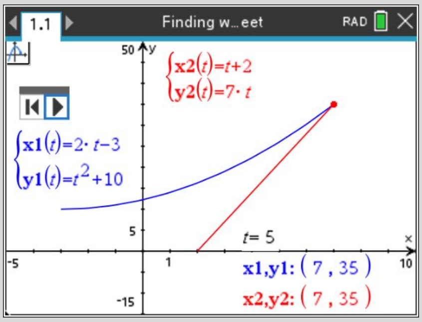
To set up the animation of the two particle's motion:

- Press menu > **Trace** > **Path Plot** > **Parametric**.

Note: The coordinates of each particle's path display at the bottom of the screen. This needs to be accounted for when positioning labels and setting suitable viewing windows.



... continued



To start the animation:

- Move the cursor over the animation start button and press enter.

To pause the animation:

- Move the cursor over the animation pause button and press enter.

To reset the animation:

- Move the cursor over the animation reset button and press enter.

The animation shows the two particles meeting when $t=5$ at $(7,35)$.

On a **Calculator** page, assign $\boldsymbol{r}_1(t)$ and $\boldsymbol{r}_2(t)$ as row vectors as follows:

- Press ctrl then the template key to access the **Assign** := command.
- Press the catalogue key then 5, select the **1-by-2 Matrix** template and enter as shown.

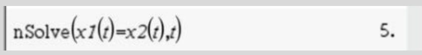
Note: *The parametric equations,* $\boldsymbol{x1}(\boldsymbol{t})$, $\boldsymbol{y1}(\boldsymbol{t})$, $\boldsymbol{x2}(\boldsymbol{t})$ *and* $\boldsymbol{y2}(\boldsymbol{t})$ *defined on the* ***Graphs*** *page are recognised on the* ***Calculator*** *page provided these two pages form part of the same problem.*

(a) Let (x, y) be any point on the curve defined by $\boldsymbol{r}(t)$.

$\boldsymbol{r}_1(t)=\boldsymbol{r}_2(t) \Rightarrow 2t-3=t+2$ and $t^2+10=7t$

Equate the $\hat{\boldsymbol{i}}$ components and solve:

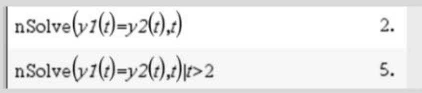
- Press menu > **Algebra** > **Numerical Solve**.
- Complete as shown



Solving $2t-3=t+2$ for t gives $t=5$.

Equate the $\hat{\boldsymbol{j}}$ components and solve:

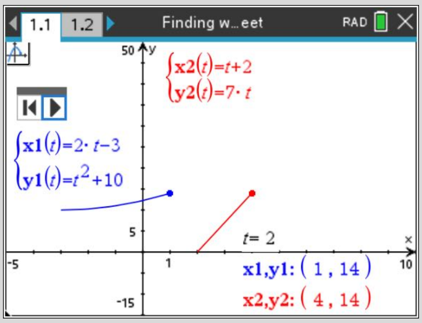
- Press ctrl = to access the ‘with’ or ‘given’ symbol |.



Solving $t^2+10=7t$ for t gives $t=2,5$.

Hence the two particles meet when $t=5$.

Note: *The paths of the particles cross at* $t=2$. *They do not meet at this time. The row vectors and the animation show when* $t=2$, *the particles are at* $(1,14)$ *and* $(4,14)$.

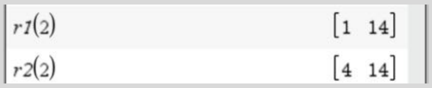


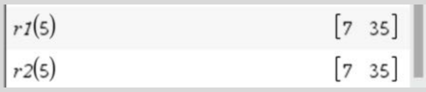
... continued

(b) $\boldsymbol{r}_1(5)=7\hat{\boldsymbol{i}}+35\hat{\boldsymbol{j}}$ and $\boldsymbol{r}_2(5)=7\hat{\boldsymbol{i}}+35\hat{\boldsymbol{j}}$

Hence the two particles meet at $(7,35)$.

Note: *If desired, press* ctrl T *to view a tabular representation of the parametric equations.*





Finding the Cartesian equation of a particle's path

The Cartesian equation of a path given as a vector equation in two dimensions include circles, ellipses and hyperbolas.

The equation of a circle is given by:

- $(x-h)^2+(y-k)^2=r^2$

The equation of an ellipse is given by:

- $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$

The equation of a hyperbola is given by:

- $\dfrac{(x-h)^2}{a^2}-\dfrac{(y-k)^2}{b^2}=1$ or $\dfrac{(y-k)^2}{a^2}-\dfrac{(x-h)^2}{b^2}=1$

Find the Cartesian equation of the path of a particle which moves such that its position vector at time t is given by

$$\boldsymbol{r}(t)=(1-2\cos(t))\hat{\boldsymbol{i}}+3\sin(t)\hat{\boldsymbol{j}} \text{ where } t\geq 0.$$

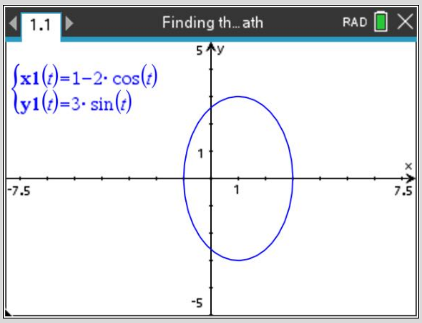
(a) Plot the path of the particle.

(b) Describe the motion of the body.

Use parametric graphing mode to plot the path of the particle.

On a **Graphs** page:

- Press menu **> Graph Entry/Edit > Parametric**.
- Enter $\boldsymbol{x1(t)=1-2\cos(t)}$
- Enter $\boldsymbol{y1(t)=3\sin(t)}$.
- Enter $\boldsymbol{0\leq t\leq 6.28}$ ***tstep*** $\boldsymbol{=0.032}$
- Press menu **> Window/Zoom > Window Settings**.



In the dialog box that follows, enter the following values:

XMin = –7.5	Xmax = 7.5	XScale = 1
YMin = –5	YMax = 5	YScale = 1

To set the path setup:

- Press menu **> Trace > Path Plot > Path Setup**.
- Complete the required fields as shown.

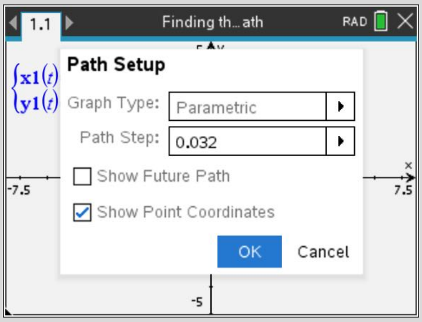
Note: *If desired, the future path can be shown on the page by checking the* ***Show Future Path*** *box.*

... continued

To set up the animation of the particle's motion:

- Press menu > **Trace** > **Path Plot** > **Parametric**.

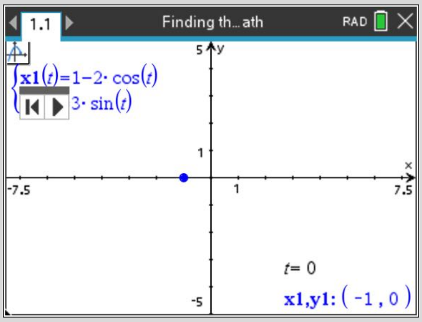
Note: *The coordinates of the particle's path display at the bottom of the screen. This needs to be accounted for when positioning labels and setting suitable viewing windows.*



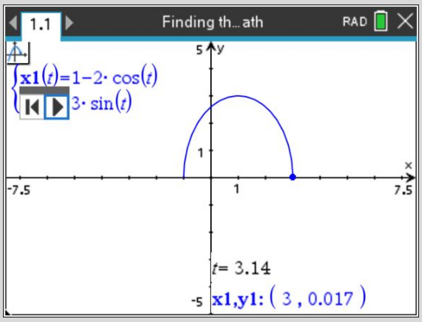
To start the animation:

- Move the cursor over the animation start button and press enter.

Note: *To pause the animation, move the cursor over the animation pause button and press* enter. *To reset the animation, move the cursor over the animation reset button and press* enter.



The animation shows the particle starting at $(-1,0)$ and travelling in a clockwise elliptical path with a period of $6.28(2\pi)$. The particle finishes its motion at $(-1,0)$.



(a) The parametric equations are:

$x = 1 - 2\cos(t)$ (1)

$y = 3\sin(t)$ (2)

From (1), $\dfrac{x-1}{-2} = \cos(t)$ and from (2), $\dfrac{y}{3} = \sin(t)$.

Squaring and adding the above two equations gives:

$\dfrac{(x-1)^2}{4} + \dfrac{y^2}{9} = 1$.

(b) An ellipse with centre $(1,0)$ and domain $-1 \le x \le 3$.

Using vector calculus to analyse the motion of a particle

The position of a particle at time t can be described by the vector function:

- $\boldsymbol{r}(t) = x(t)\hat{\boldsymbol{i}} + y(t)\hat{\boldsymbol{j}} + z(t)\hat{\boldsymbol{k}}$

The velocity of the particle at time t is given by:

- $\boldsymbol{v}(t) = x'(t)\hat{\boldsymbol{i}} + y'(t)\hat{\boldsymbol{j}} + z'(t)\hat{\boldsymbol{k}}$

The acceleration of the particle at time t is given by:

- $\boldsymbol{a}(t) = x''(t)\hat{\boldsymbol{i}} + y''(t)\hat{\boldsymbol{j}} + z''(t)\hat{\boldsymbol{k}}$

The velocity vector, $\boldsymbol{v}(t)$, has the direction of the particle's motion at time t.

The speed of a particle is given by $|\boldsymbol{v}(t)|$.

The position vector, $\boldsymbol{r}(t)$, at time t of a particle moving in a plane is given by

$$\boldsymbol{r}(t) = 60t\hat{\boldsymbol{i}} + \left(20 + 45t - 5t^2\right)\hat{\boldsymbol{j}} \text{ where } t \geq 0.$$

(a) Find the initial position of the particle.

(b) Find the initial velocity of the particle.

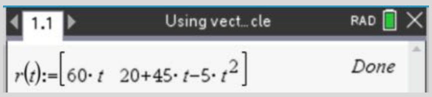
(c) Find when the particle is moving parallel to $\hat{\boldsymbol{i}}$.

(d) Find the particle's speed at $t = 1$. Give your answer correct to one decimal place.

On a **Calculator** page, assign $\boldsymbol{r}(t)$ as a row vector as follows:

- Press [ctrl] [template] to access the **Assign** [:=] command.
- Press [catalog] [5], select the **1-by-2 Matrix** template and enter as shown.

Enter $\boldsymbol{r}(t)$ as shown.



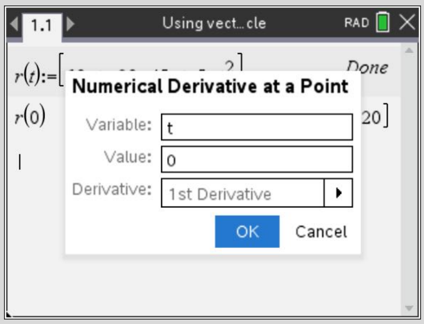
(a) The particle's initial position is given by $\boldsymbol{r}(0)$.

$\boldsymbol{r}(0) = 20\hat{\boldsymbol{j}}$

(b) The particle's initial velocity is given by $\boldsymbol{v}(0)$.

To find $\boldsymbol{v}(0)$:

- Press [menu] > **Calculus** > **Numerical Derivative at a Point**.
- Complete the required fields as shown.
- Enter $\boldsymbol{r}(t)$ and press [enter].

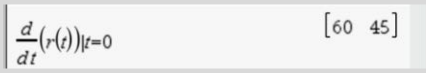


***Note:** Alternatively, to access the **Derivative** template, press [catalog] [5]. A more efficient alternative is achieved by pressing [shift] [−].*

... continued

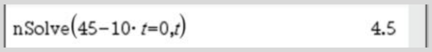
$\boldsymbol{v}(t) = 60\hat{\boldsymbol{i}} + (45 - 10t)\hat{\boldsymbol{j}}$

$\boldsymbol{v}(0) = 60\hat{\boldsymbol{i}} + 45\hat{\boldsymbol{j}}$



(c) To find when the particle is moving parallel to $\hat{\boldsymbol{i}}$, equate the $\hat{\boldsymbol{j}}$ component to zero and solve:

- Press menu > **Algebra** > **Numerical Solve**.
- Complete as shown.



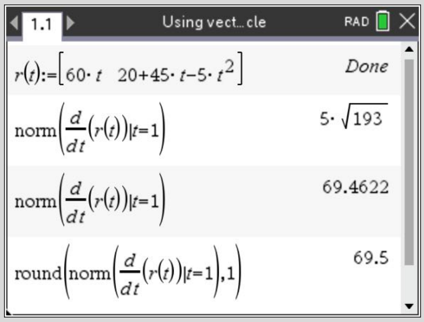
The particle is moving parallel to $\hat{\boldsymbol{i}}$ when $45 - 10t = 0$.

So $t = 4.5$.

(d) The particle's speed at $t = 1$ is given by $|\boldsymbol{v}(1)|$.

To calculate $|\boldsymbol{v}(1)|$ correct to one decimal place, enter as shown, taking note of the following instructions:

- Press menu > **Matrix & Vector** > **Norms** > **Norm**.
- Press ⇧shift − to access the **Derivative** template.
- Press ctrl = to access the 'with' or 'given' symbol |.
- Press menu > **Number** > **Number Tools** > **Round** to give $|\boldsymbol{v}(1)|$ correct to one decimal place.



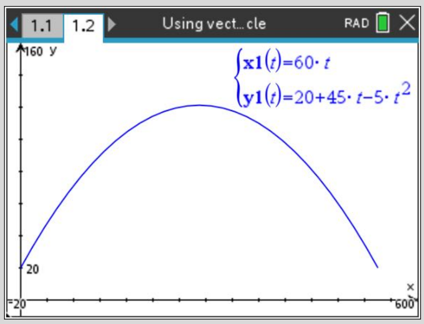
$\boldsymbol{v}(1) = 60\hat{\boldsymbol{i}} + 35\hat{\boldsymbol{j}}$

$$|\boldsymbol{v}(1)| = \sqrt{60^2 + 35^2}$$
$$= 69.4622...$$
$$= 69.5$$

The particle's path can be plotted using parametric graphing.

On a **Graphs** page:

- Press menu > **Graph Entry/Edit** > **Parametric**.
- Enter $\boldsymbol{x1(t) = 60t}$
- Enter $\boldsymbol{y1(t) = 20 + 45t - 5t^2}$.
- Enter $\boldsymbol{0 \le t \le 9}$ ***tstep*** $\boldsymbol{= 0.3}$
- Press menu > **Window/Zoom** > **Window Settings**.



In the dialog box that follows, enter the following values:

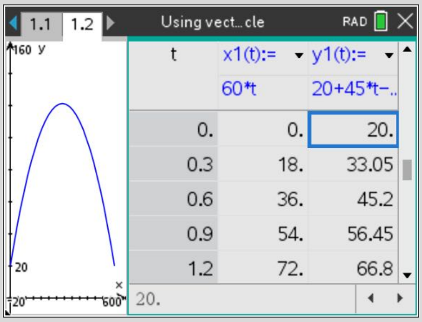
XMin = –20	Xmax = 600	XScale = 40
YMin = –10	YMax = 160	YScale = 20

Note: *If needed, it is often helpful to press* menu > ***Window/Zoom*** > ***Zoom Fit*** *to obtain a good first viewing window of the plotted graph. Otherwise, a useful guide is accessed by pressing* ctrl T *to show a tabular representation of the particle's path.*

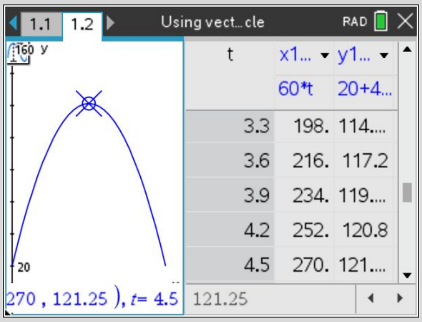
... continued

***Note:**. To edit the table settings, press* menu *> **Table** > **Edit Table Settings** and complete as desired. To resize the table's column widths, press* menu *> **Actions** > **Resize** and resize as desired.*

The screenshot at right shows that the particle is initially at $(0,20)$.



***Note:** Press* ctrl tab *to move between different applications or representations on the same page.*



To find when the particle is moving parallel to $\hat{i}$ on the **Graphs** application:

- Press menu > **Trace** > **Graph Trace**.
- Press ◀▶ to trace along the particle's path.
- If needed, press menu > **Trace** > **Trace Setup** to adjust the trace setup.

The particle is moving parallel to $\hat{i}$ at $(270,121.25)$.

3.5. Topic 5: Further matrices

3.5.1. Matrix algebra and systems of equations

Verifying properties involving the multiplicative identity and multiplicative inverse

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$ is an $n\times n$ matrix where a_{ij} is the element in row i and column j.

Recall from Section 1.5.1:

- An identity matrix, $\boldsymbol{I}$, is a square matrix that has the number 1 for all elements on the main diagonal and 0 for all other elements.
- $\boldsymbol{AI}=\boldsymbol{A}=\boldsymbol{IA}$ (multiplicative identity)
- $\boldsymbol{AA}^{-1}=\boldsymbol{I}=\boldsymbol{A}^{-1}\boldsymbol{A}$ (multiplicative inverse)

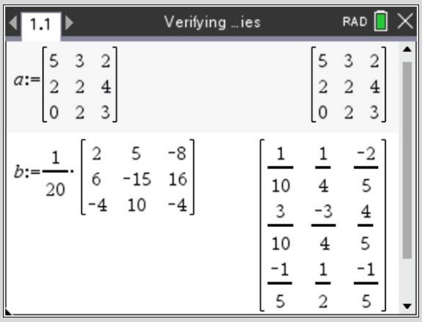
Let $\boldsymbol{A}=\begin{bmatrix} 5 & 3 & 2 \\ 2 & 2 & 4 \\ 0 & 2 & 3 \end{bmatrix}$ and $\boldsymbol{B}=\dfrac{1}{20}\begin{bmatrix} 2 & 5 & -8 \\ 6 & -15 & 16 \\ -4 & 10 & -4 \end{bmatrix}$.

Verify the following results:

(a) $\boldsymbol{AI}=\boldsymbol{A}=\boldsymbol{IA}$ **(b)** $\boldsymbol{AB}=\boldsymbol{I}=\boldsymbol{BA}$ where $\boldsymbol{B}=\boldsymbol{A}^{-1}$

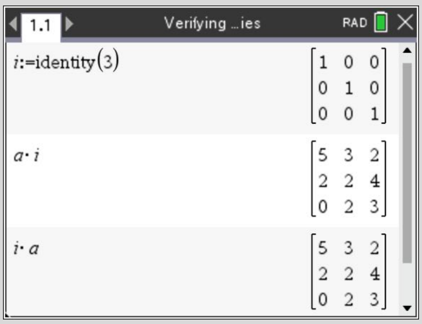
On a **Calculator** page, assign $\boldsymbol{A}$ and $\boldsymbol{B}$ as follows:

- Press [ctrl] [:=] to access the **Assign** [:=] command.
- Press [📖] [5], select the **m-by-n Matrix** template, fix the dimensions as 3-by-3 and enter as shown.



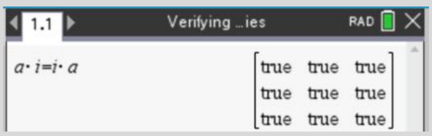
To create $\boldsymbol{I}$:

- Press [ctrl] [:=] to access the **Assign** [:=] command.
- Press [menu] > **Matrix & Vector** > **Create** > **Identity**.
- Enter as shown.



(a) $\boldsymbol{AI}=\begin{bmatrix} 5 & 3 & 2 \\ 2 & 2 & 4 \\ 0 & 2 & 3 \end{bmatrix}$, $\boldsymbol{IA}=\begin{bmatrix} 5 & 3 & 2 \\ 2 & 2 & 4 \\ 0 & 2 & 3 \end{bmatrix}$ and $\boldsymbol{AI}=\boldsymbol{A}=\boldsymbol{IA}$.

***Note:** Entering $\boldsymbol{AI}=\boldsymbol{IA}$ gives the output*
$\begin{bmatrix} \text{true} & \text{true} & \text{true} \\ \text{true} & \text{true} & \text{true} \\ \text{true} & \text{true} & \text{true} \end{bmatrix}$.



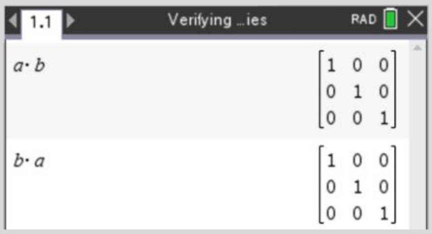
... continued

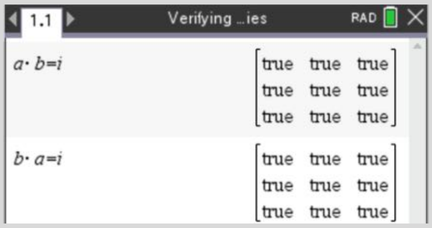
Enter $\boldsymbol{AB}$ and $\boldsymbol{BA}$ as shown.

(b) $\boldsymbol{AB} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\boldsymbol{BA} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Note: *Entering* $\boldsymbol{AB} = \boldsymbol{I}$ *and* $\boldsymbol{BA} = \boldsymbol{I}$ *both give the output* $\begin{bmatrix} \text{true} & \text{true} & \text{true} \\ \text{true} & \text{true} & \text{true} \\ \text{true} & \text{true} & \text{true} \end{bmatrix}$.

From part (b), it can be concluded that $\boldsymbol{B} = \boldsymbol{A}^{-1}$.





Calculating the determinant and inverse of 3 x 3 matrices

Consider $\boldsymbol{A}=\begin{bmatrix}1&2&4\\5&3&0\\3&2&1\end{bmatrix}$.

(a) Find $\det(\boldsymbol{A})$.

Consider $\boldsymbol{M}_1=\begin{bmatrix}3&0\\2&1\end{bmatrix}$, $\boldsymbol{M}_2=\begin{bmatrix}5&0\\3&1\end{bmatrix}$ and $\boldsymbol{M}_3=\begin{bmatrix}5&3\\3&2\end{bmatrix}$ where $\boldsymbol{M}_1$, $\boldsymbol{M}_2$ and $\boldsymbol{M}_3$ are 2×2 matrices that form part of $\boldsymbol{A}$.

(b) Find the value of $\det(\boldsymbol{M}_1)-2\det(\boldsymbol{M}_2)+4\det(\boldsymbol{M}_3)$.

(c) What do you notice about the results obtained in parts (a) and (b)?

(d) Find $\boldsymbol{A}^{-1}$.

On a **Calculator** page, assign $\boldsymbol{A}$ as follows:

- Press ctrl [template key] to access the **Assign** := command.
- Press [book key] 5, select the **m-by-n Matrix** template, fix the dimensions as 3-by-3 and enter as shown.

To find $\det(\boldsymbol{A})$:

- Press menu > **Matrix & Vector** > **Determinant** and enter as shown.

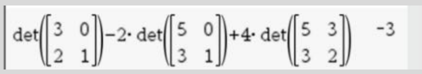
(a) $\det(\boldsymbol{A})=-3$

To find the value of $\det(\boldsymbol{M}_1)-2\det(\boldsymbol{M}_2)+4\det(\boldsymbol{M}_3)$:

- Press menu > **Matrix & Vector** > **Determinant**.
- Press [book key] 5 to select the **2-by-2 Matrix** template and enter as shown.

(b) $\det(\boldsymbol{M}_1)-2\det(\boldsymbol{M}_2)+4\det(\boldsymbol{M}_3)=-3$

(c) $\det(\boldsymbol{A})=\det(\boldsymbol{M}_1)-2\det(\boldsymbol{M}_2)+4\det(\boldsymbol{M}_3)$



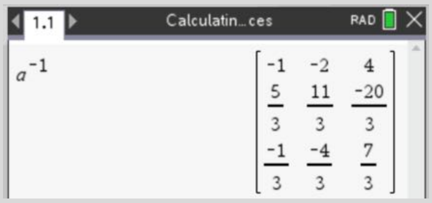
Note: *In general,*

$$\det\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}$$

$$=a_{11}\det\begin{bmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{bmatrix}-a_{12}\det\begin{bmatrix}a_{21}&a_{23}\\a_{31}&a_{33}\end{bmatrix}+a_{13}\det\begin{bmatrix}a_{21}&a_{22}\\a_{31}&a_{32}\end{bmatrix}.$$

Calculate $\boldsymbol{A}^{-1}$ as shown.

(d) $A^{-1}=\begin{bmatrix}-1&-2&4\\\frac{5}{3}&\frac{11}{3}&-\frac{20}{3}\\-\frac{1}{3}&-\frac{4}{3}&\frac{7}{3}\end{bmatrix}$



Determining whether a matrix is singular or non-singular

An $n \times n$ matrix $\boldsymbol{A}$ has an inverse if and only if $\det(\boldsymbol{A}) \neq 0$.

