

QCE Mathematical Methods Teacher Resource Book for the

TI-84 EVO graphing calculator



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Introduction

This publication, *QCE Mathematical Methods Teacher Reference Book for the TI-84 Evo*, is intended to support senior secondary school mathematics teachers in Queensland as they seek to implement the *QCAA 2025 Mathematical Methods syllabus*.

Specifically, the publication highlights ways in which *TI-84 Evo* technology might be used to assist in the teaching, learning and assessment of *QCE Mathematical Methods Units 1 to 4*.

It is not a complete manual for using this technology, rather it tries to look at each syllabus dot-point and make suggestions for possible classroom uses.

It has been developed by experienced educators and reviewed by senior mathematics teachers from Queensland schools. We hope you find this to be a useful and supportive publication.

A digital version of this publication can be found at <https://education.ti.com/aus/QLD>.

Notes for teachers

The *QCE Mathematical Methods Teacher Reference Book for the TI-84 Evo* has been designed to maximise its usefulness to teachers. The authors have provided the following notes:

- It is assumed that the user of this reference guide has a basic familiarity with navigating the *TI-84* family of calculators. Readers requiring an introduction to the basics are referred to tutorials at <https://education.ti.com/aus> and <https://www.youtube.com/@TIAustralia>
- Throughout this publication, unless otherwise specified, the default calculator document settings have been used. The calculator user interface language has been set to *English* (via `[mode]`).
- In some instances in this publication, space has been added to the syntax of commands to improve readability, even though in general spaces should **not** be used in calculator commands without a clear reason.
- There will be some variation in the formatting of commands and text to be entered, but the authors have attempted to use bold formatting when referring to commands to be entered or accessed via the calculator.
- For screenshots of function and relation graphs, grid and label settings will vary. In general, use `[mode]` and `[2nd][format]` to modify these settings.
- When catalogue commands are mentioned, pressing `[2nd][catalog]` will display catalog commands in alphabetical order. Pressing the first letter of the desired command will locate it more quickly.
- The *TI-84 Evo* includes a new “Undo Clear” command, which can be accessed by pressing `[2nd][↶clear]`. This is very useful if the user accidentally clears a line of input.
- To make this publication as practical and concise as possible, the mathematical problems considered have been restricted to those that can be completed by teachers and students via the included step by step instructions.
- For more interactive digital resources aligned to *QCE Mathematical Methods Units 1 to 4*, go to the following web page:

<https://education.ti.com/en-au/seniornspiredcurriculum/aus-nz/qld-nspire>

Unit 1: Surds, algebra, functions and probability

3.1. Topic 1: Surds and quadratic functions

3.1.1. Surds

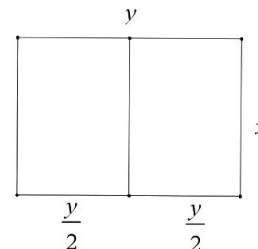
Investigating the A-series of paper sizes

Consider the paper size A0, with width x metres, and length y metres.

It has a total area of 1 square metre, so $xy = 1$ or $y = \frac{1}{x}$ (Equation 1).

Also, the A-series of paper sizes are designed so that the length-to-width ratio is the same for all sizes (i.e. A0, A1, A2, A3, A4 and so on).

This means that: $\frac{y}{x} = \frac{x}{y/2} \Leftrightarrow y^2 = 2x^2 \Leftrightarrow y = \sqrt{2}x$ (Equation 2).



Question

- Use the above two equations to find the dimensions of a sheet of A0 paper (to the nearest mm).
- Find the dimensions of a sheet of A4 paper, and its area.

Solution

(a) Equations 1 and 2 can be solved by equating their RHS,

(i.e. solve $\frac{1}{x} = \sqrt{2}x$ for x), using the numeric solver:

- Press $\boxed{\text{on}} \boxed{\Delta}$ and select the NUMERIC SOLVER application.
- For **E1**, enter $\frac{1}{X}$. For **E2**, enter $\sqrt{2}X$.
- Press **NEXT** [f5].
- Enter **X = 1** as an initial estimate, then press **SOLVE** [f5].
- Press $\boxed{\text{on}} \boxed{\Delta}$ to display the CALCULATOR application.
- Enter $\boxed{X,T,\theta,n}$ (to display the solved x value).
- Enter $\frac{1}{X} \rightarrow Y$ (to display and store y).

Answer: An A0 sheet has a width of 841 mm and a length of 1189 mm (to nearest mm).

(b) To find the dimensions and area of the sizes from A0 to A4, on the HOME screen:

- Enter $\{X, Y, XY\}$.
- Enter $\{\text{Ans}(2)/2, \text{Ans}(1), \text{Ans}(2)/2 \times \text{Ans}(1)\}$.
- Press $\boxed{\text{enter}}$ to repeat this recursive calculation to find the width, length and area of each paper size down to A4.

Answer: The recursive calculations show that an A4 sheet has width = 210 mm and length = 297 mm (to nearest mm), and area of $1/16 \text{ m}^2$ (0.0625 m^2).

Note: The recursive calculation works by simulating the size of successive pages in the A-series. For example, the width of the A1 sheet is half the length of the A0 sheet, and the length of the A1 sheet is the width of A0 sheet. This can be demonstrated by folding any A-series sheet in half.

NORMAL FIX4 AUTO REAL RADIAN MP
ENTER EQUATION E1=E2
EQUATION SOLVER

E1: $\frac{1}{X}$
 E2: $\sqrt{2}X$

NEXT

NORMAL FLOAT AUTO REAL RADIAN MP
SOLUTION IS MARKED

$\frac{1}{X} = \sqrt{2}X$
 X = 0.8408964152537
 bounds = {-1E99, 1E99}
 E1-E2 = 0

HISTORY

X
 1/X→Y

{X,Y,XY}
 {0.8409, 1.1892, 1.0000}
 {Ans(2)/2, Ans(1), Ans(2)/2·Ans(1)}
 {0.595, 0.841, 0.500}
 {Ans(2)/2, Ans(1), Ans(2)/2·Ans(1)}
 {0.420, 0.595, 0.250}
 {Ans(2)/2, Ans(1), Ans(2)/2·Ans(1)}
 {0.297, 0.420, 0.125}
 {Ans(2)/2, Ans(1), Ans(2)/2·Ans(1)}
 {0.210, 0.297, 0.063}

3.1.2. Quadratic functions

Investigating the effect of a , b , c upon the graph of $y = ax^2 + bx + c$

Question

Investigate the effect of a , b and c on the graph of the parabola with equation $y = ax^2 + bx + c$.

Solution

- Press **on** and select the TRANSFORMATION GRAPHING application.
- Press **y=** and then **clear** to clear any equations in Y1.
- Press **enter** to select the equation type.
- Press **▢** as required to select the $AX^2 + BX + C$ equation type as shown and press **enter**.
- Press **zoom** > **Zoom [-10,10]** to view the graph screen using the standard window.

The sliders for transformation parameters a , b and c will be visible at the bottom of the screen.

- Press **▼** or **▲** to move between sliders for a , b and c .
- Press **◀** or **▶** to change the value of the selected transformation parameter.