If $\det(\boldsymbol{A}) = 0$, then $\boldsymbol{A}$ is a singular matrix and $\boldsymbol{A}^{-1}$ does not exist.

Consider $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 3 & 8 \end{bmatrix}$.

Find $\det(\boldsymbol{A})$ and hence determine whether $\boldsymbol{A}$ is singular or non-singular.

To find $\det(\boldsymbol{A})$:

- Press [menu] > **Matrix & Vector** > **Determinant**.
- Press [📖][5], select the **m-by-n Matrix** template, fix the dimensions as 3-by-3 and enter as shown.

$\det(\boldsymbol{A}) = 0$ and so $\boldsymbol{A}$ is singular.

Note: *Since 2 is a common factor of the elements of the second row, the determinant can be expressed as* $2\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 8 \end{vmatrix}$.

The resulting determinant has two identical rows. If the corresponding elements of two rows (or columns) of a square matrix $\boldsymbol{A}$ *are equal, the determinant is zero. When the two equal rows of the matrix are interchanged, the matrix remains the same and hence the value of their determinants are equal. However, the values of their determinants are equal in magnitude but opposite in sign. This occurs only when the determinant of the matrix is zero.*

Solving matrix equations involving matrices beyond dimension 2 x 2

Recall from Section 1.5.1:

If $\boldsymbol{AX} = \boldsymbol{B}$, where $\boldsymbol{A}$ is a square matrix and has inverse $\boldsymbol{A}^{-1}$ such that $\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$, then the solution is $\boldsymbol{X} = \boldsymbol{A}^{-1}\boldsymbol{B}$.

Other matrix equations include $\boldsymbol{XA} = \boldsymbol{B}$ and $\boldsymbol{AX} + \boldsymbol{BX} = \boldsymbol{C}$.

Matrices can be used to encode and decode messages.

In this coding method, assign each letter of the alphabet with its position number in the alphabet.

So $A = 1$, $B = 2, \ldots, Z = 26$.

For example, to send the message *GO CATS*, write the letters in a 2×4 matrix $\boldsymbol{M}$.

$$\boldsymbol{M} = \begin{bmatrix} & G & O & \\ C & A & T & S \end{bmatrix}$$

Replace each letter of the alphabet with its position number in the alphabet and use a zero to represent a space.

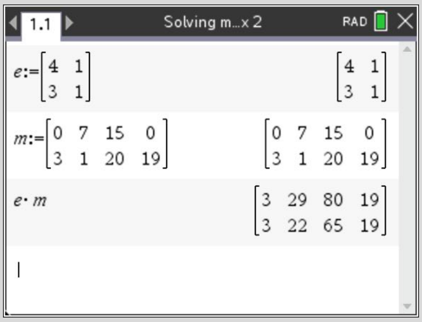
$$\boldsymbol{M} = \begin{bmatrix} 0 & 7 & 15 & 0 \\ 3 & 1 & 20 & 19 \end{bmatrix}$$

This code is fairly easy to crack. However, by multiplying $\boldsymbol{M}$ by a suitably sized encoding matrix, $\boldsymbol{E}$, this message can be made more difficult to decode.

Let $\boldsymbol{E} = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}$ and form the product $\boldsymbol{EM}$.

On a **Calculator** page, assign $\boldsymbol{E}$ and $\boldsymbol{M}$ as follows:

- Press ctrl to access the **Assign** := command.
- Press 5, select the **2-by-2 Matrix** template for $\boldsymbol{E}$ and enter as shown.
- Press 5, select the **m-by-n Matrix** template, fix the dimensions as 2-by-4 and enter as shown.



Calculate $\boldsymbol{EM}$ as shown.

$$\boldsymbol{EM} = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 0 & 7 & 15 & 0 \\ 3 & 1 & 20 & 19 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 29 & 80 & 19 \\ 3 & 22 & 65 & 19 \end{bmatrix}$$

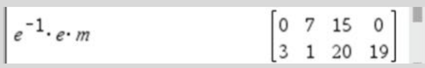
The encoded message sent to the recipient is the matrix product $\boldsymbol{EM}$.

To decode this message, the recipient must pre-multiply the matrix product $\boldsymbol{EM}$ by $\boldsymbol{E}^{-1}$.

$\boldsymbol{E}^{-1}\boldsymbol{EM} = \boldsymbol{M}$ as $\boldsymbol{E}^{-1}\boldsymbol{E} = \boldsymbol{I}$ and $\boldsymbol{IM} = \boldsymbol{M}$

Calculate $\boldsymbol{E}^{-1}\boldsymbol{EM}$ as shown.

$$\boldsymbol{E}^{-1}\boldsymbol{EM} = \begin{bmatrix} 0 & 7 & 15 & 0 \\ 3 & 1 & 20 & 19 \end{bmatrix}$$



The recipient now replaces each letter's position number in the alphabet with the corresponding letter and inserts a space for the zero.

So $\boldsymbol{M} = \begin{bmatrix} & G & O & \\ C & A & T & S \end{bmatrix}$ and the message received is *GO CATS*.

The trick to decoding messages of this type is to know $\boldsymbol{E}^{-1}$, the inverse matrix of the encoding matrix $\boldsymbol{E}$.
A good encoding matrix $\boldsymbol{E}$ is one that has $\det(\boldsymbol{E}) = \pm 1$ as this avoids the use of fractions when decoding a message.

You are encouraged to encode and decode various messages using the above approach. It is a good idea to vary $\boldsymbol{E}$.

Solving systems of linear equations involving matrices beyond dimension 2 x 2

Solve the following system of linear equations

$$\begin{aligned} 2u+4v+2z&=6\\ 3v+3w+z&=4\\ 2u+7v+9w+7z&=8\\ 6w+5z&=-4 \end{aligned}$$

using

(a) the **Reduced Row-Echelon Form** (**rref**) command.

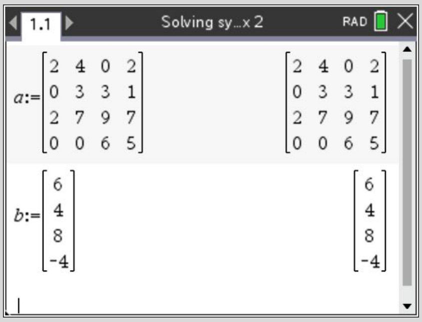
(b) the **Simultaneous** command.

In matrix form , the system of equations can be expressed as

$$\boldsymbol{AX}=\boldsymbol{B} \text{ where } \boldsymbol{A}=\begin{bmatrix}2&4&0&2\\0&3&3&1\\2&7&9&7\\0&0&6&5\end{bmatrix}, \boldsymbol{X}=\begin{bmatrix}u\\v\\w\\z\end{bmatrix} \text{ and } \boldsymbol{B}=\begin{bmatrix}6\\4\\8\\-4\end{bmatrix}.$$

On a **Calculator** page, assign $\boldsymbol{A}$ and $\boldsymbol{B}$ as follows:

- Press [ctrl] [template] to access the **Assign** [:=] command.
- Press [catalog] [5], select the **m-by-n Matrix** template, fix the dimensions as 4-by-4 and enter as shown.
- Press [catalog] [5], select the **m-by-n Matrix** template, fix the dimensions as 4-by-1 and enter as shown.



Solve $\boldsymbol{AX}=\boldsymbol{B}$ using reduced row-echelon form as follows:

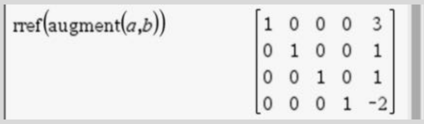
- Press [menu] > **Matrix & Vector** > **Reduced Row-Echelon Form**.
- Press [menu] > **Matrix & Vector** > **Create** > **Augment**.
- Enter as shown.

The **Reduced Row-Echelon Form** command instructs the *TI-Nspire CX II-T* to solve the system of linear equations in the form of a 4×5 augmented matrix using the method of elimination.

(a) The new augmented matrix, $\begin{bmatrix}1&0&0&0&3\\0&1&0&0&1\\0&0&1&0&1\\0&0&0&1&-2\end{bmatrix}$, is a translation of the original augmented matrix $\begin{bmatrix}2&4&0&2&6\\0&3&3&1&4\\2&7&9&7&8\\0&0&6&5&-4\end{bmatrix}$.

... continued

The matrix $\begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$ can be interpreted as the four equivalent transformed linear equations:



$1u+0v+0w+0z=3$

$0u+1v+0w+0z=1$

$0u+0v+1w+0z=1$

$0u+0v+0w+1z=-2$

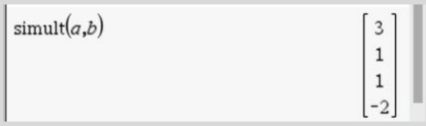
So $u=3$, $v=1$, $w=1$ and $z=-2$.

Note: *The **Augment** command is used to combine **A** and **B** so that the **Reduced Row-Echelon Form** command can be used directly without the need to create a 4 x 5 matrix from a template.*

To solve this system of linear equations using the **Simultaneous** command:

- Press menu > **Matrix & Vector** > **Simultaneous** and enter as shown.

(b) So $u=3$, $v=1$, $w=1$ and $z=-2$.



Note: *Section 1.5.1 showcases the use of the **Row Operations** menu which can be used to perform Gaussian techniques of elimination on augmented matrices.*

Examining different cases for solutions of systems of equations

There are three cases for solutions of systems of equations:

- A unique solution.
- No solution.
- Infinitely many solutions.

Consider the planes Π_1, Π_2 and Π_3 defined by the following Cartesian equations:

$$\Pi_1 : 6x - 3y - z = 3$$
$$\Pi_2 : 3x - 9y + 2z = -6$$
$$\Pi_3 : 2x + y - z = -1$$

(a) Use the **Reduced Row-Echelon Form** (**rref**) command to show that the three planes do not intersect.

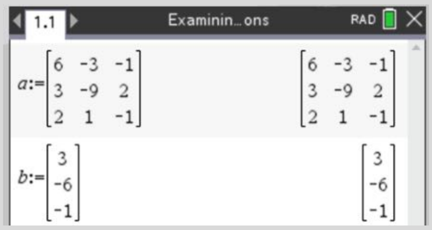
(b) Use the **3D Graphing** feature to plot the three planes.

In matrix form , the system of equations can be expressed as

$$\boldsymbol{AX} = \boldsymbol{B} \text{ where } \boldsymbol{A} = \begin{bmatrix} 6 & -3 & -1 \\ 3 & -9 & 2 \\ 2 & 1 & -1 \end{bmatrix}, \boldsymbol{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } \boldsymbol{B} = \begin{bmatrix} 3 \\ -6 \\ -1 \end{bmatrix}.$$

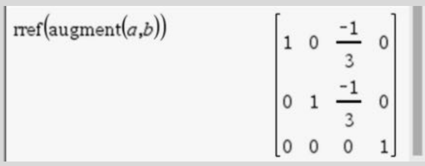
On a **Calculator** page, assign $\boldsymbol{A}$ and $\boldsymbol{B}$ as follows:

- Press ctrl to access the **Assign** := command.
- Press 5, select the **m-by-n Matrix** template, fix the dimensions as 3-by-3 and enter as shown.
- Press 5, select the **m-by-n Matrix** template, fix the dimensions as 3-by-1 and enter as shown.



Attempt to solve $\boldsymbol{AX} = \boldsymbol{B}$ using reduced row-echelon form as follows:

- Press menu > **Matrix & Vector** > **Reduced Row-Echelon Form**.
- Press menu > **Matrix & Vector** > **Create** > **Augment**.
- Enter as shown.



(a) The third row of the augmented matrix, $\begin{bmatrix} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

can be interpreted as the equation $0x + 0y + 0z = 1$.

This system of equations has no solutions and hence the three planes do not intersect.

... continued

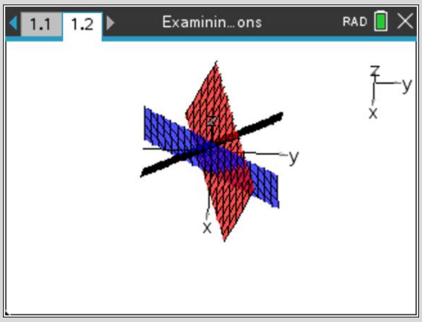
(b) Planes can be plotted using 3D graphing.

On a **Graphs** page:

- Press [menu] > **View** > **3D Graphing**.
- Enter $z1(x,y)=-3+6x-3y$.
- Enter $z2(x,y)=-3-\frac{3x}{2}+\frac{9y}{2}$.
- Enter $z3(x,y)=1+2x+y$.
- Press [menu] > **Range/Zoom** > **Range Settings**.

In the dialog box that follows, enter the following values:

XMin = –5	Xmax = 5	XScale = 5
YMin = –5	YMax = 5	YScale = 5
ZMin = –5	ZMax = 5	ZScale = 5



***Note:** Press [menu] > **Actions** or **View** or **Range/Zoom** to explore a suite of viewing alternatives. To rotate the view of the planes, for example, press [menu] > **Actions** > **Rotate** (or press [R]) and then use the arrow keys.*

For example, for the screenshot shown above right:

- Press [menu] > **View** > **Hide Box**.

The three planes do not intersect.

The three normals are coplanar but not parallel.

3.5.2. Applications of matrices

Modelling and solving problems involving Leslie matrices

$$\boldsymbol{L}=\begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_{m-1} & b_m \\ s_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & s_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & s_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & s_{n-1} & 0 \end{bmatrix}$$ is an $m \times m$ Leslie matrix where:

- m is the number of age groups being considered.
- s_i is the survival rate.
- b_i is the birth (fecundity) rate.

The state matrix, $\boldsymbol{S}_n$, is an $m \times 1$ matrix representing the number of each age group after n time periods.

- $\boldsymbol{S}_{n+1} = \boldsymbol{L}\boldsymbol{S}_n$ where $\boldsymbol{S}_0$ is the initial state (population) matrix.
- $\boldsymbol{S}_n = \boldsymbol{L}^n\boldsymbol{S}_0$ where $\boldsymbol{S}_0$ is the initial state (population) matrix.

Consider the following Leslie matrix $\boldsymbol{L}$ and initial population matrix $\boldsymbol{S}_0$:

$$\boldsymbol{L}=\begin{bmatrix} 0 & 2.5 & 0.5 \\ 0.6 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix} \text{ and } \boldsymbol{S}_0=\begin{bmatrix} 1000 \\ 500 \\ 125 \end{bmatrix}$$

(a) Determine $\boldsymbol{S}_1$, $\boldsymbol{S}_5$, $\boldsymbol{S}_{14}$ and $\boldsymbol{S}_{15}$.

(b) Find the total population after 15 time periods.

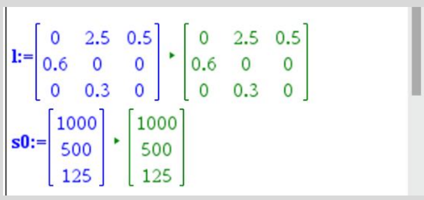
(c) Find the ratio of the total population for the 15th and 16th time periods.

(d) Plot the ratio of the total population for successive time periods.

Insert a **Maths Box** as follows:

- Press [menu] > **Insert** > **Maths Box**.

Note: *Alternatively, to insert a* ***Maths Box****, press* [ctrl] [M].



Assign $\boldsymbol{L}$ as follows:

- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [catalog] [5], select the **m-by-n Matrix** template, fix the dimensions as 3-by-3 and enter as shown.

Assign $\boldsymbol{S}_0$ as follows:

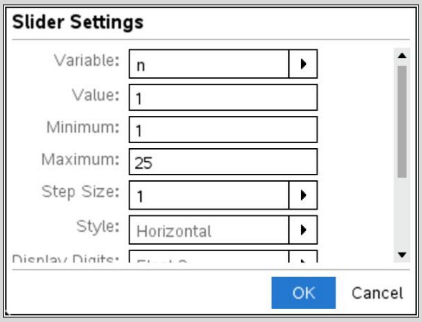
- Press [ctrl] [templates] to access the **Assign** [:=] command.
- Press [catalog] [5], select the **m-by-n Matrix** template, fix the dimensions as 3-by-1 and enter as shown.

... continued

On a **Notes** page:

Insert a **Slider** to control the value of n as follows:

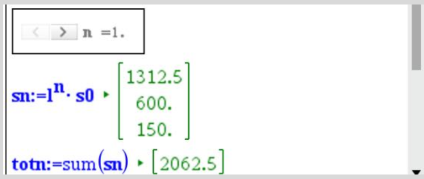
- Press menu > **Insert** > **Slider**.
- Set the **Slider Settings** as shown.
- Ensure to check the **Minimised** box.



Assign $\boldsymbol{S}_n$ as $\boldsymbol{L}^n \cdot \boldsymbol{S}_0$:

- Press ctrl M to insert a **Maths Box**
- Press ctrl to access the **Assign** := command and enter as shown.

To add the three age group populations, enter the formula $\boldsymbol{totn} := \mathbf{sum}(\boldsymbol{sn})$ as shown.



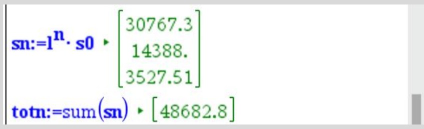
*Note: The command **Sum** can be typed in using the keypad.*

(a) Click on the slider to change the value of n and hence determine $\boldsymbol{S}_1$, $\boldsymbol{S}_5$, $\boldsymbol{S}_{14}$ and $\boldsymbol{S}_{15}$.

$$\boldsymbol{S}_1 = \begin{bmatrix} 1312.5 \\ 600 \\ 150 \end{bmatrix}, \boldsymbol{S}_5 = \begin{bmatrix} 3229.88 \\ 1488.38 \\ 370.575 \end{bmatrix}, \boldsymbol{S}_{14} = \begin{bmatrix} 23\,979.9 \\ 11758.4 \\ 2742.76 \end{bmatrix},$$

$$\boldsymbol{S}_{15} = \begin{bmatrix} 30\,767.3 \\ 14388.0 \\ 3527.51 \end{bmatrix}$$

(b) The total population after 15 time periods is found by adding the three age group populations in $\boldsymbol{S}_{15}$.



The total population is approximately 48,683.

(c) To calculate the state matrix after n + 1 time periods:

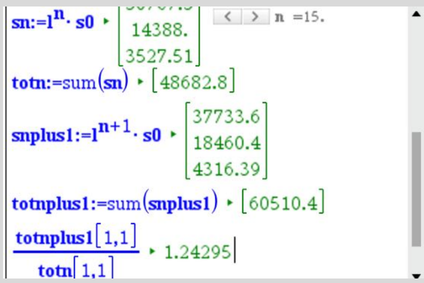
- Enter the formula $\boldsymbol{snplus1} := \boldsymbol{l}^{n+1} \times \boldsymbol{s0}$

To add the three age group populations after n + 1 time periods:

- Enter the formula $\boldsymbol{totnplus1} := \mathbf{sum}(\boldsymbol{snplus1})$

To find the successor ratio for the total population between consecutive time periods:

- Enter the formula $\dfrac{\boldsymbol{totnplus1}[1,1]}{\boldsymbol{totn}[1,1]}$.



Use the slider to set n = 15. The percentage increase between the 15th and 16th time periods is approximately 24.3%.

... continued

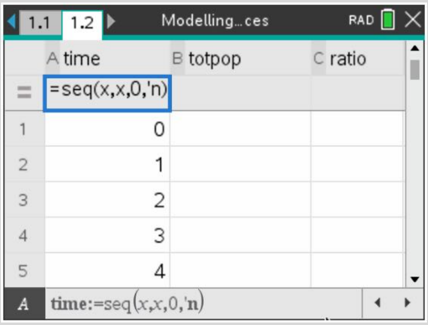
(d) To calculate the values for the successor ratio, add a **Lists & Spreadsheet** page, then:

- In the column A heading cell, enter the variable ***time***.
- In the column B heading cell, enter the variable ***totpop***.
- In the column C heading cell, enter the variable ***ratio***.

To generate the values for the time period, the column A formula cell:

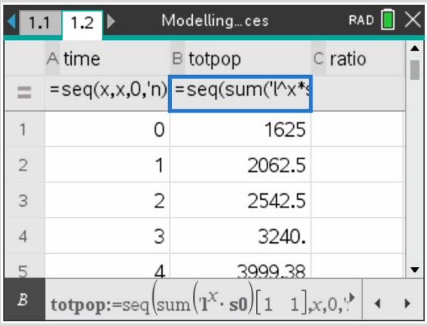
- Press [=], then press [📖][S], scroll down and select **seq(**.
- Enter $\textit{time} := \mathbf{seq}(x, x, 1, \text{'n})$.

Note: *The syntax for expressing a sequence as a list is* ***seq(Expression, Variable, Low, High[,Step])****. The default value for* ***Step*** *is 1.*
Note: *The symbol* **'** *in* **'n** *specifies n as a variable reference. (press [?!▸] to access the* **'** *symbol). Otherwise, TI-Nspire CX II-T will consider n as a column reference. If the* **'** *symbol is omitted, a prompt may appear asking whether the variable being used refers to a variable or to a spreadsheet column.*



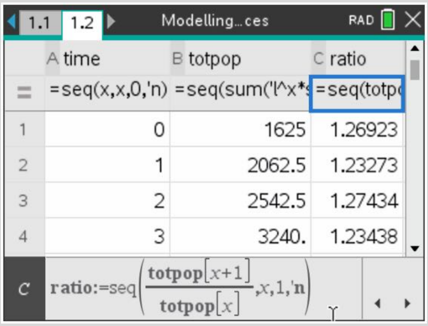
To generate the total population after n time periods, enter the following in the column B formula cell:

- Press [=], then press [📖][S], scroll down and select **seq(**.
- Enter $\textit{totpop} := \mathbf{seq}\left(\mathbf{sum}\left(\text{'l}^{x} \times s0\right)[1,1], x, 0, \text{'n}\right)$.



To generate the ratio for the $n + 1$ and n time periods, enter the following in the column C formula cell:

- Press [=], then press [📖][S], scroll down and select **seq(**.
- Enter $\textit{ratio} := \mathbf{seq}\left(\dfrac{\textit{totpop}[x+1]}{\textit{totpop}[x]}, x, 1, \text{'n}\right)$.

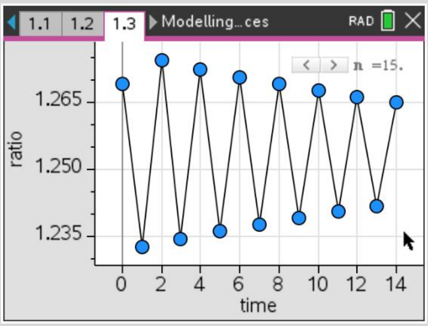


To construct a time series plot of ***ratio***, add a **Data & Statistics** page, then:

- Press [tab] and select ***time*** on the horizontal axis.
- Press [tab] and select ***ratio*** on the vertical axis.
- Press [menu] **> Plot Properties > Connect Data Points.**

To add a slider that will allow the user to alter the number of values of the successor ratio to be plotted, on the **Data & Statistics** page:

- Press [menu] **> Actions > Insert Slider**, then enter the following settings.
- Variable: **n**, Value: **1**, Min: **1,** Max: **15,** Step:**1,** minimise: ☑**.**



From the time series plot, it appears that the value of the ratio is oscillating and converging to a value of approximately 1.25 (approaching an increase of just over 25% in the total population for each time period).

Unit 4: Further calculus and statistical inference

4.1. Topic 1: Integration techniques

4.1.1. Integration techniques

Setting up a template to test antiderivatives found using substitution

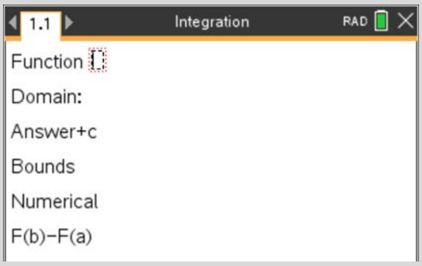
A student used the substitution $u = 1 - x$ to integrate the expression $x\sqrt{1-x}$.

The student obtained the answer $\frac{2}{5}(1-x)^{\frac{5}{2}} - \frac{2}{3}(1-x)^{\frac{3}{2}} + c$. Test the reasonableness of this answer.

The **Notes** application will be used to create an editable template to test answers obtained using integration techniques.

To enter the template headings & maths boxes on a **Notes** page:

- Enter the text shown in the screenshot.
- Move the cursor to the right of the word 'Function' and press [menu] > **Insert** > **Maths Box** (or press [ctrl] [M]).
- Repeat to insert **Maths Boxes** next to each of the other template headings.



To enter the formulas in the maths boxes on this **Notes** page, click in the relevant maths box and then:

- For 'Function', input $\boldsymbol{f(x) := x \cdot \sqrt{1-x}}$ then press [enter].

Note: *Press [ctrl] [:=] to assign a function or variable.*

- For 'Domain', input $\boldsymbol{x \le 1}$ then press [ctrl] [menu] > **Maths Box Attributes** > **Input & Output** > **No Calculation**. Then press [enter].
- For 'Answer + c', input $\boldsymbol{af(x) := \frac{2}{5}(1-x)^{\frac{5}{2}} - \frac{2}{3}(1-x)^{\frac{3}{2}}}$ then press [enter].

For the bounds, two random integers will be generated between −100 and 1, which is a subset of the domain of f.

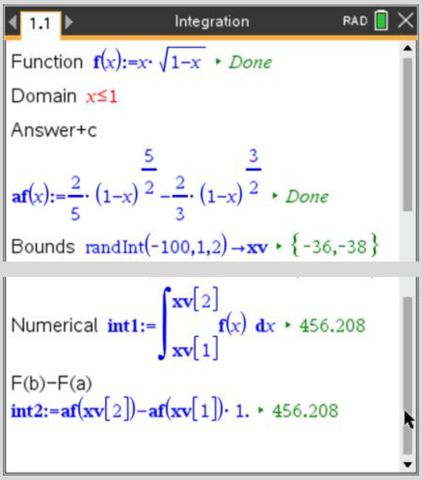
- For 'Bounds', input **randInt(–100,1,2)** then press [ctrl] [sto→] $\boldsymbol{xv}$ and then press [enter].

The numerical integral will be calculated with the first element in the list as the lower bound and second as the upper bound.

- For 'Numerical', input $\boldsymbol{\text{int1} := \int_{xv[1]}^{xv[2]} f(x)\,dx}$ and press [enter].

Note: *To access the numerical integral template either press [templates] and then select the template, or press [⇧shift] [+].*

- For 'F(b) – F(a)': input $\boldsymbol{\text{int2} := af(xv[2]) - af(xv[1]) \times 1.0}$ then press [enter]. Include '×**1.0**' to force a decimal answer.



Comparing the numerical integral with F(b) – F(a) confirms numerically the reasonableness of the student's answer.

... continued

The antiderivative will now be tested with different bounds.

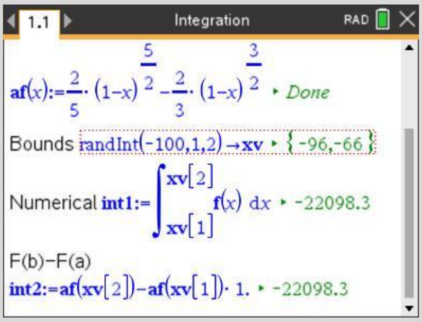
- Click the 'Bounds' **Maths Box** and press [enter] several times.

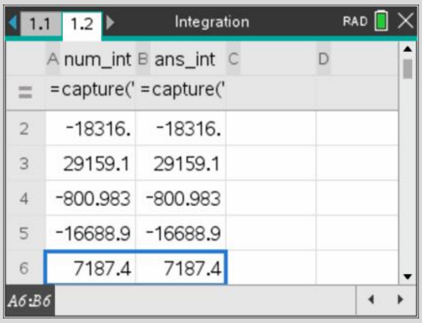
Observe that each time, a new pair of bounds is generated and the integrals **int1** and **int2** are recalculated. The recalculated values will be captured in the **Lists & Spreadsheet** application.

- Press [ctrl] [+page] > **Add Lists & Spreadsheet**
- Enter **num_int** in the Column A heading cell.
- Enter **ans_int** in the Column B heading cell.
- Use the arrow keys to navigate to the column A formula cell (second cell from the top). Press [menu] > **Data** > **Data Capture** > **Automatic**. Press [var], select **int1**, press [enter].
- Use the arrow keys to navigate to the column B and formula cell. Press [menu] > **Data** > **Data Capture** > **Automatic**. Press [var] > **int1**. Press [enter].
- Navigate to the **Notes** page 1.1, click the 'Bounds' Maths Box and press [enter] ten times.
- Navigate to the **Lists & Spreadsheet** page 1.2.

Observe that columns A and B have been populated with the values of **int1** and **int2**, and that **int1 = int2** in all cases. This provides compelling numerical support for the student's answer.

- Press [ctrl] [S] to save and name the document "Integration'.





***Note:** To clear the captured data, click on the formula cell, then press* [menu] *> **Data** > **Clear Data**.*

Applying inverse trigonometric functions to integration

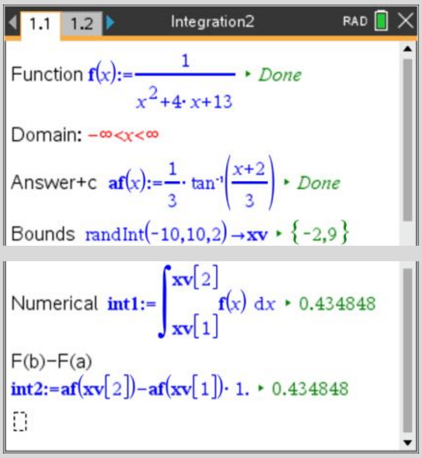
To find the indefinite integral $\int \frac{1}{x^2+4x+13}\,dx$, a student completed the square on the denominator and then used the substitution $u = x+2$. The student obtained the answer

$$\frac{1}{3}\tan^{-1}\left(\frac{x+2}{3}\right)+c,\ x\in\mathbb{R}.$$

Test the reasonableness of this answer.

To test the reasonableness of this answer, open the document from the previous question.

- On page 1.1, edit the Maths Boxes for 'Function', 'Domain' and 'Answer+c' in accordance with the information provided.
- Edit the Maths Boxes for 'Bounds' to an appropriate subset of ℝ, such as **randInt(–10,10,2)** then press [enter].
- Navigate to page 1.2 and clear the data in columns A and B: click on formula cell, then press [menu] > **Data** > **Clear Data**.
- On page 1.1, click the 'Bounds' Maths Box and press [enter] ten times. Observe the results on page 1.2, which provides compelling support for the student's answer.



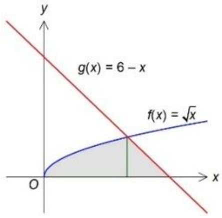
4.2. Topic 2: Applications of integral calculus

4.2.1. Applications of integral calculus

Determining the area of a region between curves defined with respect to x or y

Determine the area of the region bounded by the x-axis and the graphs of the functions $f(x)=\sqrt{x}, x\ge 0$ and $g(x)=6-x, x\in\mathbb{R}$.

The diagram on the right illustrates part of the graphs and the bounded region.



To determine the bounded area, on a **Calculator** page:

- Enter $\boldsymbol{f(x):=\sqrt{x}}$ and $\boldsymbol{g(x):=6-x}$
- Press [menu] > **Algebra** > **Numerical Solve**, enter **nSolve**$\boldsymbol{(f(x)-g(x)=0,x)}$.

Answer: Intersection at $x=4$.

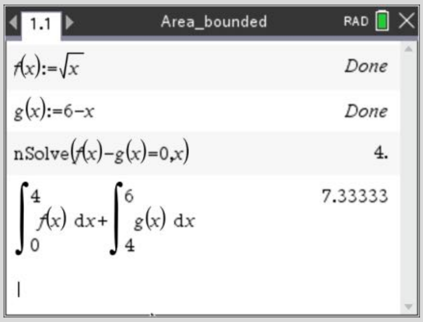
- Press either [template] or [shift][+] for the numerical integral template. Enter $\boldsymbol{\int_0^4 f(x)dx+\int_4^6 g(x)dx}$.

Alternative method – as a region defined with respect to y.

The equations of the curves can be expressed as $x=y^2, y\ge 0$ and $x=6-y, y\in\mathbb{R}$. Curves intersect at $y=f(4)=2$.

- Press either [template] or [shift][+] for the **Numerical Integral** template. Enter $\boldsymbol{\int_0^2\left((6-y)-y^2\right)dy}$.

The area is $7.\dot{3}=\dfrac{22}{3}$.



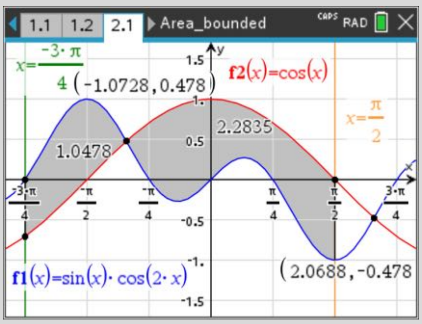
Determining the area of a region between the graphs of two trigonometric functions

Find the area of the region bounded by the graphs of the functions $f(x)=\sin(x)\cos(2x)$ and $g(x)=\cos(x)$ over the interval $x\in\left[-\dfrac{3\pi}{4},\dfrac{\pi}{2}\right]$. Give the answer correct to four decimal places.

To find the area of the region, on a **Graphs** page:

- Enter $\boldsymbol{f1(x)=\sin(x)\cdot\cos(2x)}$ and $\boldsymbol{f2(x)=\cos(x)}$.
- Press [menu] > **Graph Entry/Edit** > **Relation**.
- Enter $\boldsymbol{x=-\dfrac{3\pi}{4}}$ and $\boldsymbol{x=\dfrac{\pi}{2}}$.
- Press [menu] > **Window/Zoom** > **Window Settings**

  In the dialog box that follows, enter the following values:
  XMin = −2.6 Xmax = 2.6 XScale = π/4
  YMin = −1.7 YMax = 1.7 YScale = 0.5

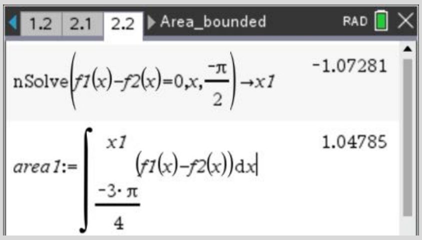


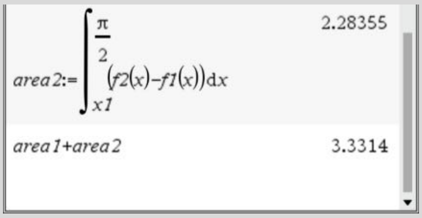
... continued

- Press [menu] > **Analyse Graph** > **Bounded Area**.
- Click on **Graph f1** then **Graph f2.** Enter lower bound point at (−3π/4, 0), and upper bound point at (−1.072 …, 0.478…).
- Repeat the above, except click lower bound point at (−1.072 …, 0.478…) and upper bound point at (π/2, 0).

Note: *To find the coordinates of multiple intersection points simultaneously, press* [menu] ***> Geometry > Points & Lines > Intersection points****. Click on* ***Graph f1*** *and then* ***Graph f2****.*

In the **Calculator** page, verify that the total bounded area is 3.3314.





Calculating volumes of solids of revolution about the x-axis

(a) The region bounded by the graph of $y = \cos(x)$, and the vertical lines with equations $x = -\dfrac{7\pi}{16}$ and $x = \pi$ is rotated 360° about the *x*-axis. Find the volume of this solid of revolution, correct to three decimal places.

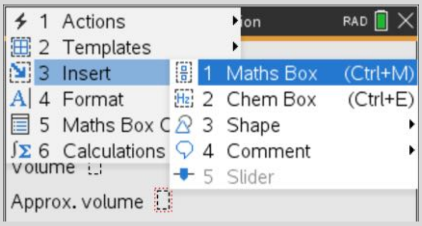
(b) The region bounded by the graphs of $y = 2^x$, $y = 1$ and $x = 2$ is rotated about the *x*-axis. Find the volume of this solid of revolution, correct to three decimal places.

(a) To find the volume of solid of revolution, on a **Notes** page:

- Enter the text shown in the screenshot.
- Move the cursor to the right of the word ‘Function’ and press [menu] > **Insert** > **Maths Box** (or press [ctrl][M]).

Repeat to insert **Maths Boxes** next to each of the other template headings.

Note: *To edit the text colour, select the text by holding* [⇧shift] *and ‘arrow’ across the text. Then press* [menu] ***> Format > Text colour****.*



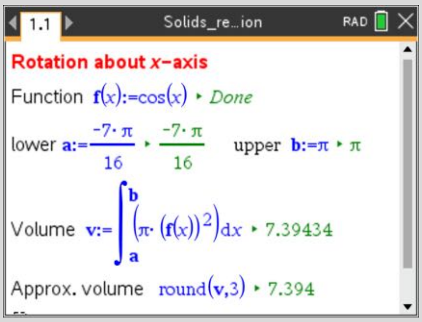
- Click on the **Maths Box** next to the word ‘Function’.
- Inside the **Maths Box**, input $\boldsymbol{f(x) := \cos(x)}$ then press [enter].

Note: *Press* [ctrl][:=] *to assign a function or variable.*

Similarly, in the other **Maths Boxes**:

- For ‘lower’, input $\boldsymbol{a := -\dfrac{7\pi}{16}}$.
- For ‘upper’, input $\boldsymbol{b := \pi}$.
- For ‘Volume’, input $\boldsymbol{v := \int_a^b \left(\pi \cdot (f(x))^2\right) dx}$.

Note: *To access the* ***Numerical Integral*** *template, either press* [templates] *then select the template, or press* [⇧shift][+].



- For ‘Approx. volume’, input **round(*v*,3)**.

Note: ‘round’ *command* [📖][R] *and down arrow to* ***round*** *.*

Answer: The volume is 7.394, correct to 3 decimal places.

… continued

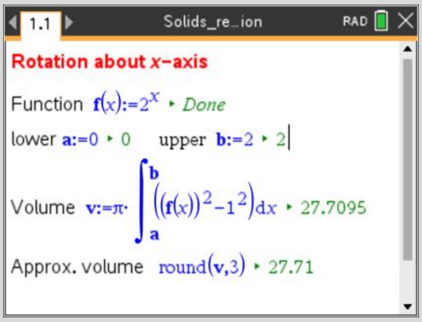
(b) Note that this region is not bounded by the x-axis. The graphs of $y = 2^x$ and $y = 1$ intersect at $x = 0$.

To find the volume of the solid of revolution:

- Open the saved document from part **(a)** above.

Click a **Maths box**, edit the input as follows.

Note: *Press* enter *after each edit to evaluate the Maths Box.*



- For 'Function', input $\boldsymbol{f(x) := 2^x}$
- For 'lower', input $\boldsymbol{a := 0}$, 'upper' $\boldsymbol{b := 2}$.
- For 'Volume', input $\boldsymbol{v := \int_a^b \left((f(x))^2 - 1^2 \right) dx}$.

Answer: The volume is 27.710, correct to 3 decimal places.

Note: *Although the Calculator application could be used to solve the problem above, it is useful to set up the solution method in the* ***Notes*** *application. The document can be saved, reopened and edited to solve similar problems in the future. Press* ctrl S *to save a document.*

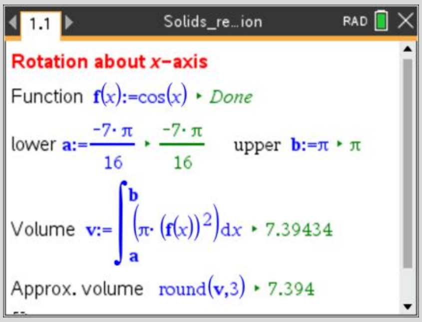
Constructing graphs associated with solids of revolution

(a) Plot a graph to display the region bounded by the graph of $y = \cos(x)$, and the vertical lines with equations $x = -\dfrac{7\pi}{16}$ and $x = \pi$.

(b) Create a 3-D graph of the surface of revolution when the region bounded by the graphs of $y = \cos(x), x = -\dfrac{7\pi}{16}$ and $x = \pi$ is rotated 360° about the x-axis.

(a) Open the saved document from the previous question above, with the Maths Box inputs set for part **(a)** of the question.

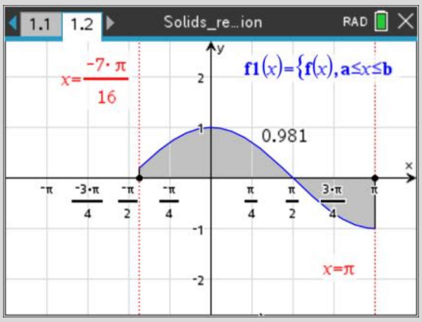
To plot a graph of the bounded region, add a **Graphs** page (press ctrl +page > **Add Graphs**.), and then:



- Enter the function $\boldsymbol{f1(x) := f(x) \mid a \le x \le b}$
- Press menu > **Graph Entry/Edit** > **Relation**, then enter the relations $\boldsymbol{x = -\dfrac{7\pi}{16}}$ and $\boldsymbol{x = \pi}$.
- Press menu > **Window/Zoom** > **Window Settings**.

  In the dialog box that follows, enter the following values:

XMin = −5π/4	Xmax = 5π/4	XScale = π/4
YMin = −2.7	YMax = 2.7	YScale = 1

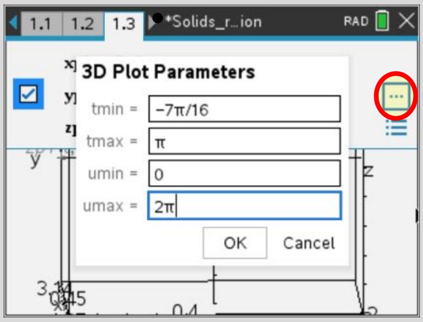


- Press menu > **Analyse Graph** > **Integral**.
  Click on the lower bound point at (−7π/16, 0)
  Click on the upper bound point at (π, 0). Press esc to exit.
- Press ctrl S to save the document.

... continued

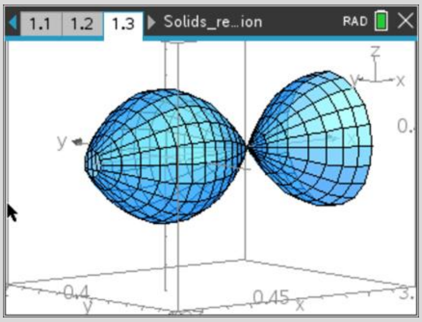
(b) If a surface is obtained by rotating a curve $y = f(x)$, $x \in [a,b]$, about the x-axis, then this surface of revolution has parametric equations:

$x = t, \quad y = f(t)\cdot\cos(u), \quad z = f(t)\cdot\sin(u)$, where $t \in [a,b]$ and $u \in [0, 2\pi]$.



To create a 3D graph of the surface of revolution, open the saved document from Part (a) above, with the **Maths Box** inputs set for the region $y = \cos(x), x = -\dfrac{7\pi}{16} \le x \le \pi$.

- Add a Graphs page: press [ctrl] [+page] **> Add Graphs**.
- Press [menu] > **View** > **3D Graphing**
- Press [menu] > **3D Graph Entry/Edit** > **Parametric**



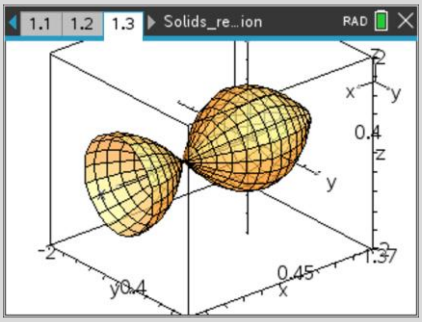
In the dialog box that follows, click the [...] icon on the right (circled in red in the screenshot above right).

In the **3D Plot Parameters** dialog box that follows, input the following values, then press **OK**.

tmin = −7π/16 tmax = π umin = 0 umax = 2π

Note*: *Clicking* OK*, causes a return to the* 3D Graph Entry/Edit *dialog box.

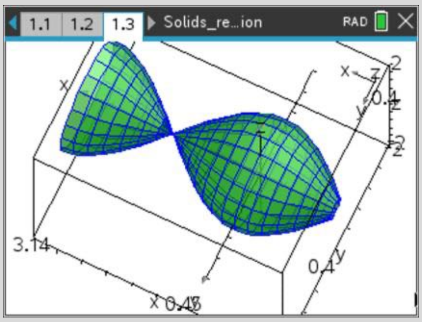
- Enter the parametric equation $\boldsymbol{xp1(t,u) = t}$, followed by
- $\boldsymbol{yp1(t,u) = f(t)\cdot\cos(u)}$ and $\boldsymbol{zp1(t,u) = f(t)\cdot\sin(u)}$



- Press [menu] **> Range/Zoom > Range Settings**

  In the dialog box that follows, enter the following values:

  XMin = −π/2 Xmax = π XScale = Auto
  YMin = −1.1 YMax = 1.1 YScale = Auto
  ZMin = −1.1 ZMax = 1.1 ZScale = Auto



- Press the multiplication key [×] repeatedly to magnify the graph size. (Press the division key [÷] to shrink graph size)
- Move the cursor to the graph and press [ctrl] [menu]. From the **Context** menu select **Colour > Fill Colour** to change colour. Select **Attributes** to change other characteristics.
- Press the arrow keys to rotate the graph in 3D.
- Press [ctrl] [S] to save the document.

Calculating the volume of a solid of revolution about the y-axis

The region bounded by the curve with equation $y = x^2 - 4$, the line with equation $y = 5$ and the y-axis is rotated by 360° about the y-axis to form a solid of revolution.

(a) Plot a graph of the region bounded the curve $y = x^2 - 4$, the line $y = 5$ and the y-axis.

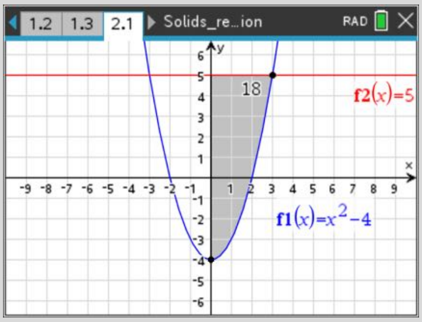
(b) Determine the volume of the solid of revolution, correct to two decimal places.

(c) Create a 3-D graph of the surface of revolution when the region bounded by the curve $y = x^2 - 4$, the line $y = 5$ and the y-axis is rotated about the y-axis.

Note: *The solution method can be set up similarly to the previous problem, either in a* ***New Document*** *or as a* ***New Problem*** *in the existing document. To add a new problem to the existing 'Solids of Revolution' document, press* doc▾ ***> Insert > Problem****.*

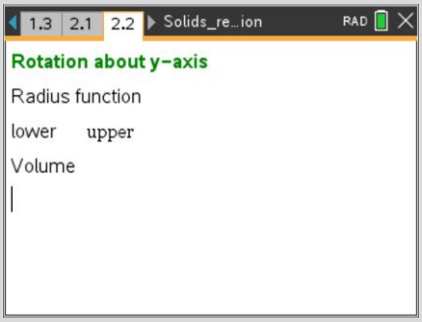
(a) To plot a graph of the bounded region, on a **Graphs** page:

- Enter $f1(x) = x^2 - 4$ and $f2(x) = 5$
- Press menu > **Analyse Graph** > **Bounded Area**. Click on the two graphs and then the bounds (i.e. the points at (0, −4), (0, 5) and (3, 5). Press esc to exit this tool.



(b) To determine the volume of the solid of revolution, insert a **Notes** page (press ctrl +page > **Add Notes**), then:

- Enter the text shown in the screenshot.
- Move the cursor to the right of the word 'Radius function' and press menu > **Insert** > **Maths Box** (or press ctrl M).
- Repeat to insert **Maths Boxes** next to each of the other template headings.



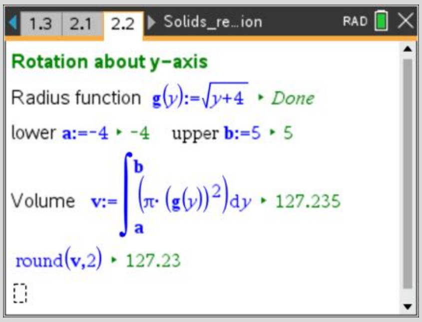
Note that if $y = x^2 - 4, 0 \le x \le 3$, then $x = \sqrt{y+4}, \ -4 \le y \le 5$

Click on the **Maths Box** next to the word 'Radius function'.

Inside the **Maths Box**, input $\boldsymbol{g(y) := \sqrt{y+4}}$ then press enter.

Similarly, in the **Maths Box** next to the word:

- For 'lower', enter $\boldsymbol{a := -4}$.
- For 'upper', enter $\boldsymbol{b := 5}$.
- For 'Volume', enter $\boldsymbol{v := \int_a^b \left(\pi \cdot (g(y))^2\right) dy}$.

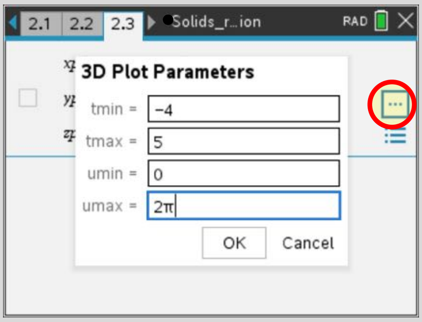


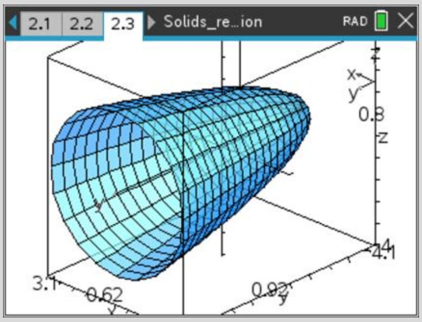
Answer: The volume is 127.23, correct to two decimal places.