As an example, set the parameters for investigation as follows.

- Set $A = 1$ and $B = 1$, then vary the value of C and observe the effect upon the graph.
- Set $B = 1$ and $C = 1$, then vary the value of A and observe the effect upon the graph.
- Set $A = 1$ and $C = 1$, then vary the value of B and observe the effect upon the graph.

It may aid the investigation to show multiple graphs when observing the effect of each parameter. For example, to keep a trailing set of the graphs produced as you vary the slider values, from the graph screen:

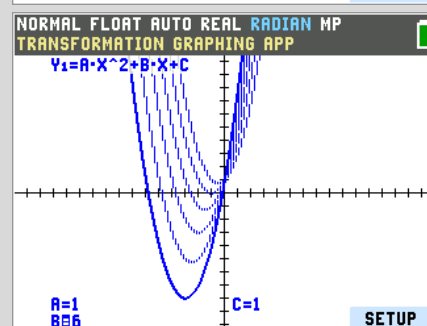
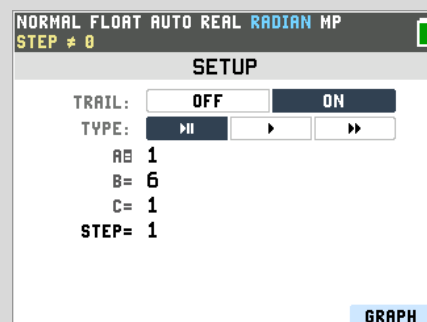
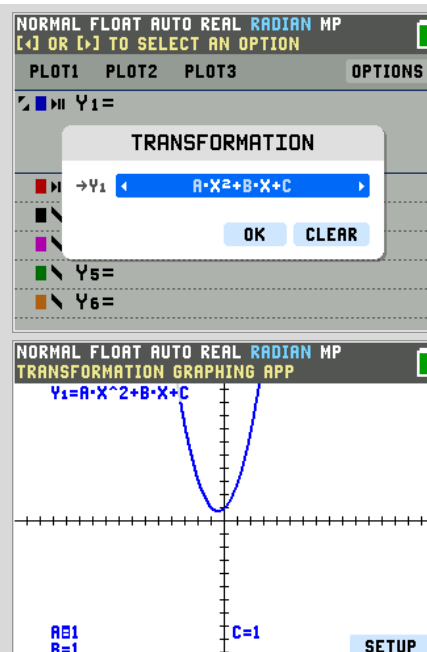
- Press **SETUP** [f5] to view and edit the options.
- Set **TRAIL** to **ON**.
- Press **graph** to return to the graph screen.

With **TRAIL** set to **ON**, varying a transformation parameter will leave a trail of graphs for any previously defined value.

For example, the screen shown right displays a trailing set of graphs for the cases where $a = 1$, $c = 1$ and the value of b has been varied from $b = 1$ to $b = 6$ with integer increments.

There are other options for animating the effect of changing the value of a transformation parameter. These options will be explored in later examples.

Note: To quit from the TRANSFORMATION GRAPHING application, press **y=** > **OPTIONS** [f5] > **Quit App**.



Using the discriminant to find the number of solutions to a quadratic equation

Question

Investigate how the value of the discriminant conveys information about the number of solutions to the equation $ax^2 + bx + c = 0$.

Solution

- Press $\boxed{\text{on}}\boxed{\text{graph}}$ and select the **TRANSFORMATION GRAPHING** application.
- Press $\boxed{y=}$ and then $\boxed{\text{clear}}$ to clear any equations in Y1.
- Press $\boxed{\text{enter}}$ to select the $\mathbf{AX^2 + BX + C}$ equation type.
- Set $\mathbf{A = 1}$, $\mathbf{B = 0}$, and $\mathbf{C = 1}$.

To calculate the value of the discriminant $\Delta = b^2 - 4ac$, for a quadratic equation with the above values:

- Press $\boxed{2\text{nd}}\boxed{\text{quit}}$ to display the **CALCULATOR** screen.
- Use the $\boxed{\alpha}$ keys as required to enter the expression $\mathbf{B^2 - 4AC}$.

For $a = 1$, $b = 0$, and $c = 1$, $\Delta = -4$.

Repeat the above to **show that**:

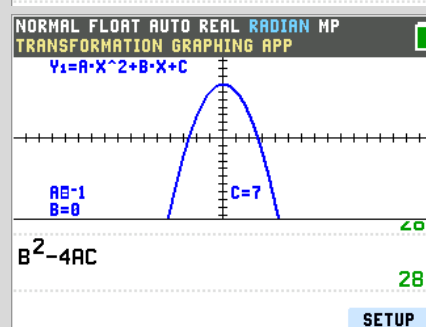
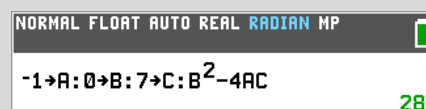
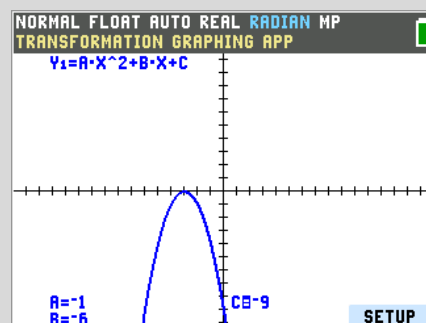
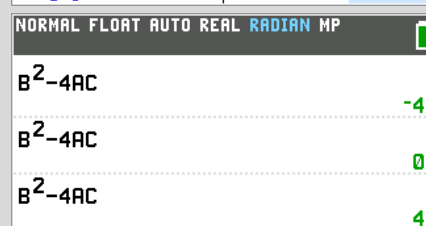
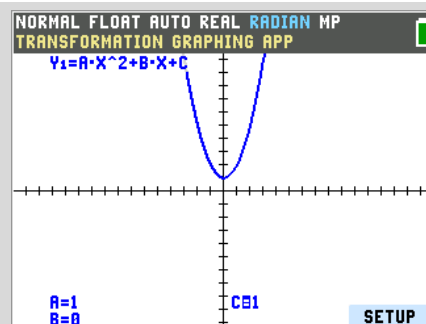
For $a = 1$, $b = 0$, and $c = 0$, $\Delta = 0$.

For $a = 1$, $b = 0$, and $c = -1$, $\Delta = 4$.

To investigate further:

- Note whether the discriminant value returned is positive, negative or zero.
- Look at the graph screen page and observe the number of times the parabola intersects the x-axis.
- Use sliders to adjust a , b , c such that the parabola intersects the x-axis i) no times ii) one time (just touching) and iii) twice. After setting up each of these 3 cases, return to the CALCULATOR screen and press $\boxed{\text{enter}}$ to recalculate the discriminant. Take note of how the sign of the discriminant relates to the number of x-intercepts on the graph and therefore the number of solutions that exist for the equation $y = 0$.