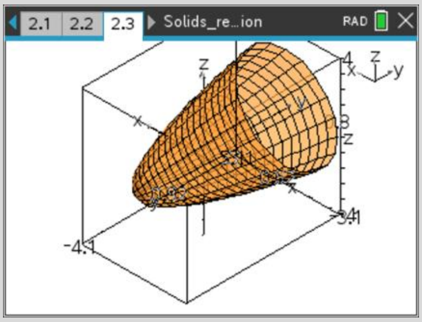
... continued

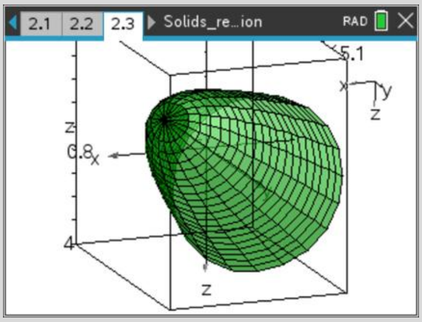
(c) If a surface is obtained by rotating a curve $x = g(y)$, $y \in [a,b]$, about the y-axis, then this surface of revolution has the following parametric equations:

$x = g(t)\cdot\sin(u), \quad y = t, \quad z = g(t)\cdot\cos(u)$, where $t \in [a,b]$ and $u \in [0, 2\pi]$.









To create a 3D graph of the surface of revolution, add a **Graphs** page (press ctrl +page > **Add Graphs**) then:

- Press menu > **View > 3D Graphing**
- Press menu > **3D Graph Entry/Edit > Parametric**

In the dialog box that follows, click the ⋯ icon on the right. (circled in red in the screenshot above right).

In the **3D Plot Parameters** dialog box that follows, input the following values, then press **OK**.

tmin = −4 tmax = 5 umin = 0 umax = 2π

Clicking **OK** causes a return to the **3D Graph Entry/Edit** dialog box. Enter the parametric equations:

- $\boldsymbol{xp1(t,u) = g(t)\cdot\sin(u)}$
- $\boldsymbol{yp1(t,u) = t}$
- $\boldsymbol{zp1(t,u) = g(t)\cdot\cos(u)}$
- Press menu > **Range/Zoom > Range Settings**

  In the dialog box that follows, enter the following values:
  XMin = −0.1 Xmax = 3.1 XScale = Auto
  YMin = −4.1 YMax = 5.1 YScale = Auto
  ZMin = −4 ZMax = 4 ZScale = Auto

- Press × repeatedly to magnify the graph size. (Press ÷ to shrink the graph size)
- Move the cursor to the graph and press ctrl menu. From the **Context** menu select **Colour > Fill Colour** to change colour. Select **Attributes** to change other characteristics.
- Press the arrow keys to rotate the graph in 3D.
- Press ctrl S to save the document.

Applying Simpson's rule to a definite integral

(a) Use Simpson's rule with four intervals to find an approximate value of $\int_3^4 \left(x^3 - x + 1\right) dx$, correct to two decimal places.

(b) Evaluate the integral using technology and determine the reasonableness of the approximation in part (a) above.

(a) To calculate the Simpson's rule approximation to the integral, on a **Calculator** page:

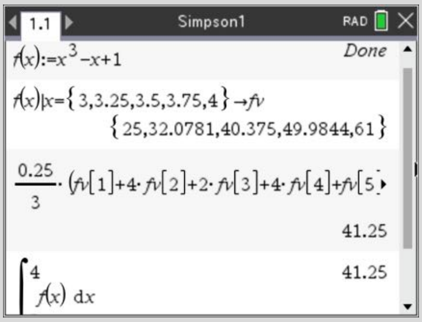
- Enter $\boldsymbol{f(x) := x^3 - x + 1}$.

Substitute the list of x-values in $f(x)$ and store as list $\boldsymbol{fv}$:

- Enter $\boldsymbol{f(x) \mid x = \{3, 3.25, 3.5, 3.75, 4\}}$ ctrl sto→ $\boldsymbol{fv}$

The individual terms in the list can be recalled as $\boldsymbol{fv[1]}, \boldsymbol{fv[2]}$ etc. Recalling values eliminates possible transcription errors.

- Enter $\boldsymbol{\frac{0.25}{3}\left(fv[1] + 4fv[2] + 2fv[3] + 4fv[4] + fv[5]\right)}$



(b) Answer: 41.25 for both integral and Simpson's rule.

Exploring Simpsons rule for the effect of interval size

Investigate the effect of increasing the number of intervals on the accuracy of the Simpson's rule approximation for the area under the graph of $f(x) = \sin(x)e^{2x}$, $0 \le x \le \pi$.

(a) Plot the graph of the function and use a graphical method to find the area under the graph bounded by the x-axis, correct to two decimal places.

(b) Explore the accuracy of the Simpson's rule approximation for a variety of interval sizes, between 2 intervals and 20 intervals.

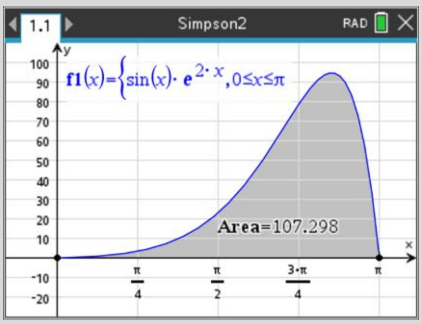
(c) Plot the value of Simpson's rule approximation against the interval size.

(a) To find the area, on a **Graphs** page:

- Enter $\boldsymbol{f1(x) = \sin(x) \cdot e^{2x} \mid 0 \le x \le \pi}$
- Press menu > **Window/Zoom** > **Window Settings**

  In the dialog box that follows, enter the following values:
  XMin = −0.5 Xmax = 3.5 XScale = π/4
  YMin = −30 YMax = 110 YScale = 10

- Press menu > **Analyse Graph** > **Integral**.
- Click on the lower bound (point at (0, 0))
- Click on the upper bound (point at (π, 0)).
- To increase the precision of the answer, hover over the answer text and press +.



Answer: Area is 107.30 (2 decimal places)

For part **(b)** of this investigation, the **Notes** application will be used to create an editable template for Simpson's rule. The **seqn** command will be used to create a list of coefficients $\{1,4,2,4,2,\ ...,1\}$ and another list $\{f(x_0), f(x_1), f(x_2),\ ..., f(x_m)\}$.

The **seqn** command generates a sequence for the variable n. E.g. $\mathbf{seqn}(\boldsymbol{n}^2,\mathbf{5})$ gives $\{1,4,9,16,25\}$.

The equation $\mathbf{mod}(\boldsymbol{n},\mathbf{2})=\mathbf{0}$ selects the even terms of a sequence (terms divisible by 2 with zero remainder. An alternative command for the remainder is '**remain(input1,input2)**'.

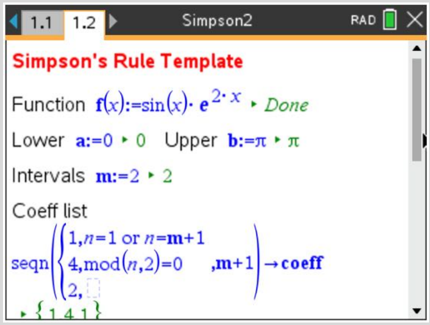
(b) To create an editable template for the Simpson's rule approximation, add a **Notes** page ([ctrl] [+page] > **Add Notes**), then:

- Enter the text as per the screenshot.
- Press [ctrl] [M] next to each text item to insert **Maths Boxes**
- Input the following in the Maths Boxes. Press [enter] each time.
- For 'Function': $\boldsymbol{f}(\boldsymbol{x}) := \mathbf{sin}(\boldsymbol{x}) \cdot \boldsymbol{e}^{2x}$
- For 'Lower': $\boldsymbol{a} := \mathbf{0}$ Upper: $\boldsymbol{b} := \boldsymbol{\pi}$ Interval: $\boldsymbol{m} := \mathbf{2}$
- For 'Coeff list': Input **seqn( )**.
- Click inside the brackets, then press [A] [÷] [÷] [×] and select **Piecewise Function** template.

  Input $\mathbf{seqn}\left(\begin{cases}\mathbf{1}, & \boldsymbol{n}=\mathbf{1} \textbf{ or } \boldsymbol{n}=\boldsymbol{m}+\mathbf{1}\\ \mathbf{4}, & \mathbf{mod}(\boldsymbol{n},\mathbf{2})=\mathbf{0}\\ \mathbf{2}, & \textbf{Else}\end{cases}, \boldsymbol{m}+\mathbf{1}\right)$, then

  [ctrl] [sto→] **coeff**, as shown.

  This generates a list $\{1,4,2,4,2,...,1\}$.

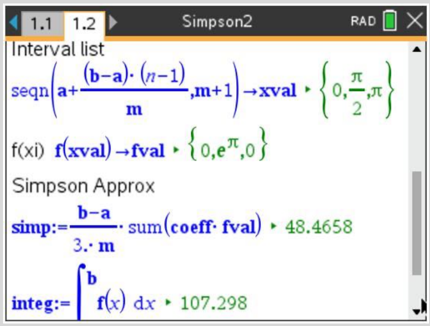


- For 'Interval list': Input $\mathbf{seqn}\left(\boldsymbol{a}+\dfrac{(\boldsymbol{b}-\boldsymbol{a})\cdot(\boldsymbol{n}-\mathbf{1})}{\boldsymbol{m}}, \boldsymbol{m}+\mathbf{1}\right)$,

  then [ctrl] [sto→] ***xval***. This generates a list $\{x_0, x_1, x_2, ..., x_m\}$.
- For 'f(*xi*)': Input $\boldsymbol{f}(\boldsymbol{xval})$ then [ctrl] [sto→] ***fxval***.

  This generates a list $\{f(x_0), f(x_1), f(x_2), ..., f(x_m)\}$.
- For 'Simpson Approx':

  $\mathbf{simp} := \left(\dfrac{\boldsymbol{b}-\boldsymbol{a}}{\mathbf{3.0}\cdot\boldsymbol{m}} \cdot \mathbf{sum}(\boldsymbol{coeff} \cdot \boldsymbol{fval})\right)$. This gives the value of Simpson's rule approximation. Including '**3.0**' guarantees a decimal answer.
- To calculate the numerical integral, press [ctrl] [M] to insert a **Maths Box**, then enter $\mathbf{integ} := \int_a^b (\boldsymbol{f}(\boldsymbol{x}))\,\boldsymbol{dx}$



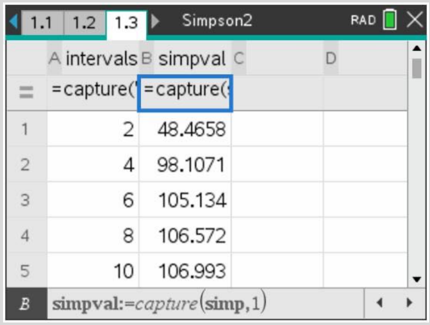
***Note:** The number of intervals can be changed by editing the value of **m** on the **Notes** page. This will automatically recalculate the value of the Simpson's rule approximation.*

... continued

(c) To plot the value of Simpson's rule approximation against the interval size, proceed as follows:

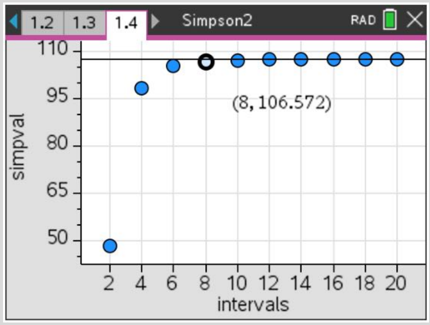
- In the **Notes** page, ensure that $\boldsymbol{m} := \mathbf{2}$.

Add a **Lists & Spreadsheet** page (Press ctrl +page > **Add Lists & Spreadsheet),** then:



- Name columns A and B as shown, to declare them as lists.
- Without clicking on the cell, use the arrow keys to select the column A formula cell (the second from the top)
- Press menu > **Data** > **Data Capture** > **Automatic**. Press var select ***m***, then press enter.
- Similarly, select the column B formula cell.
- Press menu > **Data** > **Data Capture** > **Automatic**. Press var select ***simp***, then press enter.
- On the Notes page, systematically change the value of ***m***, $\boldsymbol{m} := \mathbf{2}, \boldsymbol{m} := \mathbf{4}, \boldsymbol{m} := \mathbf{6}, \ldots, \boldsymbol{m} := \mathbf{20}$. This populates the spreadsheet columns with the values of ***m*** and ***simp***.

The data in the spreadsheet can be displayed graphically.



- Press ctrl +page > **Add Data & Statistics**
- Press tab, select ***intervals*** on the horizontal axis. Press tab, select ***simpval*** on the vertical axis.
- Press menu > **Analyse** > **Plot Function**. Press var select ***integ***.

This shows a comparison of the calculated numerical integral and the Simpson's rule values.

Note: *To reset the spreadsheet, select the formula cell for column A and B in turn and press* menu ***> Data > Clear Data***.

Press ctrl S *to save the document.*

Working with exponential random variables

The travel life of a tyre is the distance travelled before the tyre becomes unroadworthy due to wear or structural failure. For a particular brand of tyre, the travel life can be modelled as an exponential random variable X with probability density function $f(x)=\begin{cases}\dfrac{1}{40}e^{-\left(\frac{x}{40}\right)}, & x\ge 0\\ 0 & \text{otherwise}\end{cases}$,

where x is the distance travelled in thousands of kilometres.

(a) Determine the probability, correct to two decimal places, that the travel life of a randomly chosen tyre is

(i) less than 20,000 km

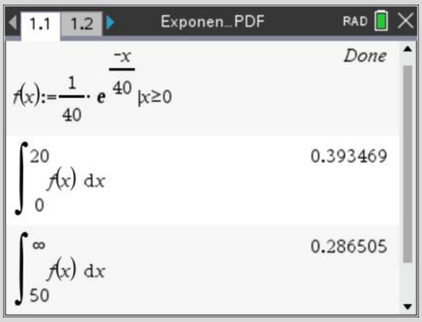
(ii) greater than 50,000 km

(b) Find $P(m < X < \mu)$ for such a randomly chosen tyre, where m and μ are the median and mean travel life, respectively. Give the answer correct to two decimal places.

(c) Eighty percent of such tyres have a travel life of at least w thousands of kilometres. Find the value of w in thousands of kilometres, correct to the nearest thousand kilometres.

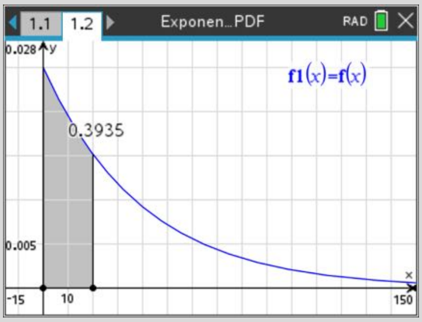
(a) To determine $P(X < 20)$, $P(X > 50)$, on a **Calculator** page:

- Enter $\boldsymbol{f(x) := \frac{1}{40}e^{-x/40} \mid x \ge 0}$
- Press [template key] or [⇧shift] [+], select **Numerical Integral** template and enter (for **(i)**) $\int_0^{20} f(x)\,dx$, then (for **(ii)**) $\int_{50}^{\infty} f(x)\,dx$.



Add a **Graphs** page (ress [ctrl] [+page] > **Add Graphs**.)

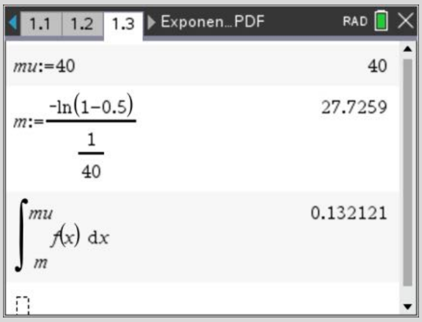
- Enter $\boldsymbol{f1(x) = f(x)}$ (or $\boldsymbol{f1(x) = \frac{1}{40}e^{-x/40} \mid x \ge 0}$).
- Press [menu] > **Window/Zoom** > **Window Settings**
- In the dialog box that follows, enter the following values:
  XMin = −15 Xmax = 150 XScl = 10
  YMin = −0.003 YMax = 0.028 YScl = 0.005
- Press [menu] > **Analyse Graph** > **Integral**.
- For part **(i)**, type **0** then [enter]. Type **20** then [enter].
- For part **(ii)**, type **50** then [enter], type **500** [enter]. This will automatically change the window settings, as a value of $x = 500$ (an arbitrary large number has been chosen here) is outside the current viewing window settings.



Answer: The probabilities are **(i)** 0.39 and **(ii)** 0.29.

(b) The mean, $\mu = \dfrac{1}{\lambda} = 40$.

For an exponential random variable X, if $\int_0^k f(x)\,dx = p$, then $k = \dfrac{-\ln(1-p)}{\lambda}, 0 \le p \le 1$. For the median, $p = 0.5$.



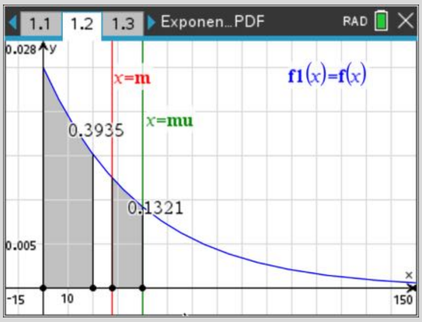
... continued

To find the probability that the travel life of a randomly chosen tyre lies between the mean and the median, on a **Calculator** page (with $f(x)$ already defined):

- Enter $\boldsymbol{mu := 40}$, $\boldsymbol{m := \dfrac{-\ln(1-0.5)}{(1/40)}}$ and $\boldsymbol{\int_m^{mu} f(x)dx}$.

On the **Graphs** page from part (a) above,

- Press [menu] > **Graph Entry/Edit** > **Relation** and enter the relations $\boldsymbol{x = m}$ and $\boldsymbol{x = mu}$.
- Press [menu] > **Analyse Graph** > **Integral**. Click on the intersection point of the x-axis and the line $x = m$, then click on the intersection point of the x-axis and the line $x = mu$.

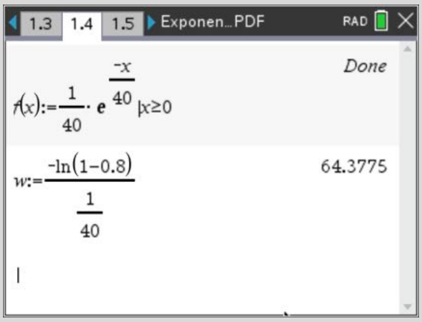


Answer: $P(m < X < \mu) = 0.13$

(c) Let $\lambda = \dfrac{1}{40}, p = 0.8$.

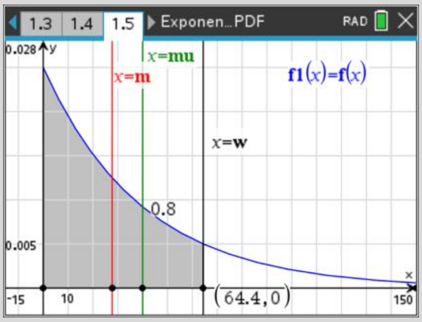
To find the value of w such that $P(X < w) = 0.8$, on a **Calculator** page (with $f(x)$ already defined):

- Enter $\boldsymbol{w := \dfrac{-\ln(1-0.8)}{(1/40)}}$.



Confirm the result on the **Graphs** page from part (a) above.

- Hide the integrals from parts **(a)** and **(b)**. (Click on the objects you wish to hide, then press [ctrl][menu] > **Hide/Show** > **Hide Selection**.
- Press [menu] > **Analyse Graph** > **Integral**. Click the intersection point at the origin, then click on the intersection point of the x-axis and the line $x = w$.



Answer: $w = 64$ (travel life is 64,000 km, correct to the nearest thousand kilometres).

Note: *To unhide objects on the* ***Graphs*** *page, press* [menu] > ***Actions*** > ***Hide/Show****. Click on the objects to unhide them. Press* [esc] *to exit the 'unhide' tool.*

4.3. Topic 3: Rates of change and differential equations

4.3.1. Rates of change

Applying implicit differentiation to curves whose equations are given in implicit form

(a) Use implicit differentiation to find $\frac{dy}{dx}$ for the relation with equation $x^2 + y^2 = 16$.

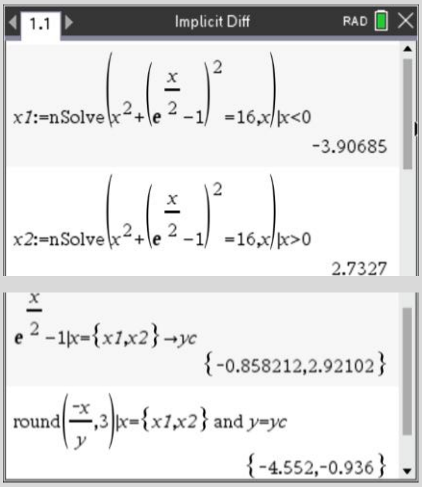
(b) Hence find the gradients of the tangents to the curve with equation $x^2 + y^2 = 16$ at the points where it intersects the graph of $y = e^{\left(\frac{x}{2}\right)} - 1$. Give your answers correct to three decimal places.

(c) Use a graphical method to verify the answer found in part (b) above.

(d) Generalise part (c) above to show graphically that the gradient of the tangent at any point $P(x, y)$ on the curve with equation $x^2 + y^2 = 16$ is given by $-\frac{x}{y}$.

(a) Using implicit differentiation:

$$\frac{d(x^2)}{dx} + \frac{d(y^2)}{dy} \times \frac{dy}{dx} = \frac{d(16)}{dx} \Leftrightarrow \frac{dy}{dx} = -\frac{x}{y}$$



(b) To find the points of intersection of the curves, and the gradient of the tangents at these points, on a **Calculator** page:

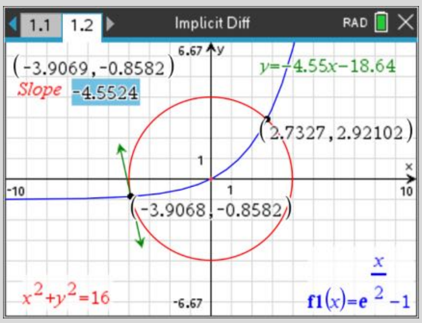
- Press [menu] > **Algebra** > **Numerical Solve**.
- Enter $\boldsymbol{x1 := \text{nSolve}\left(x^2 + \left(e^{(x/2)} - 1\right)^2 = 16, x\right) \mid x < 0}$
- Enter $\boldsymbol{x2 := \text{nSolve}\left(x^2 + \left(e^{(x/2)} - 1\right)^2 = 16, x\right) \mid x > 0}$
- Input $\boldsymbol{e^{(x/2)} - 1 \mid x = \{x1, x2\}}$ press [ctrl] [sto→] input $\boldsymbol{yc}$, and then press [enter].
- Enter $\boldsymbol{\text{round}\left(\frac{-x}{y}, 3\right) \mid x = \{x1, x2\} \text{ and } y = yc}$

Answer: At $(-3.907, -0.858), m = -4.552$

At $(2.733, 2.921), m = -0.936$

(c) To graphically verify the previous answer, proceed as follows:

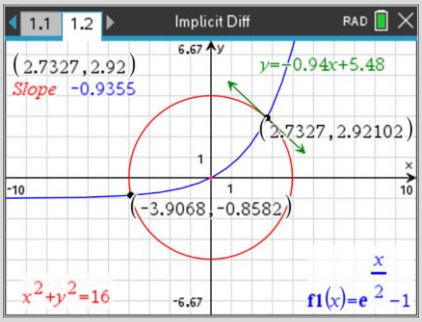
- Add a **Graphs** page and enter $\boldsymbol{f1(x) = e^{(x/2)} - 1}$.
- Press [menu] > **Graph Entry/Edit** > **Relation**.
- Enter the relation $\boldsymbol{x^2 + y^2 = 16}$.
- Press [menu] > **Geometry** > **Points & Lines** > **Intersection Points**. Click on the first graph and then on the second graph. Press [esc].



... continued

- Press menu > **Geometry** > **Points & Lines** > **Tangent**. Click the circle then press enter. Press esc to exit this tool.
- Hover the cursor over the tangent line, press menu > **Geometry** > **Measurement** > **Slope**. Press esc to exit this tool.

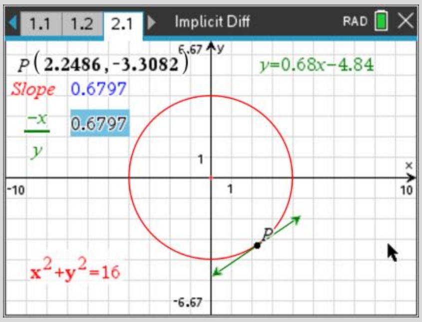
- Grab the tangent intersection point (ctrl ✋). Move the point near an intersection point, and press esc. Hover over the tangent intersection point, press ctrl menu > **Coordinates and Equations**. Click on the y-coordinates of this point until editable. Enter $(-3.9068, -0.8582)$ and note the slope measurement. Repeat for $(2.7327, 2.9210)$.

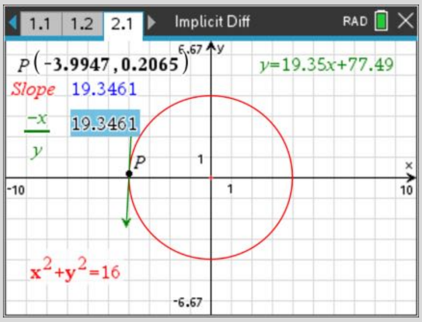


__Note:__ To increase the precision displayed for a measurement or value, hover over the text and press + (or − to decrease precision).

(d) To generalise the graphical solution, proceed as follows:

- Copy page 1.2 from part **(c)** above to a new problem: Press ctrl ▲ menu > **Insert** > **Problem**.
- Select the page 1.2 thumbnail.
- Press ctrl C then in Problem 2 ctrl V enter.
- Hover over graph $f1$, and press ctrl menu > **Delete**.
- Hover over the x-coordinate of the point of intersection of the tangent and the circle.
- Press ctrl menu > **Store.**
- Input $\boldsymbol{var := x}$, and then press enter. Repeat for the y-coordinate, except input $\boldsymbol{var := y}$.
- Label the point, P: Press ctrl menu > **Label**, then enter $\boldsymbol{P}$.
- In the **Graphs** workspace, press ctrl menu > **Text**, then enter $\dfrac{-x}{y}$.

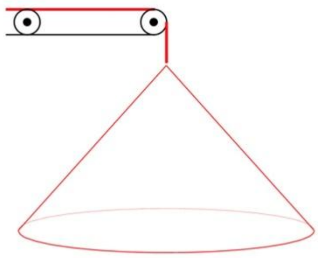




- Press menu > **Actions** > **Calculate**. Click on the text $\dfrac{-x}{y}$, then click on the x-coordinate then click on the y-coordinate.
- Grab (ctrl ✋) and move the point P around the curve and compare the measured slope value with the calculated value of $\dfrac{-x}{y}$.

__Note:__ To release/'ungrab' P, either press ✋ or esc.

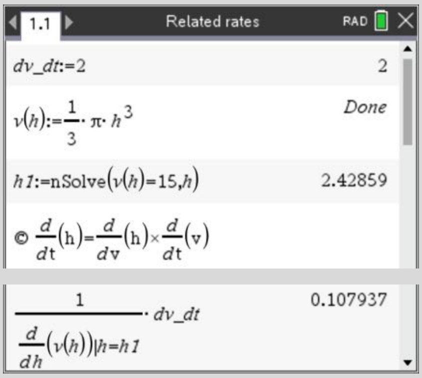
Applying related rates to the sand pile problem



Sand is dropped from a conveyor belt onto a pile at a rate of 2 m^3/min. The pile of sand is modelled as a right circular cone such that its height is always equal to the radius of its base. At time $t = 0$ minutes, the volume of the pile is zero. Give the answer to parts (a) and (b) below in m/min, correct to four decimal places.

(a) Find the rate of change of the height of the cone when the volume of sand in the pile is 15 m^3.
(b) Find the rate of change of the height of the cone at time $t = 55$ minutes.
(c) Explore graphically the rate of change in the height of the cone at time *t* minutes, for $0 \le t \le 100$. Use the graph to determine:
  (i) the volume of the pile when the rate of change in height is 0.5 m/minute and 0.1 m/minute.
  (ii) the rate of change in height at time $t = 2$ minutes and at $t = 60$ minutes.

(a) To find the rate of change of the height when the volume is 15 m^3:



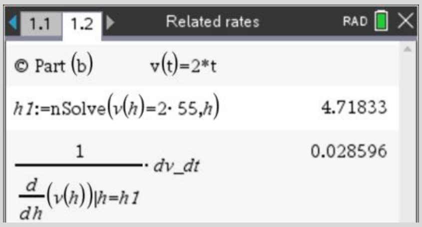
- On a **Calculator** page, enter $\boldsymbol{dv\_dt := 2}$ then enter
  $$v(h) := \frac{1}{3}\pi h^3.$$

Note: The underscore character can be found via [ctrl] [␣]

- Using [menu] > **Algebra** > **Numerical Solve**, enter $\boldsymbol{h1 := \text{nSolve}(v(h) = 15, h)}$, then enter
  $$\frac{1}{\frac{d}{dh}(v(h)) \mid h = h1} \cdot dv\_dt$$
  (using [menu] > **Calculus** > **Numerical Derivative at a Point** to input the denominator).

Answer: The height is increasing at a rate of 0.1079 m/min.

(b) To find the rate of change in height at $t = 55$ minutes:



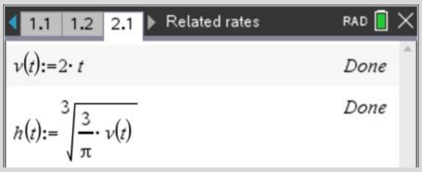
- On a **Calculator** page, enter $\boldsymbol{h1 := \text{nSolve}(v(h) = 2 \times 55, h)}$, then enter
  $$\frac{1}{\frac{d}{dh}(v(h)) \mid h = h1} \cdot dv\_dt.$$

Answer: The height is increasing at a rate of 0.0286 m/min.

Note:
1. To add a comment: [menu] > ***Actions*** > ***Insert Comment***.
2. The key sequence [⇧shift] [−] *is a shortcut to the* ***Derivative*** *template.*

... continued

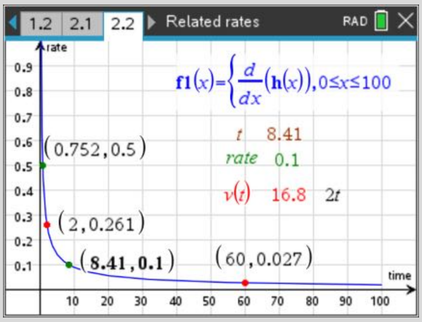
(c) To explore graphically the rate of change in the height for $0 \le t \le 100$:



- Add a **Calculator** page to a **New Problem** (Press doc▾ > **Insert** > **Problem** > **Add Calculator**)
- Enter $\boldsymbol{v(t) := 2t}$, then $\boldsymbol{h(t) := \sqrt[3]{\frac{3}{\pi} v(t)}}$
- Add a **Graphs** page and enter $\boldsymbol{f1(x) := \frac{d}{dx}(h(x)) \mid 0 \le x \le 100}$
- Press menu > **Window/Zoom** > **Window Settings**

In the dialog box that follows, enter the following values:
XMin = −10 Xmax = 110 XScale = 10
YMin = −0.1 YMax = 1.0 YScale = 0.1

- Press menu > **Geometry** > **Points & Lines** > **Point On**. Click on graph $f1$ four times to get four points on the graph and their coordinates. Press esc to exit the **Point On** tool.
- Click on the y-coordinate value of one point until the text becomes editable and edit it to be **0.5**.
- Similarly, edit the y-coordinate of another point to be **0.1**.
- Edit the x-coordinates of the remaining points to be **2** and **60**.



Answer: (i) $2 \times 0.75 = 1.5\,\text{m}^3$ and $2 \times 8.4 = 16.8\,\text{m}^3$ (to 1 d.p.)

(ii) 0.261 m/minute and 0.027 m/minute (to 3 d.p.)

4.3.2. Differential equations

Solving differential equations of the form $\frac{dy}{dx} = f(x)$

Solve the differential equation $\frac{dy}{dx} = \frac{1}{5-x}$

(a) at $x = -2$, given that $y(4) = 3$. Give the answer correct to four decimal places.

(b) at $x = 8$, given that $y(6) = 2$. Give the answer correct to four decimal places.

(c) Confirm that the answers to part (a) and part (b) above are consistent with the general solution $y = -\ln(|5-x|) + c$.

If $\frac{dy}{dx} = f(x)$ and $y = y_1$ at $x = x_1$, then the solution to the differential equation at $x = t$ is given by $\int_{x_1}^{t} f(x)\,dx + y_1$.

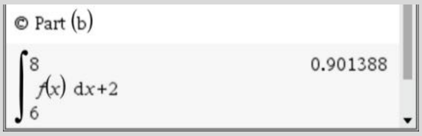
(a) To find the solution at $x = -2$, given that $y(4) = 3$:

- On a **Calculator** page, enter $\boldsymbol{f(x) := \frac{1}{5-x}}$.
- Either press [shift] [+] or [menu] > **Calculus** > **Numerical Integral** and enter $\boldsymbol{\int_4^{-2} f(x)\,dx + 3}$.

Answer: 1.0541.

(b) To find the solution at $x = 8$, given that $y(6) = 2$, on a **Calculator** page, with $f(x)$ previously defined:

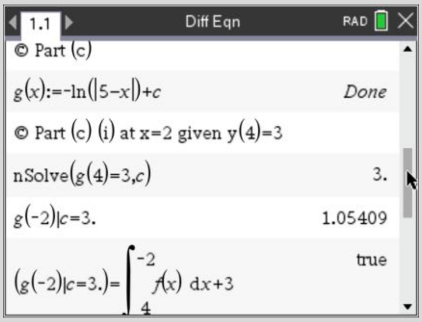
- Enter $\boldsymbol{\int_6^{8} f(x)\,dx + 2}$

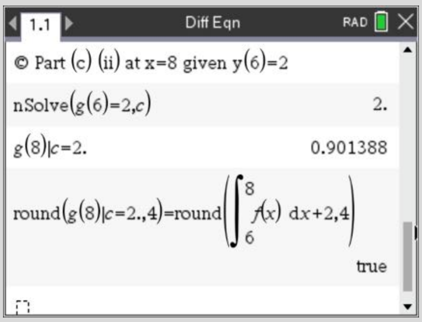


Answer: 0.9014.

(c) To confirm consistency with the general solution $y = -\ln(|5-x|) + c$:

- On a calculator page, enter $\boldsymbol{g(x) := -\ln(|5-x|) + c}$
- Press [menu] > **Algebra** > **Numerical Solve**.
- Enter $\boldsymbol{\text{nSolve}(g(4) = 3, c)}$, then enter $\boldsymbol{g(-2) \mid c = \text{ans}}$
- Arrow up and copy the previously entered expressions to input $\boldsymbol{(g(-2) \mid c = 3.0) = \int_4^{-2} f(x)\,dx + 3}$. This should return '**true**'.
- To confirm part (b), enter $\boldsymbol{\text{nSolve}(g(6) = 2.0, c)}$, then $\boldsymbol{g(8) \mid c = \text{ans}}$. This should return '**true**'.





Answer: 0.9014 for both methods confirm consistency of results.

Modelling Newton's law of cooling with a differential equation of the form dy/dx = g(y)

The rate of heat loss of an object is proportional to the difference between the temperature of the object and the ambient temperature. Assume that the ambient temperature remains constant at 20°C, and that the temperature of the object is 100°C at time $t = 0$. Let $\theta\,^\circ\text{C}$ be the temperature of the object at time, t minutes.

Given that $\theta = 70^\circ\text{C}$ at $t = 10$ minutes, find the following.

(a) The temperature of the object at time $t = 20$ minutes, in °C, correct to one decimal place.

(b) The time in minutes taken for the temperature of the object to decrease by 60°C, correct to one decimal place.

(c) Plot a graph of the object's temperature as a function of time. Hence use a graphical method to determine **(i)** the time taken to halve and to quarter the excess of the object's temperature above the ambient temperature, **(ii)** The temperature after 1 hour. Give the answers correct to one decimal place.

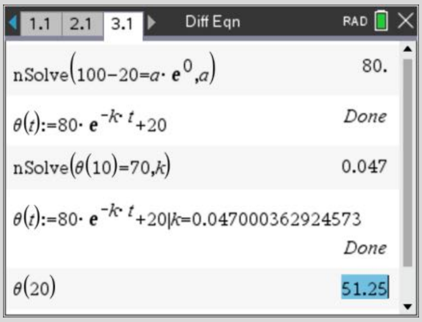
Note:

In general, if $\dfrac{dN}{dt} = kN, k \in \mathbb{R} \setminus \{0\}$, *then* $N = Ae^{kt}$.

In this case, $\dfrac{d\theta}{dt} = -k(\theta - 20), k \in \mathbb{R}^+$, *therefore* $\theta(t) - 20 = Ae^{-kt}$

(a) To find the temperature at time $t = 20$ minutes, on a **Calculator** page:

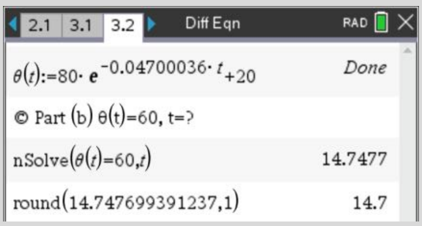
- Press [menu] > **Algebra** > **Numerical Solve** and enter $\mathbf{nSolve}\left(100 - 20 = a \cdot e^{0}, a\right)$.
- Press [ctrl] [∞β°] and select θ. Enter $\theta(t) := 80e^{-k \cdot t} + 20$.
- Enter $\mathbf{nSolve}\left(\theta(10) = 70, k\right)$ (Press [var] to select $\theta(\ )$).
- Press ▲ key to arrow up to the second entry, then press [enter].
- Complete the entry $\theta(t) := 80e^{-k \cdot t} + 20 \mid k = \mathbf{ans}$



Answer: At $t = 20$ minutes, the temperature of the object is 51.3°C.

(b) To find the time taken for the temperature to decrease by 60°C, on the **Calculator** page (with $\theta(t)$ already defined):

- Press [menu] > **Algebra** > **Numerical Solve** and enter $\mathbf{nSolve}\left(\theta(t) = 60, t\right)$.



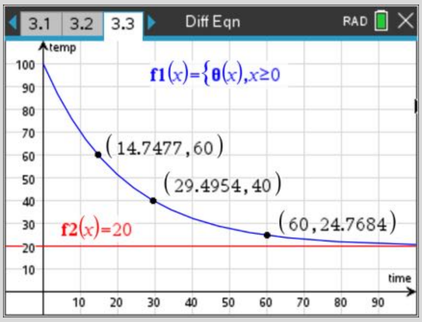
Answer: At $t = 14.7$ minutes, the temperature of the object is 60°C.

... continued

(c) To use a graphical method to determine **(i)** the halving time and **(ii)** the temperature after 1 hour:

- On the **Calculator** page, ensure that $\theta(t)$ is defined with the numerical value for k, that is $\boldsymbol{\theta(t) := 80e^{-0.0460036 \cdot t} + 20}$.
- Add a **Graphs** page and enter $\boldsymbol{f1(x) = \theta(x) \mid x \ge 0}$
- Then enter $\boldsymbol{f2(x) = 20}$
- Press menu > **Window/Zoom** > **Window Settings**

  In the dialog box that follows, enter the following values:
  XMin = −10 Xmax = 100 XScale = 10
  YMin = −10 YMax = 110 YScale = 10
- Press menu > **Geometry** > **Points and Lines** > **Point On**.
  Click on the graph of $\boldsymbol{f1}$ three times to add three points.
  Press esc to exit this tool.
- Edit the y-coordinate of one point to be **60**.
  Edit the y-coordinate of another point to be **40**.
  Edit the x-coordinate of the third point to be **60**.



Answer: The excess temperature is halved in 14.7 minutes and quartered in 29.5 minutes. The temperature after 1 hour is 24.8°C.

Modelling logistic growth with a differential equation of the form dy/dx = g(y).

Suppose that a researcher studies the population growth of an introduced species of eel in a particular stream. The findings indicate that the population grew exponentially at first, but the rate of growth of the population decreased as it stabilised to the carrying capacity of the stream.

The eel population was modelled using the logistic differential equation, $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$, where P is the population size at time t months, k is the constant of proportionality corresponding to the growth rate, and M is the carrying capacity of the stream.

The differential equation has a solution of the form $P(t) = \frac{M}{1 + Ae^{-kt}}, t \ge 0$.

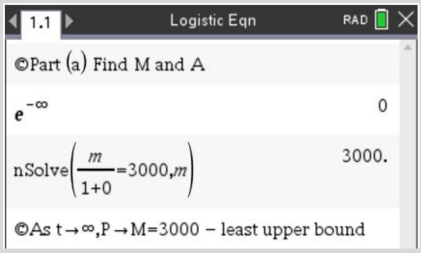
(a) If the initial population is 600 and the population stabilised over time to 3000, find
(i) the least upper bound of P.
(ii) the values of M and A. Hence verify that $A = \frac{M - P(0)}{P(0)}$.

(b) If the population was estimated to be 2000 at time $t = 18$ months, find the value of k, correct to three decimal places.

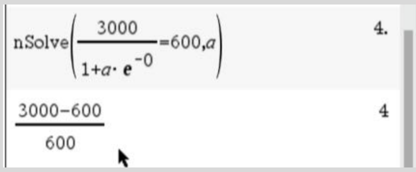
(c) Plot a graph of $P(t)$ against t for $0 \le t \le 60$. Hence use a graphical method to determine:

(i) the time taken to double the initial population, in months correct to one decimal place.
(ii) the time taken to quadruple the initial population, in months correct to one decimal place.
(iii) the population after 1 year.
(iv) the population after 3 years.

(a) To find P as $t \to \infty$ and the values of M and A, on a **Calculator** page:



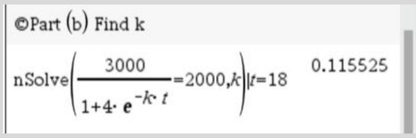
- Enter $\boldsymbol{e}^{-10000}$. *(Using $t = 10000$ to indicates $t \to \infty$).*
- Press menu > **Algebra** > **Numerical Solve** and enter $\mathbf{nSolve}\left(\dfrac{\boldsymbol{m}}{\mathbf{1+ans}} = \mathbf{3000}, \boldsymbol{m}\right)$



- Enter $\mathbf{nSolve}\left(\dfrac{\mathbf{3000}}{\mathbf{1}+\boldsymbol{a}\cdot\boldsymbol{e}^{-0}} = \mathbf{600}, \boldsymbol{a}\right)$
- To verify that $A = \dfrac{M - P(0)}{P(0)}$, enter $\dfrac{\mathbf{3000-600}}{\mathbf{600}}$

Answer:
(i) Least upper bound = 3000. **(ii)** $M = 3000$, $A = 3000$

(b) To find the value of k, on a **Calculator** page:



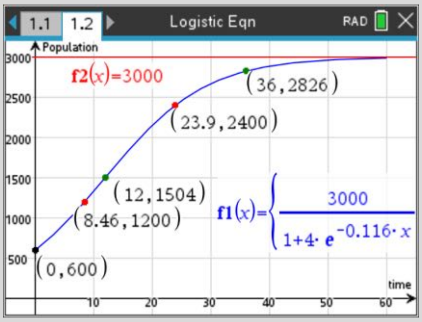
- Enter $\mathbf{nSolve}\left(\dfrac{\mathbf{3000}}{\mathbf{1+4}\cdot\boldsymbol{e}^{-k\cdot t}} = \mathbf{2000}, \boldsymbol{k}\right) \mid \boldsymbol{k} = \mathbf{18}$

Answer: $k = 0.116$

(c) To use a graphical method to determine the doubling and quadrupling times, and the population after 1 year and after 3 years: On a **Graphs** page:

- Enter $\boldsymbol{f}\mathbf{1}(\boldsymbol{x}) = \dfrac{\mathbf{3000}}{\mathbf{1+4}\cdot\boldsymbol{e}^{-0.116\cdot x}} \mid \mathbf{0} \le \boldsymbol{x} \le \mathbf{60}$ and then enter $\boldsymbol{f}\mathbf{2}(\boldsymbol{x}) = \mathbf{3000}$.
- Press menu > **Window/Zoom** > **Window Settings**

  In the dialog box that follows, enter the following values:
  XMin = −5 Xmax = 65 XScale = 10
  YMin = −200 YMax = 3200 YScale = 500



- Press menu > **Geometry** > **Points and Lines** > **Point On**. Click on graph $\boldsymbol{f}\mathbf{1}$ four times to add four points. Press esc to exit this tool.
- Edit the y-coordinate of one point to be **1200**.
- Edit the y-coordinate of another point to be **2400**.
- Edit the x-coordinate of one point to be **12**.
- Edit the x-coordinate of the fourth point to be **36**.

Answer:
(i) Population doubles in 8.5 months
(ii) Population quadruples in 23.9 months.
(iii) At $t = 12$ months, $P = 1504$.
(iv) At $t = 36$ months, $P = 2826$.

Solving differential equations of the form $\frac{dy}{dx} = f(x) \times g(y)$

Consider the differential equation $\frac{dy}{dx} = xy$.

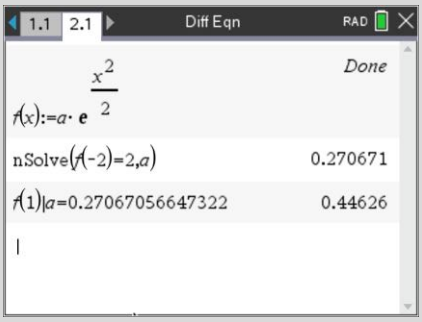
(a) Using the separation of variables method, a student obtained a general solution $y = Ae^{(x^2/2)}, A \in \mathbb{R}$. Find the particular solution at $x = 1$, given that $y(-2) = 2$. Give the answer correct to two decimal places.

(b) Compare the answer to part (a) above with the answer obtained using the default numerical method used by the *TI-Nspire CX II-T* to plot the graph of the solution on the ***slope field*** of the differential equation. Use a step size of 0.01 in the numerical calculation.

(c) Verify the result to part (b) above on the plotted graph on the slope field of the differential equation. Set a step size of 0.01 to plot the graph of the solution on the slope field.

(a) To find the solution at $x = 1$, given that $y(-2) = 2$, on a **Calculator** page in a **New Problem**:

- Enter $\boldsymbol{f(x) := a \cdot e^{x^2/2}}$
- Press [menu] > **Algebra** > **Numerical Solve**.
- Enter **nSolve**$(\boldsymbol{f(-2) = 2, a})$, then enter $\boldsymbol{f(1) \mid a = \textbf{ans}}$



Answer: $y = 0.45$

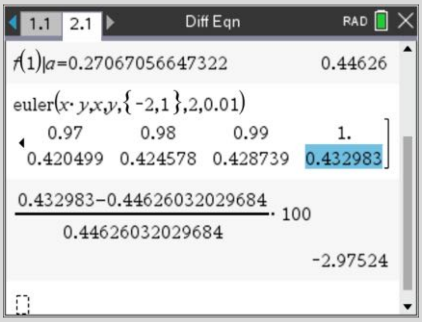
(b) The default numerical method that is used by the *TI-Nspire CX II-T* to plot graphs of particular solutions on the slope field is *Euler's method.*

To compare the answer to part (a) to Euler's method, on a **Calculator** page:

- Press [📖] [F] and press ▲ up to '**euler(**'

Note: *The general syntax for this command is as follows:* ***euler(expression,x,y,{initial x, final x},initial y, step size)***

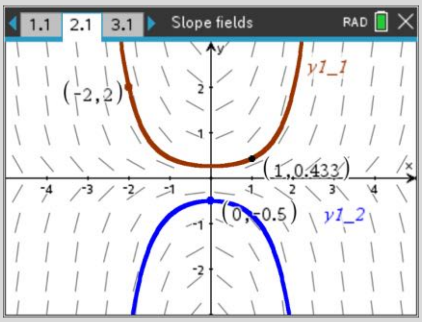
- Enter **euler**$(\boldsymbol{x \cdot y, x, y, \{-2,1\}, 2, 0.01})$



Answer: Using the Euler approximation with a step size of 0.01, $y = 0.45$ – a percentage error of approximately 3%.

(c) To verify the result to part (b) above on the plotted graph on the slope field, see 'Slope field for the differential equation $\frac{dy}{dx} = xy$' below, in the section on slope fields.

The screenshot of the plotted graph for this question is shown at right.



Note*: Euler's method is not formally part of the syllabus for QCE Specialist Mathematics.*

Using slope fields to plot solutions for differential equations of the form $\frac{dy}{dx} = f(x)$

Plot the slope field for the differential equation $\frac{dy}{dx} = 3 - x - x^2$.

(a) On the slope field, plot the graphs of the particular solutions that satisfy the following conditions.

(i) $y(0) = 3$ **(ii)** $y(0) = 0$ **(iii)** $y(0) = -2$

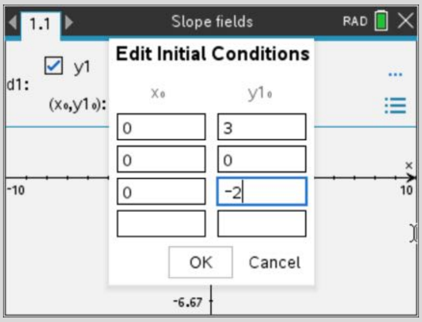
(b) Explore the effect of moving the initial point to dynamically display differing initial conditions.

(a) To plot the graphs of the particular solutions on the slope field:

- On a **Graphs** page, press [menu] > **Graph Entry/Edit** > **Diff Eq**.
- Enter the differential equation $\mathbf{y1' = 3 - x - x^2}$. (This will display the slope field.)

***Note**: The default identifier is* $y1$ *(not* y *).*

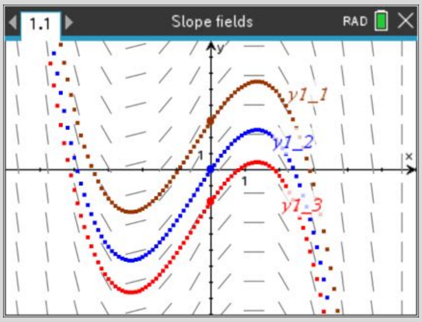
- Under the **Graph Entry** field, click the 🔍 icon.
- In the **Edit Initial Conditions** dialog box that follows, enter the initial conditions, as shown in the screenshot at right.