Note: Instead of switching between the GRAPH and CALCULATOR screens, you may prefer to just change the values of A, B and C on the CALCULATOR screen. An example of possible syntax for doing this and recalculating the value of the discriminant are shown right. It is also possible to view the GRAPH and CALCULATOR screens concurrently by pressing $\boxed{\text{mode}}$ and selecting HORIZONTAL (see example screen split 'horizontally' at right). While in this split screen mode, press $\boxed{\text{graph}}$ to jump to the GRAPH screen, and press $\boxed{\text{on}}\boxed{\text{enter}}$ to jump to the CALCULATOR screen.



Investigating the effect of a , b , and c upon the graph of $y = a(x-b)^2 + c$

Question

Investigate the effect of a , b and c on the graph of the parabola with equation $y = a(x-b)^2 + c$.

Solution

- Press **on** and select the **TRANSFORMATION GRAPHING** application.
- Press **y=** and then **clear** to clear any equations in Y1.
- Press **enter** to select the equation type.
- Press **▸** as required to select the $A(X-B)^2 + C$ equation type as shown and press **enter**.
- Press **graph** to view the graph screen.

The sliders for transformation parameters A , B and C will be visible at the bottom of the screen.

- Press **▼** or **▲** to move between sliders for A , B and C .
- Press **◀** or **▶** to change the value of the selected transformation parameter.

As an example, set $A = 1$, $B = 5$ and $C = 3$ as shown.

Note: To display the grid, press **2nd**[format] and set **GRID** option to **LINE**.

To confirm the coordinates of the turning point (B, C) in this case are $(5, 3)$:

- Press **trace** and then type **alpha**[B]**enter**.

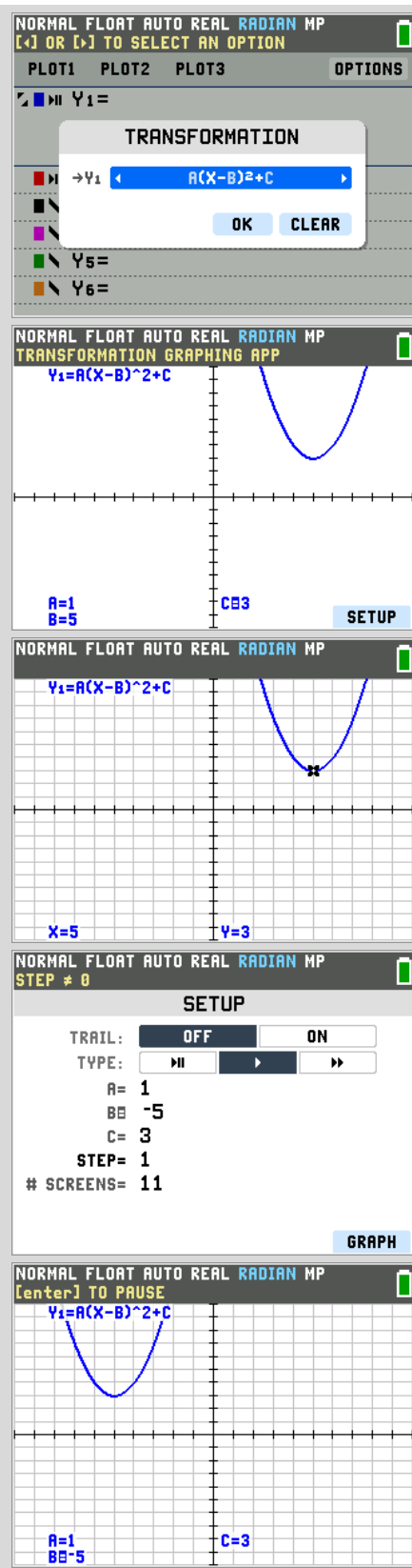
Set $A = 1$, $B = 5$ and $C = 3$. Adjust the value of C and observe the effect upon the graph, turning point and axis of symmetry. Use **trace** the grid to read the coordinates of the turning point. How does this relate to the values of h and k ?

It may aid the investigation to animate the effect of each parameter. For example, to animate a set of the graphs for $a = 1$, $b = -5$ to 5 and $c = 3$:

- Press **graph** to view and edit the **SETUP** options.
- Set the **TYPE** to the middle option ("play").
- Set $A = 1$, $B = -5$ and $C = 3$.
- Hover over the "=" to the right of the parameter b and press **enter**. (this sets b as the parameter to be varied).
- Set **STEP** = 1 and set **#SCREENS** = 11.
- Press **graph** to return to the graph screen.

The animation begins automatically, starting with the above values of a , b and c , then re-graphing for each of the 11 values of b from $b = -5$ to $b = 5$.

Note: Press **enter** to pause/continue the animation. To stop the animation, press **enter** and then **SETUP**. Change **TYPE** to the leftmost option (**▶**) and then press **graph** to return to the graph screen. To quit from the **TRANSFORMATION GRAPHING** application, press **y=** > **OPTIONS** > **Quit App**.



Using a graphical method to solve quadratic equations.

Question

Find solutions to the quadratic equation $0 = x^2 + x - 6$ using a graphical method.

Solution

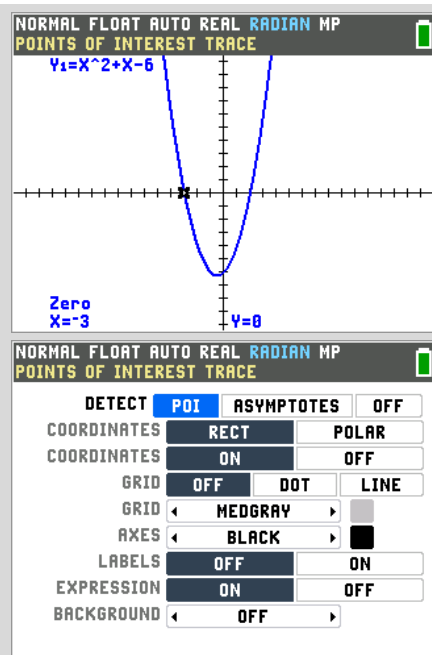
To find the solutions to the equation $0 = x^2 + x - 6$ graphically:

- Press $\boxed{Y=}$ and enter the equation **Y1 = X² + X - 6**.
- Press $\boxed{\text{trace}}$ and then $\boxed{\downarrow}$ as required to locate the trace cursor near the left x -intercept. The calculator will detect and display the cursor at the 'zero' at $x = -3$.
- Use the same process to locate the coordinates of the second x -intercept.

*Note: This 'auto-detect' trace feature is a default option. Press $\boxed{2\text{nd}}\boxed{\text{format}}$ to check that the **DETECT** option is set to **POI** (Points of Interest). This will detect the location and display more precise coordinates for axes intercepts and turning points, and points of intersection.*

Answer: The solutions to the quadratic equation $0 = x^2 + x - 6$ are $x = -3$ and $x = 2$.

Note: The 'zero' can also be found via $\boxed{2\text{nd}}\boxed{\text{calc}}$, which asks the user to indicate the approximate bounds of the x -value.



Using the polynomial roots method to solve quadratic equations.

Question

Find solutions to the quadratic equation $0 = x^2 + x - 6$ using a numeric solving method.

Solution

The roots of the polynomial $x^2 + x - 6$ can be found by finding the solutions to the equation $0 = x^2 + x - 6$ as follows:

- Press $\boxed{\text{on}}\boxed{\text{arrow}}$ and select the **POLYNOMIAL ROOT FINDER** application.
- In the **SETUP** screen, set **ORDER = 2** (as we are solving a polynomial of order 2) and then press **NEXT** $\boxed{\text{f5}}$.

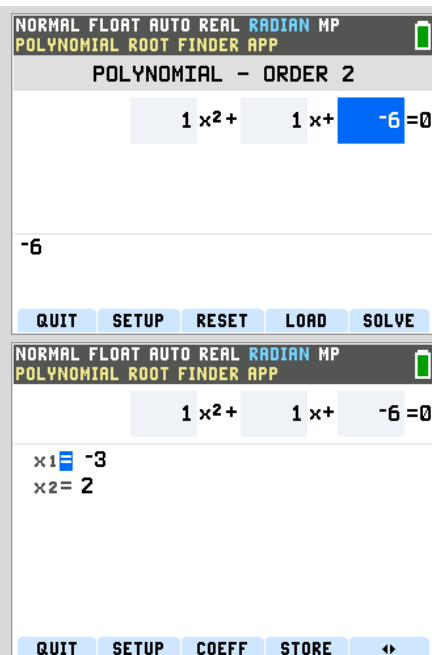
You will be prompted for the coefficients and independent term of the polynomial expression.

- Enter the values 1, 1 and -6.
- Press **SOLVE** $\boxed{\text{f5}}$ to reveal the solutions.

Answer: $x = -3$ and $x = 2$.

*Note: (1) This method is usually better for solving polynomial roots than using the **Numerical Solver** tool as the numeric solver will only reveal one of the roots at a time depending on the starting value that is currently stored for x .*

(2) When entering negative coefficients/constants, use $\boxed{(-)}$, alternatively use the subtraction operator symbol $\boxed{-}$ rather than the addition operator symbol $\boxed{+}$.



3.2. Topic 2: Binomial expansion and cubic functions

3.2.1. Binomial expansion

Using the combinations formula and combinations command

If a working team of four students is to be selected from a class of 20 students, how many different teams could be selected?

Question

- (a) Calculate the number of teams by using the combinations formula: ${}^nC_r = \frac{n!}{r!(n-r)!}$
- (b) Calculate the number of teams by using the combinations command on calculator.

Solution

(a) Use the combinations formula as follows:

- Press the fraction template key.
- Enter 20! as the numerator and 4!(20-4)! as the denominator.

*Note: The factorial command is found via $\boxed{\text{math}}$ > **PROB** > ! or via $\boxed{\alpha}$ [f2].*

(b) Use the combinations command:

- Press $\boxed{\text{math}}$ > **PROB** > nCr.
- Complete the command 20^C4.

Answer: 4845 different teams could be selected.

NORMAL FLOAT AUTO REAL RADIAN MP	
$\frac{20!}{4!(20-4)!}$	4845
20^C4	4845
$20 \rightarrow N: 4 \rightarrow R: \frac{N!}{R!(N-R)!}$	4845

Note: It is also possible to enter the combinations formula more generally, and store values of n and r (as shown).

Combinations formula in Pascal's triangle

Question

Use the combinations command to list the values of row 5 in Pascal's Triangle and hence list the coefficients in the expansion of $(x + y)^5$. Show this by two different methods.

Solution

To find the set of coefficients in row 5 of Pascal's triangle:

- Press $\boxed{\text{math}}$ > **PROB** > nCr (or via $\boxed{\alpha}$ [f2]).
- Enter nCr(5,{0,1,2,3,4,5}) as shown.

*Note: The nCr command is accessed via $\boxed{\text{math}}$ > **PROB** > nCr. To access "{", press $\boxed{2nd}$ [{] and $\boxed{2nd}$ [}].*

The resulting 6 values are from of Row 5 in Pascal's triangle.

The expansion of $(x + y)^5$ is:

$$x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$$

Note: Another way to generate a line of Pascal's triangle is to incorporate the sequence command (with syntax seq(expression,variable,begin,end). The commands shown right generates the list of coefficients for specific values of n. For each new line, press $\boxed{\uparrow}$ and then $\boxed{\text{enter}}$ to copy and paste the previous command, then edit it as required.

Answer: The 6 coefficients in row 5 are 1, 5, 10, 10, 5, and 1.

NORMAL FLOAT AUTO REAL RADIAN MP	
$5^C(\{0,1,2,3,4,5\})$	{1,5,10,10,5,1}

NORMAL FLOAT AUTO REAL RADIAN MP	
$1 \rightarrow N: N^C(\text{seq}(R,R,0,N))$	{1,1}
$2 \rightarrow N: N^C(\text{seq}(R,R,0,N))$	{1,2,1}
$3 \rightarrow N: N^C(\text{seq}(R,R,0,N))$	{1,3,3,1}
$4 \rightarrow N: N^C(\text{seq}(R,R,0,N))$	{1,4,6,4,1}
$5 \rightarrow N: N^C(\text{seq}(R,R,0,N))$	{1,5,10,10,5,1}

3.2.2. Cubic functions

Exploring graphs of cubic functions

Question

- (a) Investigate the features of the graph of $y = x^3$ including shape, intercepts, and behaviour as $x \rightarrow \infty$ and $x \rightarrow -\infty$.
- (b) Investigate the features of the graph of $y = a(x-b)^3 + c$ including shape and the effect of altering the parameters a , b , and c .

Solution

- (a) To investigate the features of the graph of $y = x^3$:

- Press $\boxed{Y=}$ and enter the equation $Y1 = X^3$.
- Take note of the x -intercept, y -intercept, the point of rotational symmetry and the direction in which the graph is headed in the 1st and 3rd quadrants.

Note: To display the grid, press $\boxed{2nd}\boxed{format}$ and set **GRID** option to **LINE**.

- (b) To investigate features of the graph of $y = a(x-b)^3 + c$:

- Press $\boxed{on}\boxed{\Delta}$ and select the **TRANSFORMATION GRAPHING** application.
- Press $\boxed{Y=}$ and then $\boxed{clear}$ to clear any equations in $Y1$.
- Press $\boxed{enter}$ to select the equation type.
- Press $\boxed{\triangleright}$ as required to select the $A(X-B)^3 + C$ equation type as shown and press $\boxed{enter}$.
- Press $\boxed{graph}$ to view the graph screen.

The sliders for transformation parameters a , b and c will be visible at the bottom of the screen.