- Press [menu] > **Window/Zoom** > **Window Settings**

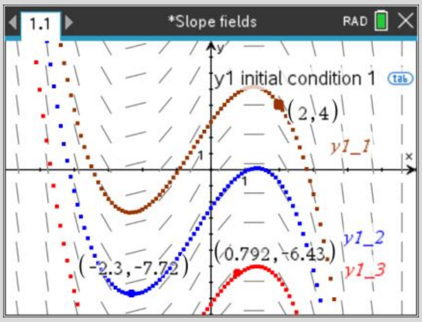
In the dialog box that follows, enter the following values:

XMin = −6 Xmax = 6 XScale = 1
YMin = −9 YMax = 8 YScale = 1



(b) To explore the effect of moving the initial point:

- Hover over the point for '**initial condition 1**' until the ✋ icon appears.
- Press [ctrl] [menu] > **Coordinates and Equations**. Repeat for the points of '**initial condition**' **2** and **3**, so that the coordinates of the three points are displayed.
- Grab ([ctrl] [✋]) the point for '**initial condition 1**' and move it so that the graphs for differing initial conditions are dynamically displayed. Repeat for the points for '**initial condition 2**' and '**initial condition 3**'.



Click on the x-coordinate value of the point for '**initial condition 1**' until it is editable and enter the value **2**. Similarly, edit the x-coordinate value to be **4**, so that the initial condition is now $y(2) = 4$.

Using slope fields to plot solutions for differential equations of the form $\frac{dy}{dx} = xy$

(a) On the slope field, plot the graphs of the particular solutions that satisfy the following conditions:

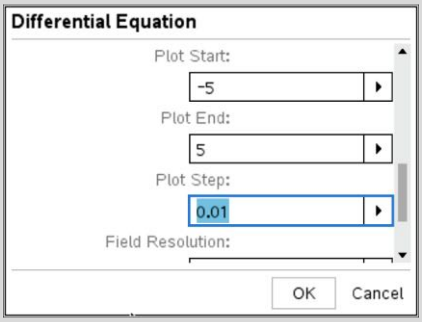
(i) $y = 1$ when $x = 0$. **(ii)** $y = -\frac{1}{2}$ when $x = 0$.

(b) Plot a graph on the slope field to find an approximate solution, correct to two decimal places, to the differential equation $\frac{dy}{dx} = xy$ at $x = 1$, given that $y = 2$ when $x = -2$.

(a) To plot the graphs of the particular solutions on the slope field:

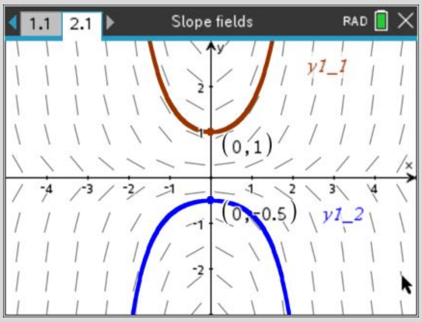
- Add a **Graphs** page to a **New Problem**: Press doc▾ > **Insert** > **Problem** > **Add Graphs**.
- Press menu > **Graph Entry/Edit** > **Diff Eq**
- Enter the differential equation $\boldsymbol{y1' = x \cdot y1}$.

Note: *The default identifier is* $y1$ *(not* y *).*



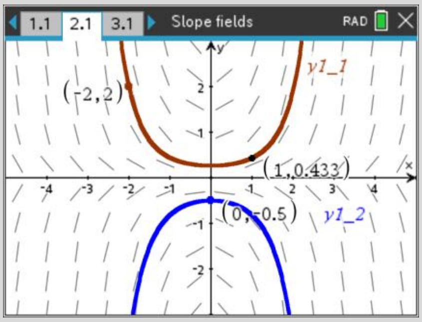
- Click the  icon on the right of the graph entry. In the dialog box that follows, enter **Plot step**: **0.01**, then press enter.
- Under the **Graph Entry** field, click the  icon.
  In the **Edit Initial Conditions** dialog box that follows, enter the initial conditions.
- Press menu > **Window/Zoom** > **Window Settings**

  In the dialog box that follows, enter the following values:
  XMin = −5 Xmax = 5 XScale = 1
  YMin = −3 YMax = 3 YScale = 1
- Hover over the point for '**Initial Condition 1**' and press ctrl menu > **Coordinates and Equations**. Repeat for '**Initial Condition 2**'.



(b) To find an approximate solution from a graph on the slope field:

- Edit '**Initial condition 1**' to $(-2, 2)$ (i.e. $y = 2$ when $x = -2$).
- Press menu > **Geometry** > **Points and Lines** > **Point On**.
- Click on the graph of $\boldsymbol{y1\_1}$, then press esc to exit the **Point On** tool.
- Move the point to the value $\boldsymbol{x = 1}$



Answer: If $\frac{dy}{dx} = xy$ at $x = 1$, given $(-2, 2)$ then $y \approx 0.43$.

4.4. Topic 4: Modelling motion

4.4.1. Modelling motion

Modelling the acceleration of an object moving in a straight line

A body is moving in a straight line so that at time t its displacement from a fixed origin is x, its velocity is v and its acceleration is a.

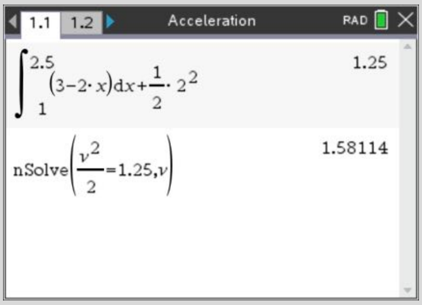
(a) If $a = 3 - 2x$ and $v = 2$ at $x = 1$, then find v when $x = 2.5$, correct to two decimal places.

(b) **(i)** If $a = \dfrac{1+v^2}{2}$ and $v = 1$ at $x = 0$, then find x when $v = \dfrac{11}{2}$, correct to two decimal places.

(ii) Suppose that a student carries out the initial part of the solution to part **(i)** above by integration and finds $x = \ln\left(1+v^2\right) + c$. Use this result to complete the solution.

(Note for part (a). Since $a = f(x)$, use $\dfrac{d\left(\frac{1}{2}v^2\right)}{dx}$)

(a) *Method 1*. If $\dfrac{d(y)}{dx} = f(x)$ and $y = y_1$ at $x = x_1$, then the solution to the differential equation at $x = k$ is given by $\int_{x_1}^{k} f(x)\,dx + y_1$.



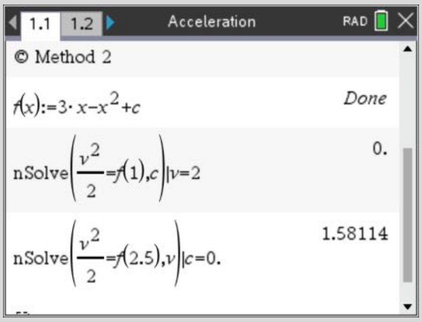
To find v when $x = 2.5$ (using Method 1), on a **Calculator** page:

- Either press [shift] [+] or [menu] > **Calculus** > **Numerical Integral** then enter $\int_1^{2.5}(3-2x)\,dx + \left(\frac{1}{2}\cdot 2^2\right)$.
- Press [menu] > **Algebra** > **Numerical Solve**.
- Enter **nSolve**$\left(\left(v^2/2\right) = 1.25, v\right)$

Answer: v = 1.58 correct to two decimal places.

***Note**: Method 1 is not on the QCE syllabus)*

Method 2. Use $\frac{1}{2}v^2 = \int(3-2x)\,dx = 3x - x^2 + c$



To find v when $x = 2.5$ (Method 2), on a **Calculator** page:

- Enter $\boldsymbol{f(x) := 3x - x^2 + c}$.
- Press [menu] > **Algebra** > **Numerical Solve.**
- Enter **nSolve**$\left(\left(v^2/2\right) = f(1), c\right) \mid v = 2$.
- Press ▲ then [enter]. Edit the previous input to **nSolve**$\left(\left(v^2/2\right) = f(2.5), v\right) \mid c = \text{ans}$

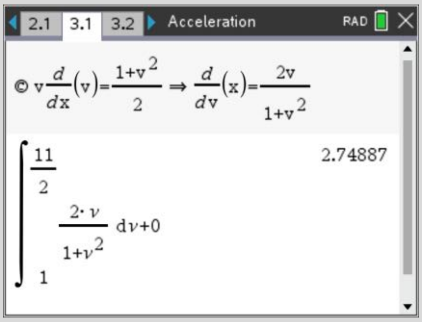
Answer: v = 1.58 correct to two decimal places.

(b) (i) Since $a = g(v)$ and initial condition (x, v), use $v\dfrac{d(v)}{dx}$.

To find x when $v = \dfrac{11}{2}$, on a **Calculator** page:

- Press either [⇧shift] [+] or [menu] > **Calculus** > **Numerical Integral** and enter $\int_1^{11/2}\left(\dfrac{2v}{1+2v}\right)dx + 0$.

Answer: $x = 2.75$, correct to two decimal places.

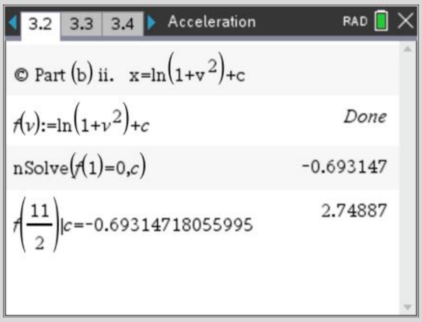


(b) (ii) To find x when $v = \dfrac{11}{2}$ using the result

$x = \log_e(1+v^2) + c$, on a **Calculator** page:

- Enter $\boldsymbol{f(x) := \ln(1+v^2) + c}$.
- Press [menu] > **Algebra** > **Numerical Solve**.
- Enter **nSolve**$(\boldsymbol{f(1) = 0, c})$.
- Enter $\boldsymbol{f\left(\frac{11}{2}\right) \mid c = \text{ans}}$

Answer: This confirms the answer is $x = 2.75$, correct to two decimal places.



Solving motion problems using graphical methods

A particle falls vertically from its initial rest position at the origin, *O*, in a resisting medium. At time t its velocity is given by $v(t) = 40\left(1 - e^{-\frac{t}{4}}\right)$, $t \geq 0$. Find the following graphically, correct to two decimal places.

(a) The particle's position x relative to O at time $t = 7$.

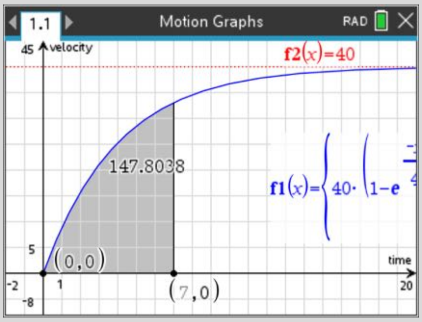
(b) The distance travelled by the particle in the interval $t = 5$ to $t = 12$.

(c) The particle's terminal speed.

(a) To find graphically the position of the particle at $t = 7$, on a **Graphs** page:

- Enter $\boldsymbol{f1(x) = 40\left(1 - e^{-\frac{x}{4}}\right) \mid x \geq 0}$, then $\boldsymbol{f2(x) = 40}$.
- Press [menu] > **Window/Zoom** > **Window settings**.
  In the dialog box that follows, enter the following values:
  XMin = –2 XMax = 20 XScale = 1
  YMin = –8 YMax = 45 YScale = 1.
- Press [menu] > **Analyse Graph** > **Integral**. Click graph $f1$, then for '**lower bound**', click '**point on**' the x-axis. For '**upper bound**', click '**point on**' the x-axis again.
- Hover over the lower bound point, press [ctrl] [menu] > **Coordinates & Equations**. Repeat for the upper bound.
- Edit the x-coordinate (i.e. the time)of the lower bound to be **0** and the x-coordinate of the lower bound to be **7**.

Answer: $x = 147.80$

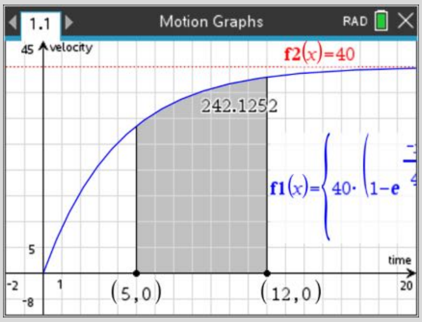


... continued

(b) To find graphically the distance travelled in the interval $t=5$ to $t=12$, on the **Graphs** page from part (a) above:

- Edit the x-coordinate (time) of the lower bound to be **5** and the x-coordinate of the lower bound to be **12**.

Answer: distance = 242.13 units, correct to two d.p.



(c) To find the terminal speed, consider the velocity-time graph: as $t \to \infty,\ e^{-t/4} \to 0, v(t) \to 40$. It is apparent that $v=40$ is a horizontal asymptote to the velocity-time graph, so that the terminal speed is $v=40$.

Solving problems involving vertical motion under gravity

An object is projected vertically upwards with a speed of 25 m/s from the top of a watchtower that is 20 m high. The watchtower is located on the edge of the top of a cliff that is 30 m high. The object ends its flight when it strikes the ground at the bottom of the cliff. Take the origin as the top of the cliff and the positive direction upwards. Assume that air resistance is negligible and that $g = 9.8\,\text{m/s}^2$.

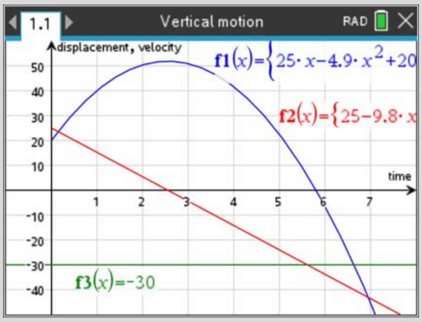
(a) Plot graphs of the displacement and velocity of the object as a function of time.

(b) Use graphical methods to determine the following.
 (i) The maximum height reached above the origin and the time taken to reach this height.
 (ii) The velocity with which the object strikes the ground and the time taken for the object to strike the ground.

(c) Use the constant acceleration formulas to verify the graphical solutions.

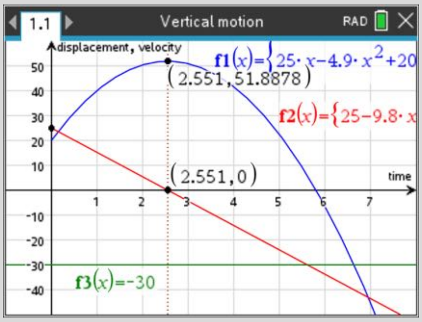
(a) To obtain $s-t$ and $v-t$ graphs based on the equations $s = ut + \frac{1}{2}at^2 + s_0$ and $v = u + at$, on a **Graphs** page:

- $\boldsymbol{f1(x) = 25x - 4.9x^2 \mid x \ge 0}$, $\boldsymbol{f2(x) = 25 - 9.8x \mid x \ge 0}$, and $\boldsymbol{f3(x) = -30}$.
- Press menu > **Window/Zoom** > **Window settings**.
 In the dialog box that follows, enter the following values:
 XMin = −1 Xmax = 8 XScale = 1
 YMin = −50 YMax = 60 YScale = 10
- Press menu > **View** > **Grid** > **Lined Grid**



(b)(i) To find the maximum height and time taken to reach maximum height using the $s-t$ graph, on the **Graphs** page from part **(a)** above:

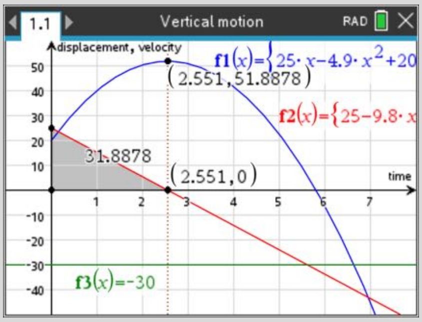
- Press menu > **Analyse Graph** > **Maximum**.
- Click Graph ***f*1**, then click **lower** and **upper bounds** to the left and right of the local maximum.
- To increase/decrease the precision of the coordinate values, hover over the value and press + or −.



Answer: Time to reach maximum height is 2.55 s.
Maximum height is 51.89 m (correct to two decimal places)

... continued

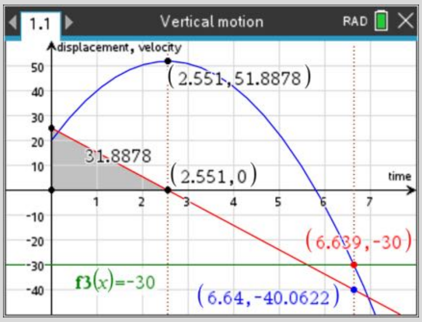
(b) (i) *Alternative method*. To find the maximum height and time taken to reach the maximum height using the area under the $v-t$ graph, on the **Graphs** page from part **(a)** above:



- Press menu > **Geometry** > **Points & Lines** > **Intersection Point(s)**. Click on graph $f2$, then on the x-axis (the time axis). Then press esc.
- Hover over this intersection point and press ctrl menu > **Coordinates & Equations**.
- Press menu > **Analyse Graph** > **Integral**.
- Click on graph $f2$, then click on the origin (lower bound), then click on the point where the graph $f2$ intersects the x-axis (upper bound).

Answer: Time to reach maximum height 2.55 s. Maximum height above tower is 31.89 m. Maximum height above the origin is 31.89 + 20 = 51.89 m.

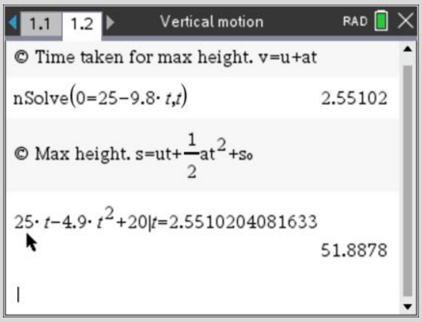
(b) (ii) To determine the velocity on striking the ground and the time taken to strike the ground:



- Press menu > **Analyse Graph** > **Intersection**. Click on graph $f1$ then on graph $f3$. The x-coordinate of the intersection point is 6.639….
- Press menu > **Geometry** > **Points & Lines** > **Point On**. Click on graph $f2$, then press esc, then click on the x-coordinate of the point until it is editable. Edit the x-coordinate to be 6.639.

Answer: Strikes ground at -40.06 m/s at time 6.64 s, correct to two decimal places.

(c) (i) To find the maximum height and the time to reach maximum height taken using formulas, on a **Calculator** page:

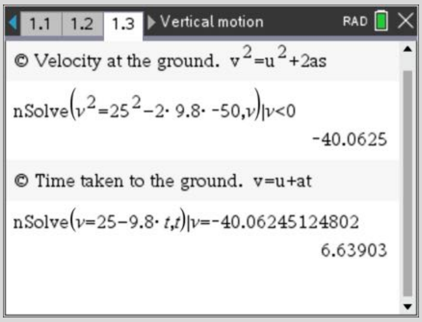


- Press menu > **Algebra** > **Numerical Solve**, then enter $\mathbf{nSolve}(\mathbf{0=25-9.8}\boldsymbol{t}\mathbf{,}\boldsymbol{t})$.
- Enter $\mathbf{25}\boldsymbol{t}\mathbf{-4.9}\boldsymbol{t}^{\mathbf{2}}\mathbf{+20}|\boldsymbol{t}=\mathbf{ans}$.

*(**Note:** Press ctrl ans to input the previous answer in current entry line).*

Answer: Time to reach maximum height is 2.55 s. Maximum height 51.89 m, correct to two decimal places.

(c) (ii) To find the velocity on striking the ground and time taken - using formulas - on a **Calculator** page:



- Press menu > **Algebra** > **Numerical Solve**, then enter $\mathbf{nSolve}(\boldsymbol{v}^{\mathbf{2}}=\mathbf{25}^{\mathbf{2}}\mathbf{-2\times9.8\times-50,}\boldsymbol{v})\,|\,\boldsymbol{v}<\mathbf{0}$
- Enter $\mathbf{nSolve}(\boldsymbol{v}=\mathbf{25-9.8}\boldsymbol{t}\mathbf{,}\boldsymbol{t})\,|\,\boldsymbol{v}=\mathbf{ans}$

***Note:** To add a comment, press menu > **Actions** > **Comment**.*

Answer: Strikes ground at -40.06 m/s at time 6.64 s, correct two decimal places.

***Note:** Motion on a smooth inclined plane problems can generally be solved in a similar manner to the above, using acceleration $a = g\sin(\theta)$, where θ is the angle of inclination of the plane.*

Exploring the graphs of a particle moving with simple harmonic motion

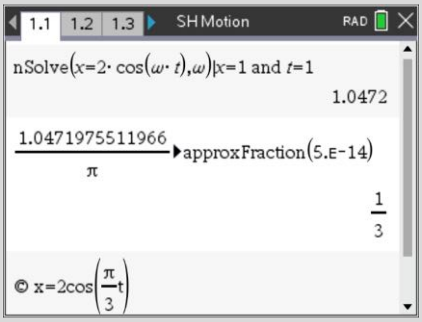
A particle is moving with SHM about a point $x = 0$. At time $t = 0$, its position is $x = 2$ and its velocity is $v = 0$. At time $t = 1$, its position is $x = 1$.

(a) Find x as a function of time, t.

(b) Compare the graphs of position-time, velocity-time and acceleration-time for the particle.

(c) Explore the relationship between the area under the velocity-time graph and the position.

(a) Note that at $t = 0, v = 0$ and $x = 2$, therefore $A = 2$ and $x = 2\cos(\omega t)$



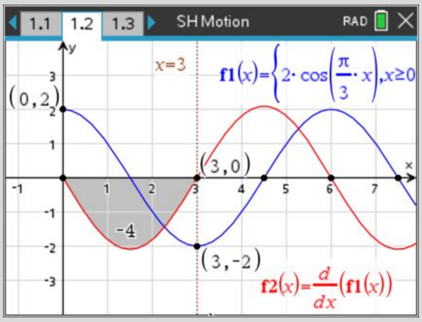
To find x as a function of time, on a **Calculator** page:

- Press [menu] > **Algebra** > **Numerical Solve**.
- Enter $\mathbf{nSolve}(x = 2\cos(\omega \cdot t), \omega) \mid x = 1 \text{ and } t = 1$.
- Input $\mathbf{ans} / \pi$, then press [menu] > **Number** > **Approximate to Fraction**.
- Press [enter].

Note: *To select* ω *from the symbols list, press* [ctrl] [∞β°] *or* [📖] [4].

Answer: $\omega = \dfrac{\pi}{3}$, $x = 2\cos\left(\dfrac{\pi}{3}t\right)$

(b) To compare position-time and velocity-time graphs, on a **Graphs** page:



- Enter $f1(x) = 2\cos\left(\dfrac{\pi}{3}x\right) \mid x \geq 0$ and

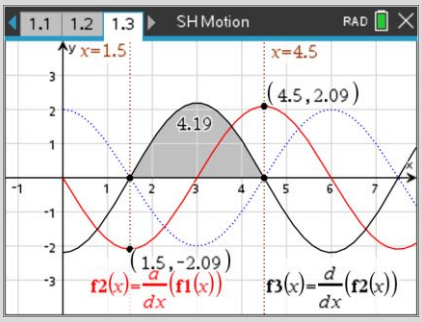
  $f2(x) = \dfrac{d}{dx}(f1(x))$ pressing [⇧shift] [−] as a shortcut to the **Derivative** template.

  (or alternatively $f2(x) = -\dfrac{2\pi}{3}\sin\left(\dfrac{\pi}{3}x\right) \mid x \geq 0$)
- Press [menu] > **Window/Zoom** > **Window settings**.
  In the dialog box that follows, enter the following values:
  XMin = –1.2, XMax = 8, XScale = 1,
  YMin = – 4, YMax = 4, YScale = 1.
- Press [menu] > **View** > **Grid** > **Lined Grid**
- Press [menu] > **Analyse Graph** > **Integral**, then click on graph $f2$, then click on the point with coordinates $(0,0)$ then click on the point at $(3,0)$.

The area, A, enclosed by the v-t graph on the interval $[0,3]$ is $A = 4$. This corresponds with change in position on $[0,3]$: $2-(-2) = 4$.

... continued

To compare velocity-time and acceleration-time graphs, on the **Graphs** page:



- Enter $\boldsymbol{f3(x)=\frac{d}{dx}(f2(x))}$ (or alternatively $\boldsymbol{f3(x)=-2\left(\frac{\pi}{3}\right)^2\cos\left(\frac{\pi}{3}x\right)|x\ge 0}$)
- Press menu > **Analyse Graph** > **Integral**
- Click on graph $\boldsymbol{f3}$, then click on the point with coordinates $(1.5,0)$ then on point at $(4.5,0)$.
- Press menu > **Trace** > **Graph Trace**. Select graph $\boldsymbol{f2}$, then type **1.5** and press enter twice.
- Type **4.5** and press enter twice. Then press esc.

Note: *To hide any unneeded objects on the graph workspace, press* menu *> **Actions > Hide/Show**, then click on the items to be hidden.*

The area, *A*, enclosed by the *a*-*t* graph on the interval $[1.5,4.5]$ is $A\approx 4.19$, corresponding with the change in velocity: $2.09...-(-2.09...)$. This also shows that acceleration $\frac{d^2x}{dt^2}=-\left(\frac{\pi}{3}\right)^2x=-\omega^2x$.