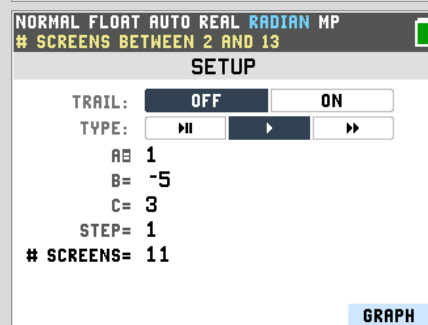
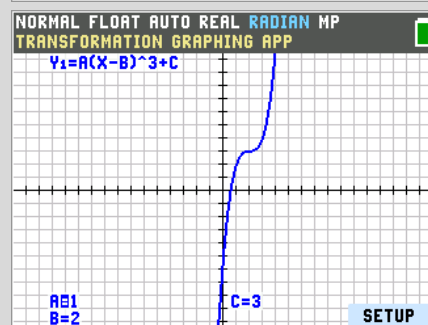
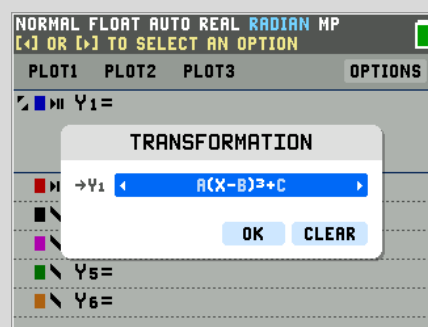
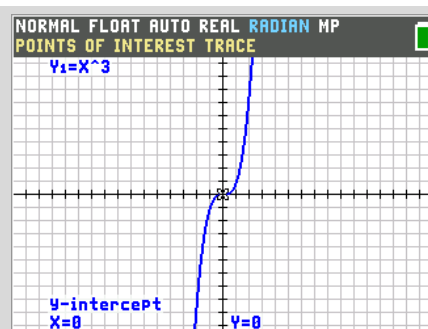
- Press $\boxed{\downarrow}$ or $\boxed{\uparrow}$ to move between sliders for a , b and c .
- Press $\boxed{\leftarrow}$ or $\boxed{\rightarrow}$ to change the value of the selected transformation parameter.

It may aid the investigation to animate the effect of each parameter. For example, to animate between a set of the graphs for $a = 1$, $b = -5$ to 5 and $c = 3$:

- Press $\boxed{graph}\boxed{f5}$ to view and edit the **SETUP** options.
- Set the **TYPE** to the middle option (“play”).
- Set **A = 1**, **B = -5** and **C = 3**.
- Hover over the “=” to the right of the parameter b and press $\boxed{enter}$. (this sets b as the parameter to be varied).
- Set **STEP = 1** and set **#SCREENS = 11**.
- Press $\boxed{graph}$ to return to the graph screen.

The animation begins automatically, starting with the above values of a , b , and c , then re-graphing for each of the 11 values of b from $b = -5$ to $b = 5$.

Note: Press $\boxed{enter}$ to pause/continue the animation. To stop the animation, press $\boxed{enter}$ and then **SETUP**. Change **TYPE** to the leftmost option ($\blacktriangleright$) and then press $\boxed{graph}$. To quit from the **TRANSFORMATION GRAPHING** application, press $\boxed{Y=}$ > **OPTIONS** > **Quit App**.



Solving cubic equations with the polynomial roots command

Question

Use a numerical solving method to solve the equation $x^3 - 6x^2 + 11x - 6 = 0$.

Solution

The roots of the polynomial $x^3 - 6x^2 + 11x - 6$ can be found by finding the solutions to the above equation as follows:

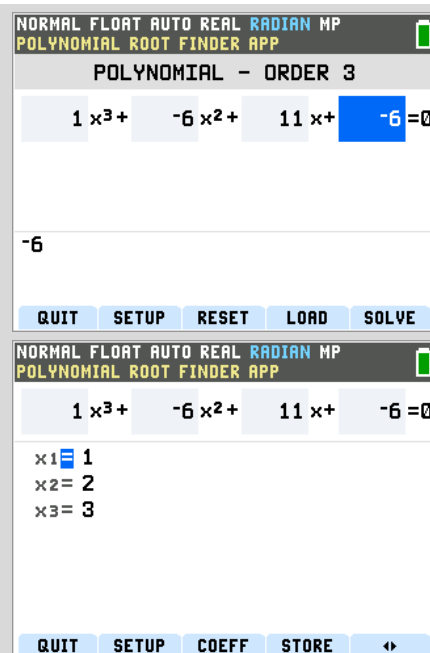
- Press $\boxed{\text{on}} \boxed{\Delta}$ and select the **POLYNOMIAL ROOT FINDER** application.
- In **SETUP** screen, set **ORDER** = 3 and press **NEXT** $\boxed{\text{f5}}$. You will be prompted for the coefficients and independent term of the polynomial expression.

Note: When entering negative coefficients/constants, use $\boxed{(-)}$, alternatively use the subtraction operator symbol $\boxed{-}$ rather than the addition operator symbol $\boxed{+}$.

- Enter the values 1, -6, 11 and -6.
- Press **SOLVE** $\boxed{\text{f5}}$ to reveal the solutions.

Answer: $x = 1$, $x = 2$ and $x = 3$.

Note: This method is better than using the **Numerical Solver** tool as that tool will only reveal one of the roots at a time depending on the starting value that is currently stored for x .



Solving cubic equations graphically

Question

Use a graphical method to find real solutions to the cubic equation $x(x-2)(x+2) = x$.

Solution

To find real solutions for the above equation, graph the LHS and RHS, and find the x -coordinate of any points of intersection, as follows:

- Press $\boxed{\text{Y=}}$ and enter the equation **Y1 = X(X-2)(X+2)** and **Y2 = X**.
- Press $\boxed{\text{trace}}$ and then press $\boxed{\text{right}}$ as required to locate the trace cursor over the rightmost intersection point.

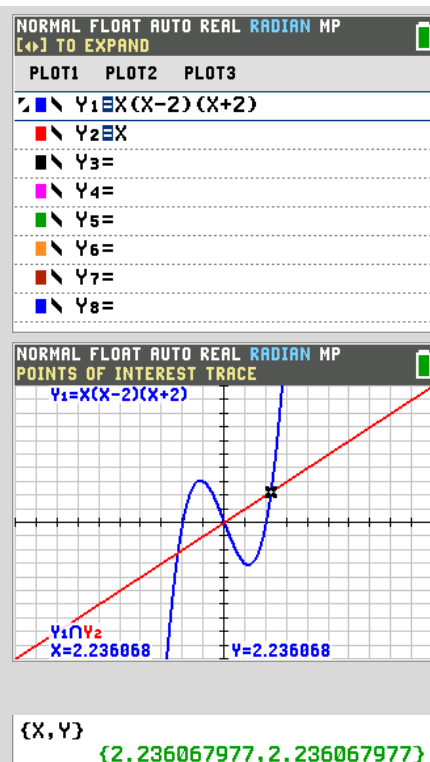
The calculator will detect and display '**Y1 $\cap$ Y2**' at the point where $x \approx 2.24$. Alternatively, use $\boxed{2\text{nd}} \boxed{\text{calc}} \boxed{>} \boxed{\text{intersect}}$

- Repeat the above process to locate the coordinates of the second and third intersection points.

Note: This 'auto-detect' trace feature is a default option. Press $\boxed{2\text{nd}} \boxed{\text{format}}$ to check that the **DETECT** option is set to **POI** (Points of Interest).