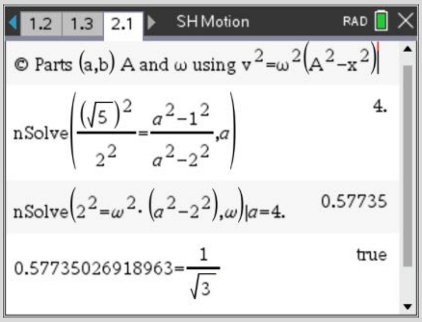
Applying the relationship $v^2=\omega^2\left(A^2-x^2\right)$ to a particle moving with simple harmonic motion

A particle is moving with SHM about a point $x=0$. Assume that $x=0$ at $t=0$ and that when $x=1$, $v=\sqrt{5}$ and when $x=2$, $v=2$.

(a) Find the amplitude and period of motion. Hence find *x* as a function of time, *t*.

(b) Graph displacement vs. time and velocity vs. time. Hence use graphical methods to determine the velocity at $x=2.5$ and $x=-3$, and the time taken to travel between these points. Give answers correct to two decimal places.

(a) To find the amplitude and period using the relationship $v^2=\omega^2\left(A^2-x^2\right)$, on a **Calculator** page:



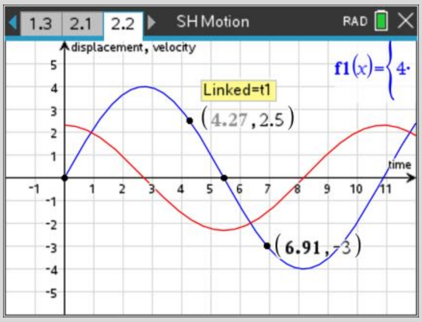
- Press menu > **Algebra** > **Numerical Solve**, enter $\textbf{nSolve}\left(\frac{(\sqrt{5})^2}{2^2}=\frac{a^2-1^2}{a^2-2^2},a\right)$ then $\textbf{nSolve}\left(2^2=\omega^2\cdot\left(a^2-2^2\right),\omega\right)|a=\textbf{ans}$.

Answer: $A=4$ and $\omega=\frac{1}{\sqrt{3}}\approx 0.5774$.

Therefore period, $T=2\pi\sqrt{3}$ and $x(t)=4\sin\left(\frac{1}{\sqrt{3}}t\right)$.

... continued

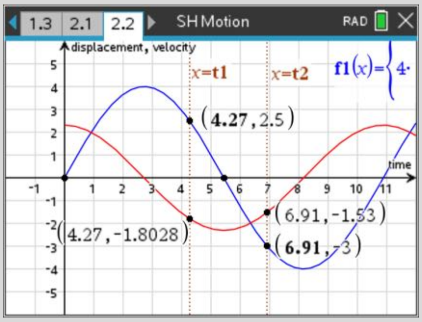
(b) To graph displacement vs. time and velocity vs. time. on a **Graphs** page in a **New Problem**:



- Enter $f1(x)=4\sin\left(\frac{x}{\sqrt{3}}\right)|x\geq 0$ then

  $f2(x)=\frac{4}{\sqrt{3}}\cos\left(\frac{x}{\sqrt{3}}\right)|x\geq 0$.
- Press menu > **Window/Zoom > Window settings**.
  In the dialog box that follows, enter the following values:
  XMin = –2, XMax = 12, XScale = 1,
  YMin = – 6, YMax = 6, YScale =1.
- Press menu > **Geometry > Points & Lines > Point On**.
  Click on each graph twice, to add two points to each graph, and then press esc.
- Edit the *y*-coordinates of the points on graph $f1$ to **2.5** and to **3**.

To determine graphically the velocity at $x=2.5$ and $x=-3$, and the time taken to travel between these points, on the **Graphs** page:



- Hover over the *x*-coordinate (the time coordinate) of the first point, press ctrl menu > **Store**. Enter $t1$ as the 'var' name.
  Likewise, enter $t2$ for the other *x*-coordinate.
- Edit the *x*-coordinate of the first point on graph $f2$ to $t1$ and edit the *x*-coordinate of the second point on graph $f2$ to $t2$.

Answer:

When $x=2.5$, $v=-1.80$. When $x=-3$, $v=-1.53$

Time taken $(6.91-4.27)=2.64$

4.5. Topic 5: Statistical inference

4.5.1. Sample means

The activities for this sub-topic focus on using dynamic features of the *TI-Nspire CX II-T* to explore how sampling behaves in different situations and help to develop a sound conceptual understanding of the distribution of sampling means.

Sampling from the Normal Distribution

The published 2024 NAPLAN Numeracy results for participating Year 9 students in Queensland showed a mean score of 552 and a standard deviation of 81.
Assume that the NAPLAN scores for this population are normally distributed.
Set up a simulation to take random samples of size $n = 50$ from the $\mathrm{N}\left(552, 81^2\right)$ distribution to model taking random samples from this population and analyse the behaviour of repeated sampling.
Let the random variable X denote the Year 9 NAPLAN score. The true population mean is $\mu = \mathrm{E}(X) = 552$ and the population standard deviation is $\sigma = \sqrt{\mathrm{var}(X)} = 81$.

Note: *To 'Seed' or initialise the pseudo-random number generator, on a* ***Calculator*** *page, press* menu *>* ***Probability*** *>* ***Random*** *>* ***Seed*** *then enter, say, the last few digits of your mobile phone number. (e.g.* ***RandSeed*** *74839)*

Using simulation to demonstrate variability between samples

In this exploration, a simulation will be set up to observe variability between samples and build up a model of the sampling distribution of sample means, building it up **one sample at a time**.
The aim is to illustrate that the sample mean is itself a random variable that varies between samples and observe the behaviour of the expected value of the sample mean as the number of samples taken increases.

Example: Select a random sample of size $n = 50$ from the normal distribution with population parameters $\mu = \mathrm{E}(X) = 552$ and $\sigma = \sqrt{\mathrm{var}(X)} = 81$, and find the sample mean, $\overline{X}$.

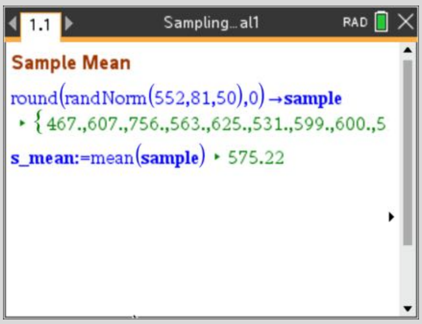
Use the command **randNorm(*μ*, *σ* [,#Trials])**.

To select a random sample of size $n = 50$ from $\mathrm{N}\left(552, 81^2\right)$ and calculate the sample means, on a **Notes** page:

- Press ctrl M to insert a **Maths Box**.
- In the **Maths Box**, type **round** (*or select* **round** *from* 📖) then press menu > **Calculations** > **Probability** > **Random** > **Normal** and input **round(randNorm(552,81,50),0)**.
- Press ctrl sto→, type **sample** and then press enter.
- Insert another **Maths Box** and enter: ***s_mean***:**= mean(*sample*)**.

Note:
1. Press var *to input the variable* ***sample****.*
2. To input the 'underscore' character, press ctrl ␣*.*

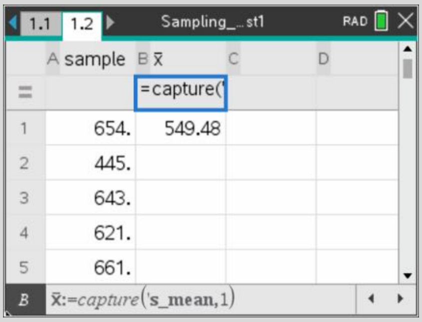
- Click on the first **Maths Box** and press enter several times. You will observe that after each press a new sample is taken.



... continued

To build up a model of the sampling distribution of sample means, one sample at a time, the sample mean for each new sample will be captured in the **Lists & Spreadsheet** application, as follows.

- Add a **Lists & Spreadsheet** page to the document.
- In the Heading row (the top row) enter the following column titles:
  - For Column A – title: **sample** (*select* **sample** *from* [var] *menu.*)
  - For Column B – title: $\bar{x}$ (*select* $\bar{x}$ from the [ctrl] [∞β°] *symbols.*)

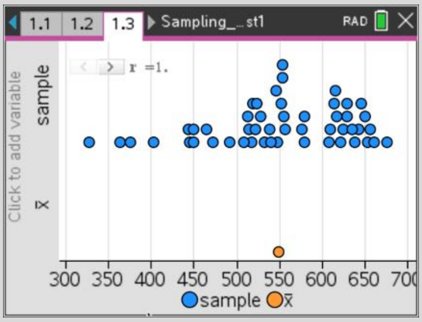


To enter the formula to capture the sample means:

- Navigate to the Column B formula cell (second cell from the top) without clicking on it. Press [menu] > **Data** > **Capture** > **Auto**.
- Press [var], select **s_mean** then press [enter].

To obtain plots of the sample and captured sample means, add a **Data & Statistics** page to the document, then:

- Press [tab] and select **sample** on the horizontal axis.
- Press [menu] > **Plot Properties** > **Add X variable**, then select $\bar{x}$.
- Hover over the plotted $\bar{x}$ point, press [ctrl] [menu] > **Colour** > **Fill Colour**. Select a contrasting colour.



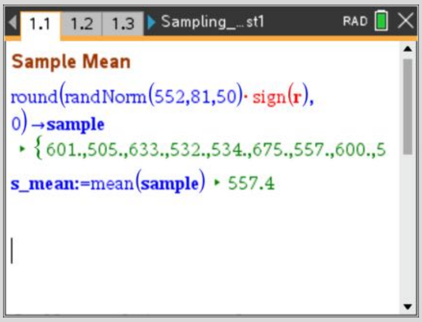
To add a slider that will allow the sampling to be repeated, on the **Data & Statistics** page:

- Press [menu] > **Actions** > **Insert Slider**, then enter the following settings.
- variable: ***r*****,** min: **1,** max: **200,** step:**1,** minimise : ☑**.**

To edit the sampling formula so the slider allows repeated sampling, navigate back to the **Notes** page, **page 1.1**.

- Edit the first **Maths Box** to:

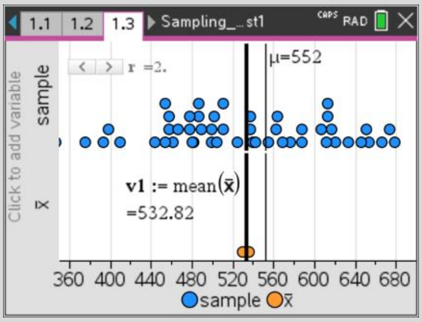
$$\textbf{round}\left(\textbf{randNorm}\left(552,81,50\right)\cdot\textbf{sign}\left(r\right),0\right)\rightarrow\textbf{sample}$$



Note**:** $\textbf{sign}(r) = +1$ *for all* $r > 0$. *Therefore, multiplying by* $\textbf{sign}(r)$ *doesn't alter the result. However, each time the value of* ***r*** *changes, it triggers a new sample to be taken.*

... continued

To add plot values for the mean of $\bar{X}$ for repeated sampling, and the population mean, navigate back to the **Data & Statistics** page, then:



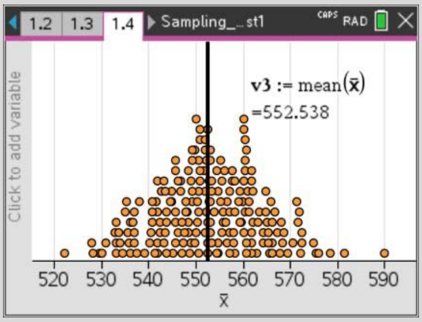
- Press menu > **Analyse** > **Plot Value**, and enter $\mathbf{v1 := mean}(\bar{\boldsymbol{x}})$.
- Press menu > **Analyse** > **Plot Value**, and enter $\mathbf{v2 := 552}$.

Note: *Press* var *to select* $\bar{\boldsymbol{x}}$.
Note: *These plot values allow a comparison of* $\mathrm{E}(\bar{X})$ *and the population mean,* $\mu = 552$, *as the number of samples taken increases.*

- Click or animate the slider to observe the building up of a model of the distribution of $\bar{X}$ for 200 samples.

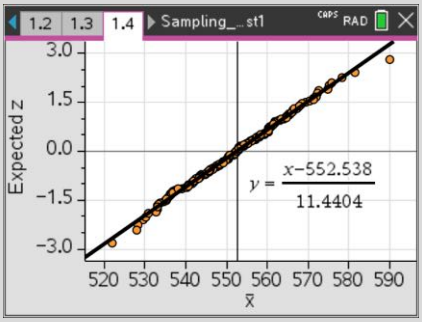
Note*: To animate, click on the slider. Press* ctrl menu *>* ***Animate****. To stop,* ctrl menu *>* ***Stop Animate****.*

To focus on dot plots of a model of the distribution of $\bar{X}$ (with 200 samples showing), add a new **Data & Statistics** page, then:



- Press tab and select $\bar{\boldsymbol{x}}$ on the horizontal axis.
- Press menu > **Analyse** > **Plot Value**.
- Enter $\mathbf{v3 := mean}(\bar{\boldsymbol{x}})$.

To compare the distribution of $\bar{X}$ with the normal distribution using a normal probability plot, on **Data & Statistics page 1.4**:

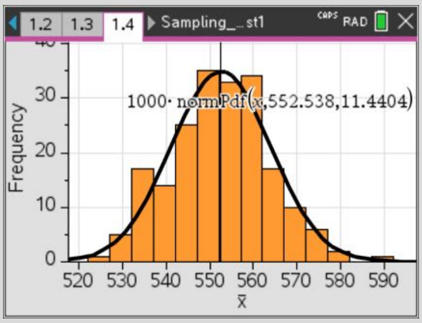


- Press menu > **Plot Type** > **Normal Probability Plot.**

The normal probability plot should show a very high correlation between the actual values of $\bar{X}$ and the expected values for $\mathrm{N}(0,1^2)$.

- Press menu > **Plot Type** > **Dot Plot** to revert to the dot plot.

To compare the distribution of $\bar{X}$ with the normal distribution using a normal pdf curve, on **Data & Statistics page 1.4**:



- Press menu > **Plot Type** > **Histogram**.
- Press menu > **Analyse** > **Show Normal PDF**.

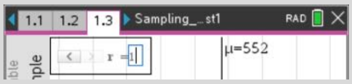
The normal pdf curve should show a good approximate fit to the histogram for $\bar{X}$. The curve parameters also give the mean and standard deviation of $\bar{X}$.
These can be compared with the theoretical values using

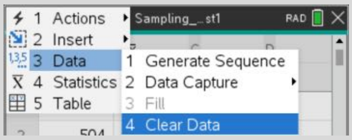
$$\mathrm{E}(\bar{X}) = \mu = 552 \text{ and } \mathrm{sd}(\bar{X}) = \frac{\sigma}{\sqrt{n}} = \frac{81}{\sqrt{50}} \approx 11.46.$$

... continued

To clear the sample data and reset the simulation, navigate back to **Data & Statistics page 1.3**, then:

- Click on the value of ***r*** on the slider until it is editable.
- Reset value to ***r* = 1**.
- Navigate back to **Lists & Spreadsheet page 1.2**.
- Click the column B formula cell, and press menu > **Data** > **Clear Data**. (Alternatively, press ctrl menu > **Clear Data**).





Exploring the effect of sample size on the distribution of sample means through simulation

Set up a simulation to observe a model of the sampling distribution of sample means for sample sizes $10 \le n \le 100$. In each case the sampling will be repeated 500 times.

To set up the simulation, carry out the following steps.

- Add a **Notes** page in a **New Document**.
- Add a **Data & Statistics** page.
- Press menu > **Actions** > **Insert Slider**. Enter the slider settings: variable: ***n*,** min: **10,** max: **100,** step:**10,** minimise: ☑**.**
- Navigate back to the **Notes page 1.1**.
- Press ctrl M to insert a **Maths Box** and input: $\mathbf{seq(mean(randNorm(552,81,\mathit{n})),\mathit{k},1,500)}$ (do not press enter yet.)
- Press ctrl sto→ followed by ctrl ∞β° and select $\bar{x}$, then press enter.

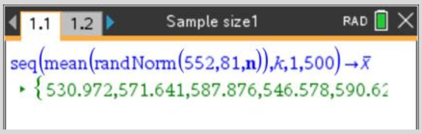
This will generate the sample means of 500 samples, with the sample size, *n*, determined by the slider value. The results are stored in list $\bar{x}$.

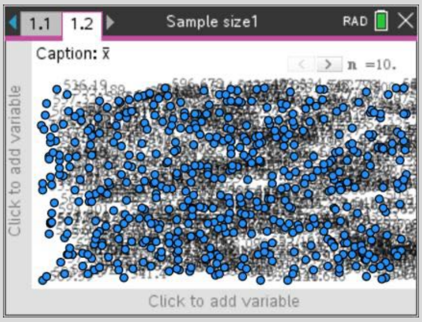
To observe the effect of changing the sample size, on the **Data & Statistics page 1.2**, click below the horizontal axis to select the list $\bar{x}$.

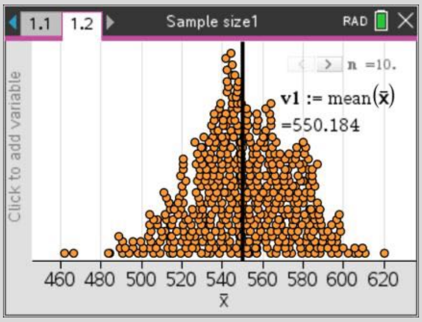
- Press menu > **Analyse** > **Plot Value**, enter $\mathbf{v1 := mean(\bar{x})}$

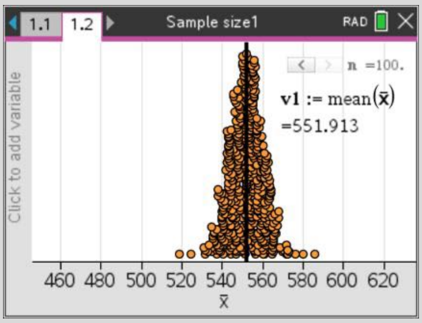
(***Note***: *Select* $\bar{x}$ *from the* var *menu).*

- Change the slider value and observe how the dot plots of the distribution of $\bar{X}$ changes as the value of n increases.









... continued

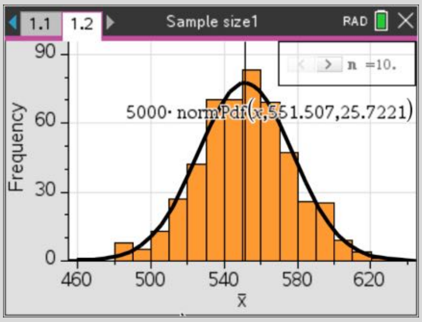
To observe the effect of changing the sample size on the sampling distribution of sample means, on **Data & Statistics page 1.2**:

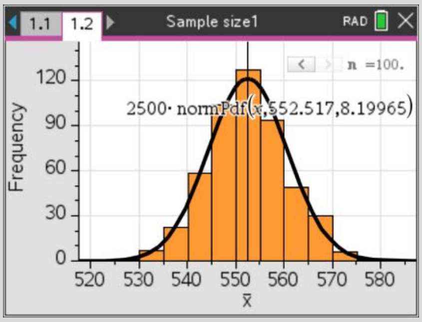
- Set the slider to $n = 10$, then press [menu] > **Plot Type** > **Histogram**.
- Press [menu] > **Analyse** > **Show Normal PDF**.
- Change the value of n from 10 through to 100 in steps of 10.
- As the value of n changes, it may be necessary to zoom (press [menu] > **Window/Zoom** > **Zoom – Data**) to the data and change **Bin Settings** (via [ctrl] [menu] > **Bin Settings**).

Observe the changes to the parameters shown in the equation of the normal pdf curve as the value of *n* changes. Compare the curve parameters with the theoretical values using

$\mathrm{E}(\bar{X}) = \mu = 552$ and $\mathrm{sd}(\bar{X}) = \dfrac{\sigma}{\sqrt{n}} = \dfrac{81}{\sqrt{10}} \approx 25.61$ through

to $\mathrm{sd}(\bar{X}) = \dfrac{\sigma}{\sqrt{n}} = \dfrac{81}{\sqrt{100}} = 8.1$





Note: *The results from these simulations should demonstrate that for a random sample of size* n, *taken from the normal distribution* $X \sim \mathrm{N}(\mu, \sigma^2)$, *the sampling distribution of the sample means is itself normal with* $\bar{X} \sim \mathrm{N}\left(\mu, \dfrac{\sigma^2}{n}\right)$.

The spread of the distribution decreases as the sample size increases is consistent with $\mathrm{sd}(\bar{X}) = \dfrac{\sigma}{\sqrt{n}}$. *As* $n \to \infty$, $\dfrac{\sigma}{\sqrt{n}} \to 0$

Sampling from the Uniform Distribution

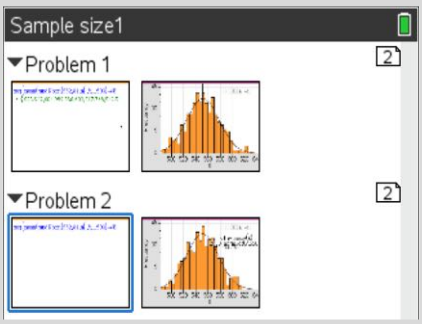
Use simulation to explore a model of the sampling distribution of the mean random number when a sample of n random numbers is taken from the continuous uniform distribution $X \sim \mathrm{U}(0,10)$.

For this distribution, $\mu = \mathrm{E}(X) = 5$ and $\sigma = \sqrt{\dfrac{(10-0)^2}{12}} \approx 2.89$. Explore the shape of the distribution of sample means and the effect of sample size.

For each sample size, model repeating the sampling 500 times.

To set up the simulation, start by copying the previous problem to a new problem as follows.

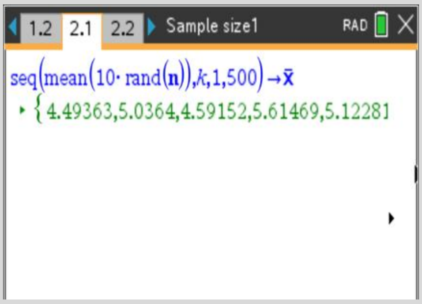
- Press [ctrl] ▲.
- In the thumbnail view that follows, click **Problem 1** and press [ctrl] [C] then [ctrl] [V]. This creates a copy, called **Problem 2**.
- Click on the first page of the copied **Problem 2** to open it.



... continued

To modify the previous problem to sample from the uniform distribution, on the **Notes page 2.1**, modify as follows.

- Delete the following text shown in strikethrough font
  seq(mean(~~**randNorm(552,61,*n*)**~~ **),** $\boldsymbol{k,1,500) \to \bar{x}}$
- Edit the command input to now be:
  $\mathbf{seq(mean}\left(\mathbf{10\cdot rand}\left(\boldsymbol{n}\right)\right)\mathbf{,}\boldsymbol{k}\mathbf{,1,500)} \to \bar{\boldsymbol{x}}$

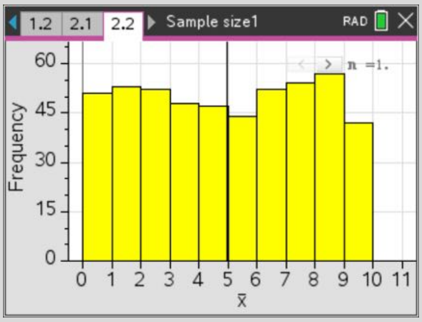


To observe the effect of changing the sample size on the model of the distribution of sample means for samples taken from the uniform distribution, navigate to the **Data & Statistics page 2.2**, then:

- Hover over one of the dots in the dot plot.
  Press ctrl menu **> Colour**. Change the colour.

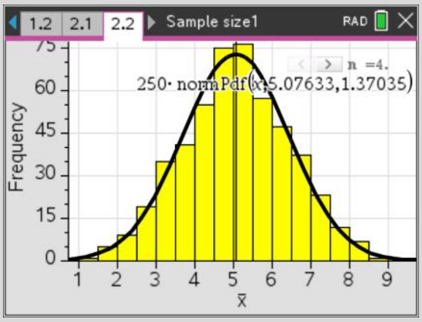
Observe the shape of the distribution with sample size $\boldsymbol{n = 1}$.

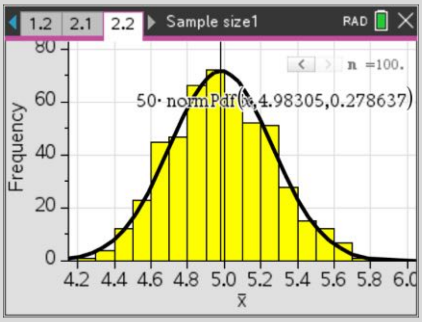
- Click on the slider value until it is editable and set it to $\boldsymbol{n = 1}$.
- Press menu > **Plot Type > Histogram**.
- **Press** menu **> Plot Properties > Histogram Properties > Bin Settings > Equal Bin Width.**
- Press menu > **Window/Zoom > Zoom – Data**



Observe the shape of the distribution with sample size $\boldsymbol{n = 4}$.