Answer: The real solutions to the cubic equation $x(x-2)(x+2) = x$ are $x = 0$ and $x \approx \pm 2.24$.

Note: The **Trace** feature, stores the current trace coordinates as X and Y , which can be used for calculations (see right).



Using cubic equations to model and solve problems

Question

Determine the maximum possible volume of an open box made from a rectangular piece of cardboard $21\text{ cm} \times 30\text{ cm}$.

Solution

Consider that squares of dimension x by x need to be removed from each corner of the rectangular piece of cardboard to make the net for the box. This means that the box will have dimensions: $\text{height} = x$; $\text{width} = 21 - 2x$; $\text{length} = 30 - 2x$.

Since all dimensions must be positive:

$$\text{height} : x > 0.$$

$$\text{width} : 21 - 2x > 0 \Leftrightarrow -2x > -21 \Leftrightarrow x < 10.5.$$

$$\text{length} : 30 - 2x > 0 \Leftrightarrow -2x > -30 \Leftrightarrow x < 15.$$

We can define a rule for the volume of the box with a restricted domain $0 < x < 10.5$.

$$\text{volume} = \text{height} \times \text{width} \times \text{length}$$

$$= x(21 - 2x)(30 - 2x), \quad 0 < x < 10.5.$$

- Press $\boxed{y=}$ and then $\boxed{\text{math}} > \text{MATH} > \text{piecewise}(\cdot$.
- Set **pieces** = 1 and press $\boxed{\text{enter}}$.

This will paste the piecewise template for one piece in Y1 (as shown right).

- For the equation, enter **Y1 = X(21 - 2X)(30 - 2X)**.
- For the domain, enter **0 < X < 10.5**.

Note: Press $\boxed{2\text{nd}}\boxed{\text{math}}$ to access the inequality symbols.

- Press $\boxed{\text{window}}$ and enter the values.
Xmin = 0 Xmax = 10.5 Xscl = 1
(ignore Ymin, Ymax and Yscl values)
- Press $\boxed{\text{zoom}} > \text{Zoom Fit}$.

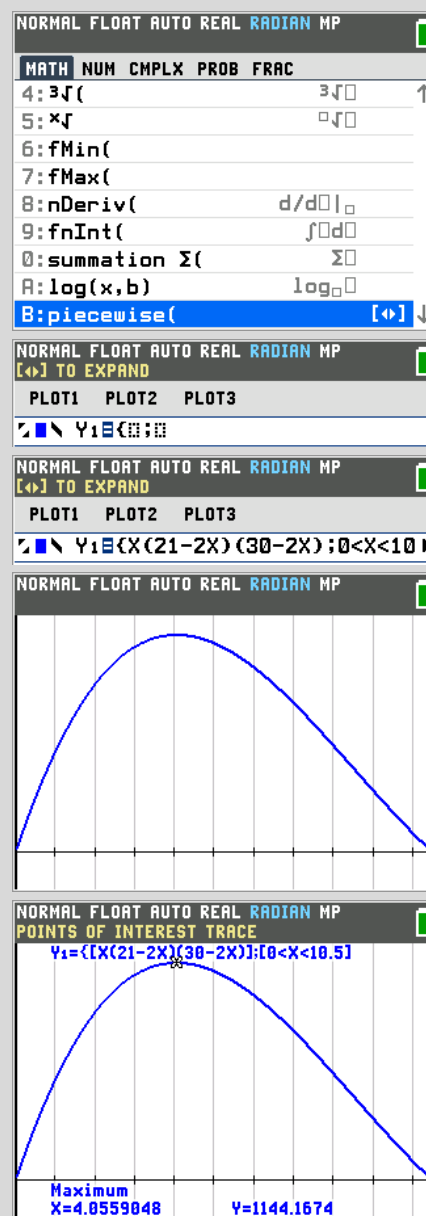
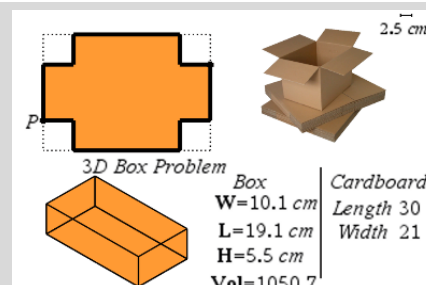
Note: The **Zoom Fit** command will adjust the values of Ymin and Ymax so that the graph is visible for the x-values from Xmin to Xmax.

To locate the local maximum near $x = 4$:

- Press $\boxed{\text{trace}} > \boxed{4} > \boxed{\text{enter}}$, and then press $\boxed{\blacktriangleright}$ as required to locate the cursor at the local maximum.

The x-coordinate of this represents the dimension x of the square to be cut from each corner and the y-coordinate represents the volume at that x value.

Answer: The maximum volume (approximately 1144 cm^3) is obtained by cutting a square from each corner of the cardboard sheet with a side length of approximately 4.1 cm.



3.3. Topic 3: Functions and relations

3.3.1. Introduction to functions and relations

Playing Guess the Rule

Question

Play “Guess the rule” game with a partner.

Solution

Define a rule for your partner to figure out. As an example:

- Press $\boxed{y=}$ and enter the equation **Y1 = 2X + 3**.

This defines a rule that will double any input value and add 3.

- Press **on** and **enter** to display the home screen.
- Press **clear** to clear the home screen.
- Press **vars** > **Y-VARS** > **Function** > **Y1** (or use **2nd** > **Y-VARS** > **Function** > **Y1**).
- Complete the command **Y1(5)**.

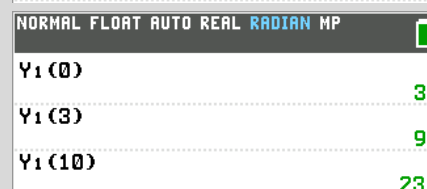
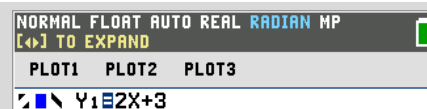
This will clear the screen and produce the result $Y1(5) = 13$.

- Now pass the calculator to your partner and ask them to figure out the hidden rule by testing various input values (independent variable) and observing the resulting output value (dependent variable).
- Your partner can test the rule for various input values by entering such things as **Y1(3)** etc.
- When your partner has tested enough input variables (two tests should be sufficient if you tell them that the rule is linear), they state what they believe the rule to be and then test their assertion by predicting the output for their next input value.

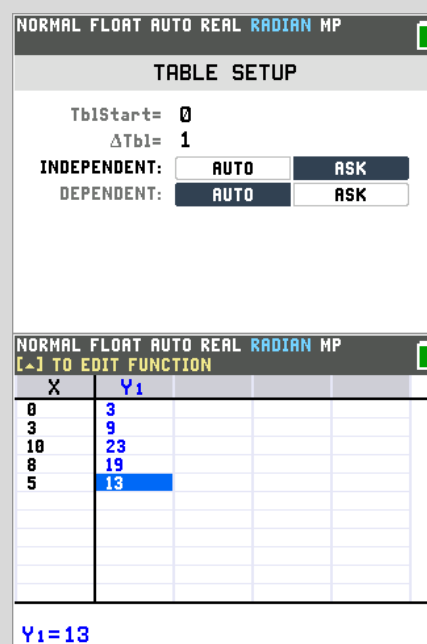
As an alternative for producing function values, consider using the TABLE feature. To do this, assuming that **Y1=2X+3**:

- Press $\boxed{2\text{nd}}\boxed{[tblset]}$.
- Set INDEPENDENT to **ASK**.
- Press $\boxed{2\text{nd}}\boxed{[table]}$.
- Enter the guesses for **X** (see example right).

Using the TABLE feature like this simplifies the substitution of x-values into the function rule.



Note: To make the challenge achievable, it is best to first agree upon some guidelines. For example, agree that the function will be linear with a restricted set of integer coefficient and constant terms.



Exploring domain and range

Question

Graph $y = \frac{1}{x}$ for the restricted domain $-2 \leq x < 2$ and state its range.

Solution

- Press $\boxed{y=}$ and then $\boxed{\text{math}} > \text{MATH} > \text{piecewise(}$
- Set **pieces** = 1 and press $\boxed{\text{enter}}$.

This will paste the piecewise template for one piece in Y1.

- For the equation, enter **Y1 = 1 / X**.
- For the domain, enter $-2 \leq X < 2$.

Note: Press $\boxed{2\text{nd}}\boxed{\text{math}} < \text{TEST}$ to access the inequality symbols, or $\boxed{2\text{nd}}\boxed{\text{math}} < \text{CONDITIONS}$ to enter the domain restrictions using a template.

- Use the **Trace** tool to explore endpoints of the graph within the specified domain.

Answer: The graph shows that the domain is

$$x \in [-2, 0) \cup (0, 2) \text{ and the range is } y \in (-\infty, -0.5] \cup (0.5, \infty).$$

The TABLE feature can be used to explore limiting behaviour near $x = 0$. For example, to use a table of values for the above equation, assuming **Y1 = 1 / X**:

- Press $\boxed{2\text{nd}}\boxed{\text{tblset}}$.
- Set INDEPENDENT to **AUTO**.
- Set **TblStart** = 0.
- Set **ΔTbl** = 1.
- Press $\boxed{2\text{nd}}\boxed{\text{table}}$ to view table values for these settings.
- In the column for **X**, scroll up to **X = -5** (see example right).

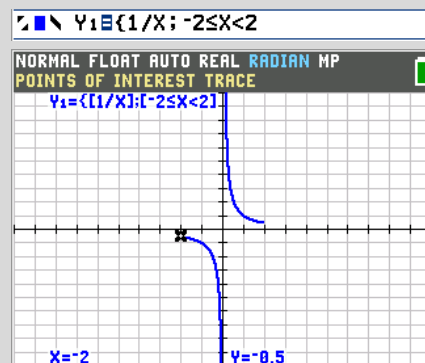
The increment for x values (i.e. **ΔTbl**) can be changed as follows (e.g. to set **$\Delta\text{Tbl} = 0.001$**):

- Press $\boxed{2\text{nd}}\boxed{\text{table}}$ to view table values.
- Press $\boxed{+}$ to be prompted for a new value for **ΔTbl** .
- Enter **$\Delta\text{Tbl} = 0.001$** .

The result is shown right.

Alternatively, press $\boxed{2\text{nd}}\boxed{\text{tblset}}$ to edit the values of **TblStart** and **ΔTbl** , and set the INDEPENDENT variable x to permit the user to enter the x values manually. In the final screen shown right,

- Press $\boxed{2\text{nd}}\boxed{\text{tblset}}$.
- Set INDEPENDENT to **ASK**.
- Press $\boxed{2\text{nd}}\boxed{\text{table}}$.
- Enter values for **X** to view the associated function value (see example right).



NORMAL FLOAT AUTO REAL RADIAN MP

TABLE SETUP

TblStart= 0
 ΔTbl = 1

INDEPENDENT: **AUTO** **ASK**
 DEPENDENT: **AUTO** **ASK**

NORMAL FLOAT AUTO REAL RADIAN MP

[+] FOR ΔTbl

X	Y1
-5	-0.2
-4	-0.25
-3	-0.333
-2	-0.5
-1	-1
0	ERROR
1	1
2	0.5
3	0.3333
4	0.25
5	0.2

X=1

NORMAL FLOAT AUTO REAL RADIAN MP

[+] FOR ΔTbl

X	Y1
-0.005	-200
-0.004	-250
-0.003	-333.3
-0.002	-500
-0.001	-1000
0	ERROR
0.001	1000
0.002	500
0.003	333.33
0.004	250
0.005	200

X=0.001

NORMAL FLOAT AUTO REAL RADIAN MP

X	Y1
0.01	100
0.009	111.1111
0.008	125
0.005	200
0.001	1000
-0.001	-1000
-0.005	-200

X=

Defining a piecewise function

Question

Produce graphs for each of the following piecewise functions:

$$(a) \ y = \begin{cases} \frac{1}{x}, & x < -1 \\ x^2 - 2, & -1 \leq x \leq 2 \\ x, & x > 2 \end{cases}$$

$$(b) \ y = \begin{cases} 2x^3 - 3x^2 - 2x + 4, & -1 \leq x \leq 2 \\ x^2 - 5, & x < -1 \text{ or } x > 2 \end{cases}$$

Solution

(a) To construct this piecewise function which has three components:

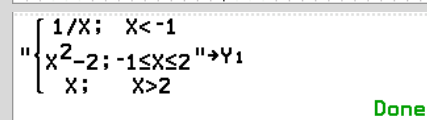
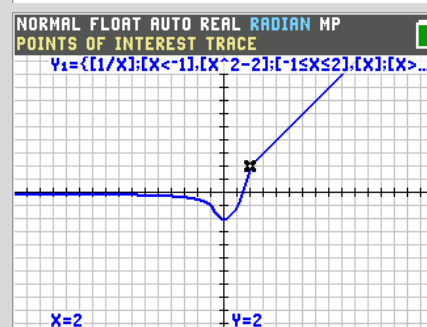
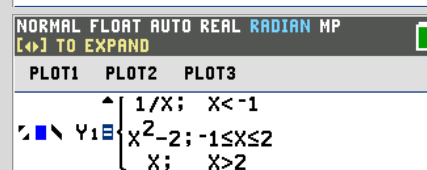
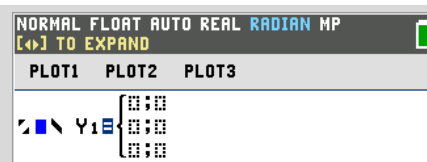
- Press $\boxed{y=}$ and then $\boxed{\text{math}} > \mathbf{MATH} > \mathbf{piecewise(}$
- Set **pieces** = 3 and press $\boxed{\text{enter}}$.

This will paste the piecewise template three pieces in Y1 (as shown right).

- Enter the three components of the piecewise function with their domain restrictions into the appropriate spaces on the template (use $\boxed{2nd}\boxed{\text{math}}$ to access the inequality symbols).
- Press $\boxed{\text{graph}}$ to display the graph of the piecewise function.
- Press $\boxed{\text{enter}}$ to observe graph of the piecewise function.