- Edit the slider value to be $\boldsymbol{n = 4}$.
- Set **Bin Width = 0.5** and zoom to the data.
- Press menu > **Analyse > Show Normal PDF**.
- Edit the value of ***n*** to larger sample sizes and observe the increase on the shape and spread of the histogram and the normal pdf parameters.
- Adjust the bin width and press **Zoom – Data** as necessary.





This should demonstrate that, unsurprisingly, with a sample size $n = 1$ *the distribution of sample means is uniform. However, with a sample size as small as* $n = 4$*, the distribution of sample means is approximately normal.*

The normal pdf parameters for $n = 4$ *should be consistent with* $\mathrm{E}\left(\bar{X}\right) = \mu = 5$ *and* $\mathrm{sd}\left(\bar{X}\right) = \dfrac{\sigma}{\sqrt{n}} = \dfrac{2.88675\ldots}{\sqrt{4}} \approx 1.44$.

Texas Instruments

Sampling from an asymmetric distribution

(a) Use simulation to explore the distribution of the number of rolls of a six-sided die needed to get the first ‘six’.

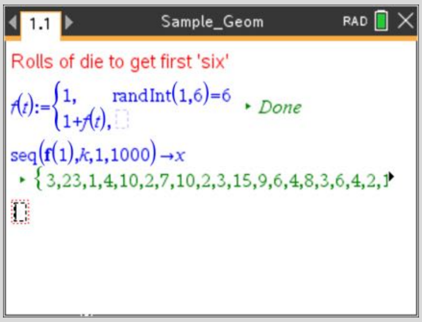
(b) Use simulation to explore the sampling distribution of the sample mean when samples of size n are taken from the asymmetric distribution generated in part (a) above.

(a) To set up a simulation for the number of rolls of a die needed to get the first ‘six’, on a **Notes** page define a piecewise recursive function, $f(t)$, as follows.

- Press [ctrl] [M] to insert a **Maths Box**.
- In the **Maths Box**, enter

$$\boldsymbol{f(t):=\begin{cases}1, & \text{randInt}(1,6)=6\\ 1+f(t), & \text{else}\end{cases}}.$$

- Insert a new **Maths Box** and input $\mathbf{seq}(\boldsymbol{f}(\mathbf{1}),\boldsymbol{k},\mathbf{1},\mathbf{1000})$
- Press [ctrl] [sto→], input ***x*** then press [enter].

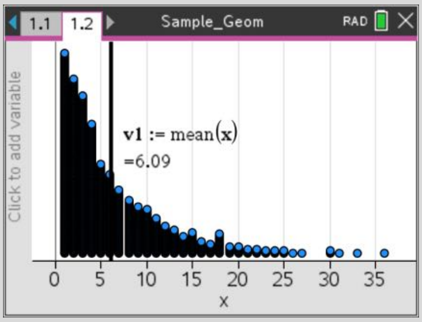


The list of ‘number of rolls required for first ‘six’’ is stored as list ***x***.

Note: *Press [template] to select the* ***Piecewise*** *template.*
Note: *The ‘t’ in the recursive function is a ‘dummy’ input. The function adds 1 to the previous value until the random integer is a ‘6’.*

To obtain a plot of the distribution of the number of rolls for the first ‘six’, *X*, add a **Data & Statistics** page, then:

- Press [tab] and select ***x*** on the horizontal axis.
- Press [menu] **> Analyse > Plot Value** and enter $\boldsymbol{v1 := \text{mean}(x)}$



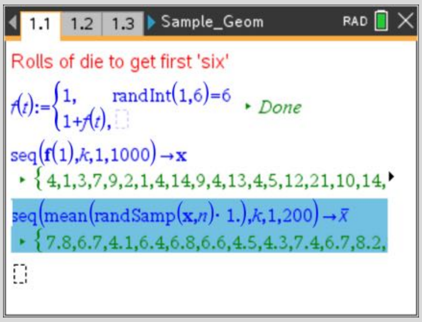
Note: *This simulation models the geometric distribution for the number of Bernoulli trials until the first ‘success’, with parameter* $p=\frac{1}{6}$. *The geometric distribution is the discrete analogue of the exponential distribution, which is covered in Topic 2.*

$$\text{E}(X)=\tfrac{1}{p}=6 \text{ and } \sigma=\text{sd}(X)=\frac{\sqrt{1-p}}{p}=\frac{\sqrt{\frac{5}{6}}}{\frac{1}{6}}\approx 5.48$$

(b) To explore the sampling distribution of the sample mean when samples of size n are taken from the distribution generated in part **(a)** above, add a new **Data & Statistics** page, then:

- Press [menu] > **Actions** > **Insert Slider**.
- Enter slider settings: Variable: ***n***, Value: **20**, Min.: **5**, Max.: **100**, Step Size: **10**, Minimise ☑
- Navigate to the **Notes page 1.1** and in a new **Maths Box** input $\mathbf{seq}(\mathbf{mean}(\mathbf{randSamp}(\boldsymbol{x},\boldsymbol{n})\cdot\mathbf{1.0}),\boldsymbol{k},\mathbf{1},\mathbf{200})$

  (don’t press [enter] yet.)
- Press [ctrl] [sto→], then press [ctrl] [∞β°], select $\bar{x}$ then press [enter].

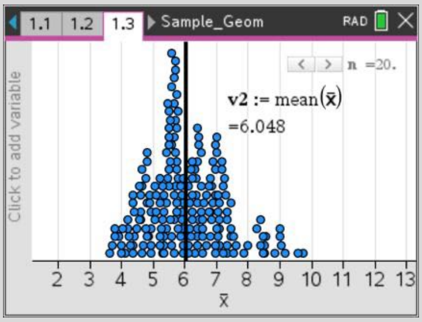


... continued

***Note**: This will select 200 samples of size n (slider value) from list* $\boldsymbol{x}$*, calculate the sample means and store the means as list* $\bar{\boldsymbol{x}}$.

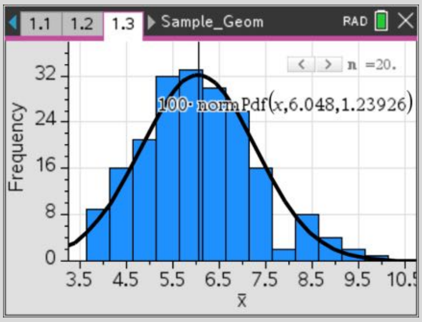
To plot a model of the sampling distribution of sample means taken from the asymmetric distribution, navigate to **Data & Statistics page 1.3**, then:

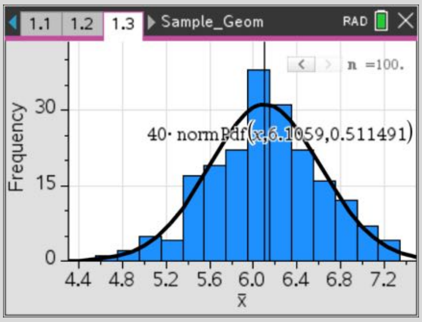
- Click below the horizontal axis and select the variable $\bar{x}$.
- Press [menu] > **Analyse** > **Plot Value** and enter $\mathbf{v2} := \mathbf{mean}(\bar{x})$ (select $\bar{x}$ from the [var] menu).
- Use the slider to observe the effect of increasing the sample size, $\boldsymbol{n}$.



To observe the approximate normality of the sampling distribution of sample means using a Normal PDF curve, on the **Data & Statistics page 1.3**:

- Set the slider value to $\boldsymbol{n} = \mathbf{20}$.
- Press [menu] > **Plot Type** > **Histogram**.
- Press [menu] > **Window/Zoom** > **Zoom – Data**.
- Press [menu] > **Analyse** > **Show Normal PDF**.
- Edit the value of $\boldsymbol{n}$ to larger sample sizes and observe the increase on the shape and spread of the histogram and the normal pdf parameters.
- Adjust the **Bin Settings** and **Zoom – Data** as necessary.





Note: This simulation should demonstrate that, despite the samples being drawn from such a skewed distribution, the distribution of sample means is approximately normal.
The normal pdf parameters for $n = 20$ *should be consistent with* $\mathrm{E}(\bar{X}) = \mu = 6$ *and* $\mathrm{sd}(\bar{X}) = \frac{\sigma}{\sqrt{n}} = \frac{5.46}{\sqrt{20}} \approx 1.22$.

***Note:** The suite of simulations in this sub-topic should demonstrate that for repeated sampling, the expected value of the sample statistic,* $E(\bar{X})$*, tends to the value of the population parameter, μ, so that in the long run,* $\mathrm{E}(\bar{X}) = \mu$.

Students can also verify that the standard deviation of the simulated sample means for samples of size, n, tends to $\frac{\sigma}{\sqrt{n}}$*, so that in the long run* $\mathrm{sd}(\bar{X}) = \frac{\sigma}{\sqrt{n}}$*, as predicted by the theory.*

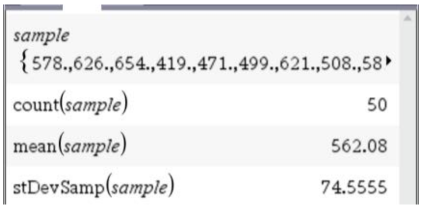
*These simulations should serve as good illustrations of the **central limit theorem**, which establishes the approximate normality of the sampling distribution of the sample means in many situations, even if the original variables themselves are not normally distributed.*

4.5.2. Confidence intervals for means

Finding confidence intervals for the mean

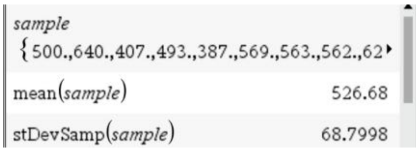
Suppose that prior to a full analysis of the Year 9 NAPLAN numeracy results, a researcher uses a random sample of 50 scores to estimate the population parameters. The sample mean is a point estimate that is unlikely to be exactly equal to the population mean. The researcher therefore finds it useful to calculate an interval that is likely to contain the true population mean, with a high level of confidence.

(a) Assume that for this sample of size $n = 50$, the sample mean is $\bar{x} = 562$ and the sample standard deviation is $s = 74.6$, as shown in the screenshot. Determine an approximate 95% confidence interval, correct to one decimal place, for the mean NAPLAN numeracy score, using $s = 74.6$ as an estimate of the population standard deviation.



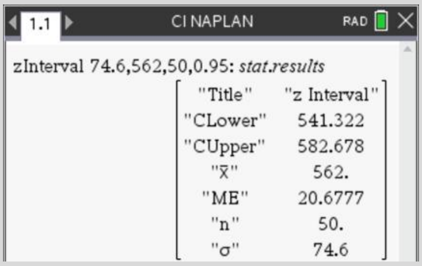
(b) It subsequently becomes known that the true population standard deviation is $\sigma = 81$. Determine an approximate 95% confidence interval for the mean NAPLAN score using $\bar{x} = 562$ and the actual value of the population standard deviation $\sigma = 81$. Compare this interval with the interval determined in part (a) above.

(c) Suppose that another researcher selected a different random sample of size $n = 50$, with sample mean $\bar{x} = 527$ and sample standard deviation $s = 68.8$. Find an approximate 95% C.I. for the mean NAPLAN score using these known sample statistics, and comment on any notable aspect of this result.



(a) To find an approximate 95% confidence interval, on a **Calculator** page:

- Press menu > **Statistics > Confidence Intervals > z Interval**.
- Select **Data Input Method: Stats**. In the dialog box enter: $\boldsymbol{\sigma = 74.6, \bar{x} = 562, n = 50, C\ Level = 0.95}$

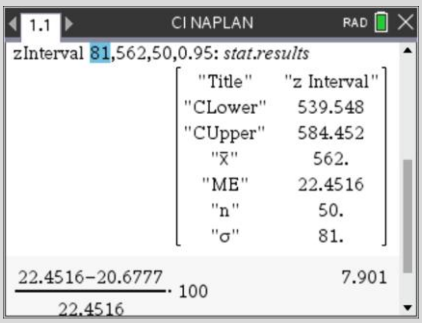


Answer: The approximate 95% confidence interval is $(541.3, 582.7)$.

The margin of error for this interval is 20.7, and represents the radius of the confidence interval, i.e. half the interval length.

(b) To find an approximate 95% confidence interval using the actual population standard deviation, on the **Calculator** page from part **(a)** above:

- Press ▲ up to the **zInterval** input as shown, then press enter.
- Edit the pasted input to **zInterval 81,562,50,0.95: *stat.results***.

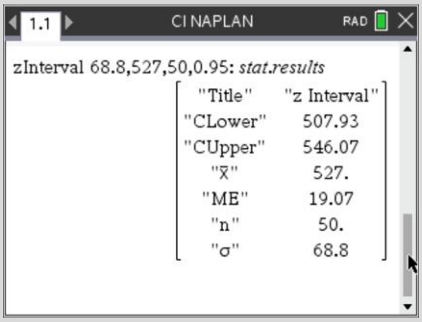


The approximate 95% confidence interval is $(539.5, 584.5)$ using $\sigma = 81$, compared with $(541.3, 582.7)$ using $s = 74.6$ as an estimate of the population standard deviation.
This is a 7.9% difference in the margin of error.

... continued

(c) To find an approximate 95% confidence interval using $\bar{x} = 527$ and $s = 68.8$, on the **Calculator** page from part (b) above:

- Press ▲ up to the previous input, then press [enter].
- Edit the pasted input to **zInterval 68.8,527,50,0.95: *stat.results***



Answer: The approximate 95% confidence interval, using the sample statistics for this sample, is $(507.9, 546.1)$. It is notable that this interval does **not** contain the true population mean, i.e. $\mu \notin (507.9, 546.1)$.

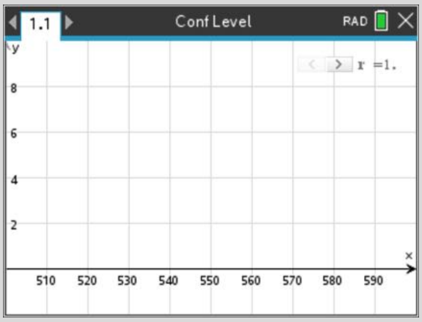
Note**:** *This illustrated that as the sample mean and margin of error vary from sample to sample, most* ***but not all*** *95% confidence intervals contain the true population mean,* μ.

Exploring the trade-off between level of confidence and margin of error

Use simulation to explore the effect of variability on confidence intervals, and the trade-off between level of confidence and margin of error. For the simulation, assume that random samples of size $n = 100$ are drawn from a population with a population mean $\mu = 552$ and a population standard deviation $\sigma = 81$. Compare confidence intervals with confidence levels of 99%, 95%, 90% and 50% for the same sample, and repeat the sampling many times.

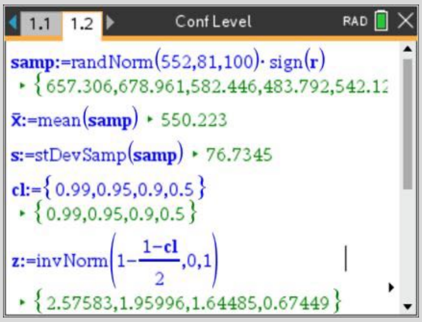
To set up a slider that will be used to trigger the taking of a new random sample, on a **Graphs** page:

- Press [menu] > **Window/Zoom** > **Window Settings**.
  In the dialog box that follows, enter the following values:
  XMin = 500 Xmax = 600 XScale = 10
  YMin = −2 YMax = 10 YScale = 2
- Press [menu] > **Actions** > **Insert Slider**.
  Enter the slider settings:
  variable: ***r*****,** min: **1,** max: **200,** step:**1,** minimise: ☑.



To set up the first part of the simulation, add a **Notes** page, then:

- Press [ctrl] [M] to add a **Maths Box.**
- In the Maths Box, enter:
  $\textbf{samp} := \textbf{randNorm}(552, 81, 100) \cdot \textbf{sign}(r)$.
- Enter the following, with each entry in a new **Maths Box**.
- $\bar{x} :=$ **mean**(**samp**) (select $\bar{x}$ from the [∞β°] menu).
- $s :=$ **stDevSamp**(**samp**) (**stDevSamp** is in [📖] list).
- $cl := \{0.99, 0.95, 0.90, 0.5\}$ (this sets the confidence level)
- $z := \textbf{invNorm}\left(1 - \dfrac{1 - cl}{2}, 0, 1\right)$ (z-scores for conf. levels)

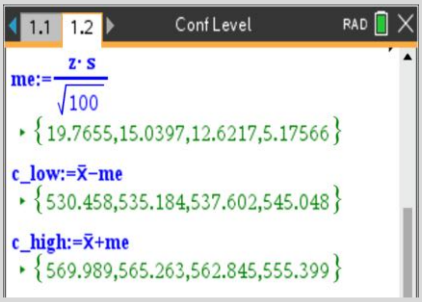


Note: *All the commands for this activity can be selected from the catalog by pressing* [📖].

... continued

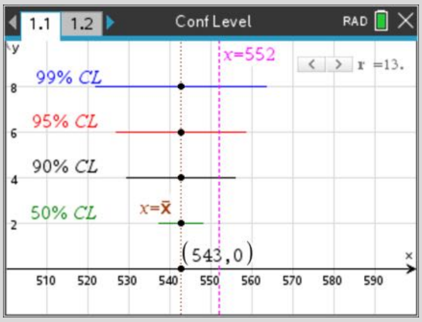
To set up the second part of the simulation on **Notes page 1.2**:

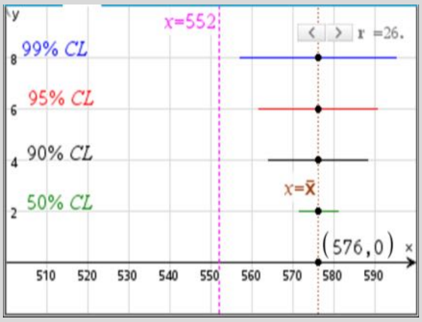
- Enter the following, with each entry in a new **Maths Box**.
- $\boldsymbol{me} := \dfrac{z \cdot s}{\sqrt{100}}$ (Calculates margins of error for each CI.).
- $\boldsymbol{c\_low} := \bar{x} - \boldsymbol{me}$ (Lower fence for each CI.).
- $\boldsymbol{c\_high} := \bar{x} + \boldsymbol{me}$ (Upper fence for each CI.).



To complete setting up the graphical representations of the confidence intervals, navigate to **Graphs page 1.1**, then:

- Press ctrl G and enter:
- $f1(x) = 8 \mid c\_low[1] \le x \le c\_high[1]$
- $f2(x) = 6 \mid c\_low[2] \le x \le c\_high[2]$
- $f3(x) = 4 \mid c\_low[3] \le x \le c\_high[3]$
- $f4(x) = 2 \mid c\_low[4] \le x \le c\_high[4]$
- Press menu > **Graph Entry/Edit** > **Relation**.
- Enter $x = \bar{x}$ then $x = 552$.
- Press menu > **Geometry** > **Points & Lines** > **Intersection Point(s)**. Click on line with equation $x = \bar{x}$, and then click on each horizontal interval.
- Press menu > **Show/Hide**. Click to hide unwanted labels, then press esc.
- Use the slider to draw samples and observe which intervals contain the population mean (i.e. intersect the line $x = 552$).





***Note**: The simulation should demonstrate that to have a greater level of confidence that the interval will contain the population mean requires a greater margin of error.*
Conversely, a smaller margin of error results in less confidence that the interval will contain μ.
The simulation can also demonstrate that for intervals with a $C\%$ confidence level, $C\%$ of all possible samples of a given size from this population will result in an interval that contains the unknown population mean, and that $(100 - C)\%$ of intervals will not contain μ.

Appendix: TI-Nspire shortcuts and tips

(With thanks to Neale Woods. Note that a tick (✓) indicates where the shortcut is applicable. MacOS users – substitute CMD for ctrl*.)*

Shortcut	Handheld	Computer	Glob	Calc	Gra	Geo	L&S	D&S	Note	Data	Prog	Pyth	Result
ctrl A	✓	✓		✓					✓	✓	✓	✓	Select all text
ctrl B	✓	✓									✓	✓	Check Syntax & Store/Save
ctrl C	✓	✓	✓										Copy text
ctrl E	✓	✓							✓				Insert a Chem Box
ctrl F	✓	✓									✓	✓	Find
ctrl G	✓	✓			✓								Toggle Graph entry line
ctrl G	✓	✓					✓				✓	✓	Go to line/cell
ctrl H	✓	✓									✓	✓	Find and replace
ctrl I	✓	✓	✓										Insert page
ctrl J		✓	✓										Take screen capture
ctrl K	✓	✓	✓										Select page (in split screen)
ctrl L	✓	✓		✓	✓	✓			✓		✓		Display variable list
ctrl L	✓	✓					✓						Store or link to variable
ctrl M	✓	✓							✓				Insert Math Box
ctrl N	✓	✓	✓										New document
ctrl O	✓	✓	✓										Open document
ctrl P		✓	✓										Print document
ctrl R	✓	✓					✓						Recalculate
ctrl R	✓	✓									✓	✓	Checks syntax/run program
ctrl S	✓	✓	✓										Save document
ctrl T	✓	✓			✓		✓						Toggle Table/No Table
ctrl T	✓	✓									✓	✓	Comment/Uncomment
ctrl V	✓	✓	✓										Paste text
ctrl W	✓	✓	✓										Close current document
ctrl X	✓	✓	✓										Cut text
ctrl Y	✓	✓	✓										Redo
ctrl Z	✓	✓	✓										Undo

Appendix: TI-Nspire shortcuts and tips (continued)

(Note that a tick (✓) indicates where the shortcut is applicable. MacOS users – substitute CMD for ctrl*.)*

Shortcut	Handheld	Computer	Global	Calc	Graph	Geom	L&S	D&S	Notes	DataQ	Prog	Python	Result
ctrl 1	✓	✓		✓			✓		✓	✓	✓	✓	Jump to last line
ctrl 2	✓	✓		✓			✓		✓	✓	✓	✓	Jump to end of line/last cell
ctrl 3	✓	✓		✓			✓		✓	✓	✓	✓	Page down
ctrl 4	✓	✓	✓										Merge two pages into split screen.
ctrl 6	✓	✓	✓										Convert split screen into two pages
ctrl 7	✓	✓		✓			✓		✓	✓	✓	✓	Jump to first line
ctrl 8	✓	✓		✓			✓		✓	✓	✓	✓	Jump to start of line/first cell
ctrl 9	✓	✓		✓			✓		✓			✓	Page up
ctrl ␣	✓	✓	✓										Underscore
ctrl tab	✓		✓										Toggle b/w split screen windows
ctrl tab		✓	✓										Toggle b/w open documents
tab	✓	✓		✓	✓		✓	✓	✓	✓	✓		Move through fields or zones
tab	✓	✓			✓								Display graph entry line
tab	✓	✓										✓	Indent
⇧shift ◀▶▲▼	✓	✓	✓										Highlight selected text
⇧shift tab	✓	✓			✓		✓	✓	✓	✓			Move back through fields or zones
⇧shift tab	✓	✓										✓	Remove indent
⇧shift +	✓			✓	✓		✓	✓	✓		✓		Derivative
⇧shift −	✓			✓	✓		✓	✓	✓		✓		Integral
⇧shift ↵	✓			✓	✓			✓	✓		✓		Add a column to a matrix
⇧shift enter		✓		✓	✓			✓	✓		✓		Add a column to a matrix
↵	✓			✓	✓			✓	✓		✓		Add a row to a matrix
Alt enter		✓		✓	✓			✓	✓		✓		Add a row to a matrix
P	✓	✓			✓	✓							Add a point

Appendix: TI-Nspire Shortcuts and Tips (continued)

From the Handheld or Computer Keyboard

To enter this:	Type this shortcut:
π	**pi**
θ	**theta**
∞	**infinity**
≤	**<=**
≥	**>=**
≠	**/=**
⇒ (logical implication)	**=>**
⇔ (logical double implication, XNOR)	**<=>**
→ (store operator)	**=:**
| | (absolute value)	**abs(...)**
√()	**sqrt(...)**
Σ() (Sum template)	**sumSeq(...)**
Π() (Product template)	**prodSeq(...)**
$\sin^{-1}()$, $\cos^{-1}()$, ...	**arcsin(...), arccos(...), ...**
ΔList()	**deltaList(...)**
ΔtmpCnv()	**deltaTmpCnv(…)**

From the Computer Keyboard

To enter this:	Type this shortcut:
e (natural log base e)	**@e**
E (scientific notation)	**@E**
^{T} (transpose)	**@t**
^{r} (radians)	**@r**
° (degrees)	**@d**
^{g} (gradians)	**@g**
∠ (angle)	**@<**
▶ (conversion)	**@>**
▶ Decimal, ▶ approxFraction(), and so on.	**@>Decimal, @>approxFraction(), and so on.**
*c*1, *c*2, … (constants)	**@c1, @c2, …**
*n*1, *n*2, … (integer constants)	**@n1, @n2, …**
i (imaginary constant)	**@i**

Useful functions/commands available in the Catalog not available in the menus.

Function/Command name	Function/Command purpose
and	Boolean ‘and’, useful for specifying restrictions.
domain(expr,var)	Displays the domain of a function.
euler(…)	Generates a table of values using Euler’s method.
isPrime(…)	Displays ‘true’ if prime and ‘false’ if composite.
true	Displays ‘true’ if two expressions are equivalent.

Texas Instruments Australia

email: teacher-support@list.ti.com

Website: **education.ti.com/australia**