Notes: (1) In this example the three sections form a continuous function. However, this does not necessarily need to be the case for a piecewise function. A discontinuous function is shown in the next example.

(2) The piecewise function can be added in the **CALCULATOR** application. This improves the readability of the equations and domain restrictions as you are entering them. Remember that the RHS of the function rule must be enclosed by double quotes as shown right.

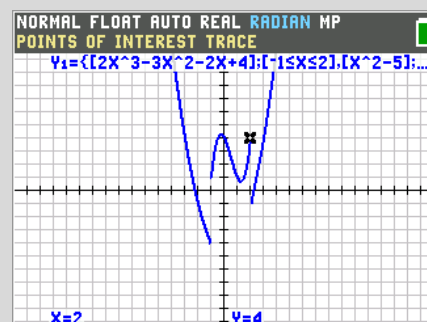
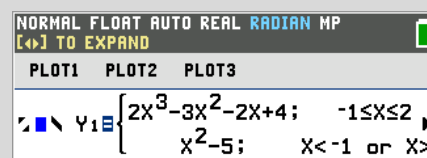


(b) To construct this piecewise function which has two components, follow the same process as for part (a) but specify 2 components for the piecewise function (see selected option from the template right).

Notes:

(1) Press $\boxed{2nd}\boxed{\text{math}} > \mathbf{TEST}$ to access the inequality symbols. The 'or' operator can be found by pressing $\boxed{2nd}\boxed{\text{math}} > \mathbf{LOGIC} > \mathbf{or}$. To enter a domain restriction using a template, press $\boxed{2nd}\boxed{\text{math}} < \mathbf{CONDITIONS}$.

(2) An alternative to entering the piecewise function via the template would be to enter the separate components as separate functions (specifying their respective domain restrictions) and to graph them on the same set of axes.



3.3.2. Graphs of relations

Graphing circles

Question

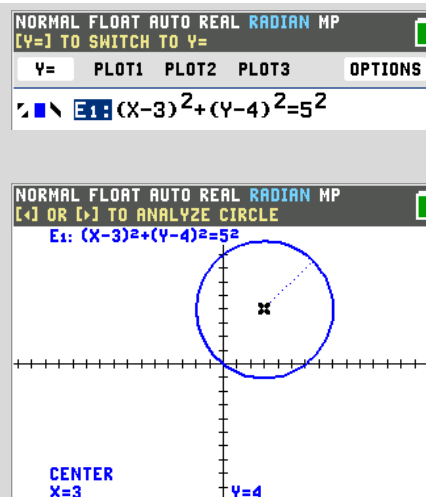
Use the relations template to plot the graphs of $(x-3)^2 + (y-4)^2 = 25$.

Solution

- Press $\boxed{Y=}$ and use $\boxed{\text{clear}}$ to remove any previously defined function rules.
- Press $\boxed{\text{on}} \boxed{\wedge}$ and select the **LINES AND CONICS** application.

This application permits the plotting of up to six relations, denoted equations E1 to E6.

- In the **E1** line, press $\boxed{\text{enter}}$ to **Add Equation ...**
- Press $\boxed{\blacktriangleright}$ to locate the circle template, and press $\boxed{\text{enter}}$.
- Use the template to enter $(X-3)^2 + (Y-4)^2 = 5^2$.
- Press $\boxed{\text{zoom}} \blacktriangleright$ **Zoom [-10,10]** to view the circle in the standard viewing window.
- Press $\boxed{\text{zoom}} \blacktriangleright$ **Zoom Square** to view the circle in a modified viewing window with equal scales on the axes..
- Press $\boxed{2\text{nd}} \boxed{\text{calc}} \blacktriangleright$ **Conic Analysis** to view the radius and centre of the circle.



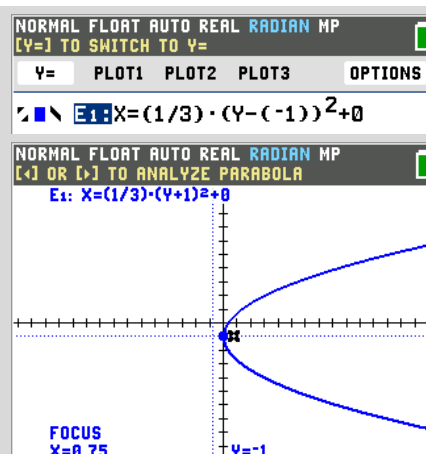
Graphing parabolas

Question

Use the relations template to produce graphs of the relation $x = \frac{(y+1)^2}{3}$, taking note of the shape and axis of symmetry.

Solution

- Press $\boxed{\text{on}} \boxed{\wedge}$ and then select the **LINES AND CONICS** application. This application permits the plotting of up to six relations, denoted equations E1 to E6.
- In the **E1** line, press $\boxed{\text{enter}}$ to **Add Equation ...**
- Press $\boxed{\blacktriangleright}$ as required to locate the parabola template with equation for $X = A(Y - K)^2 + H$, and press $\boxed{\text{enter}}$.
- Use the template to enter $A = 1/3$, $K = -1$, $H = 0$.
- Press $\boxed{\text{zoom}} \blacktriangleright$ **Zoom [-10,10]** to view the parabola in the standard viewing window.
- Press $\boxed{2\text{nd}} \boxed{\text{calc}} \blacktriangleright$ **Conic Analysis** and then use the left and right cursor keys to view features of this parabola, such as the directrix, focus and to display the axis of symmetry.



Answer: Note that parameter A affects the aperture of the parabola and that the turning point of the parabola is located at (H, K) . The line $Y = K$ defines the parabola's axis of symmetry. The equation of the axis of symmetry for this example is $Y = -1$.

3.3.3. Reciprocal functions

Graphing reciprocal functions with the TRANSFORM application

Question

Construct a command to graph hyperbolas of the form $y = \frac{A}{x-B} + C$, permitting the user to enter and store values for the parameters A , B , and C .

Solution

To enter hyperbolas of the above form, using the sample values $A = 1$, $B = 3$, and $C = 5$:

- Press $\boxed{\text{on}} \boxed{\text{graph}}$ and select the TRANSFORMATION GRAPHING application.
- Press $\boxed{Y=}$ and then $\boxed{\text{clear}}$ to clear any equations in $Y1$.
- Enter the equation $Y1 = \frac{A}{X-B} + C$.
- Enter the equation $Y2 = \frac{1}{X}$ (as a reference graph).

Note: As in previous questions, the TRANSFORMATION GRAPHING application will automatically interpret the capital letters A , B and C (and D if present) as parameters that can be modified, or treated as a 'slider'.

The sliders for transformation parameters A , B and C will be visible at the bottom of the screen.

- Press $\boxed{\downarrow}$ or $\boxed{\uparrow}$ to move between sliders for A , B and C .
- Press $\boxed{\leftarrow}$ or $\boxed{\rightarrow}$ to change the value of the selected transformation parameter.
- Modify the parameter values so that $A = 1$, $B = 3$, $C = 5$.

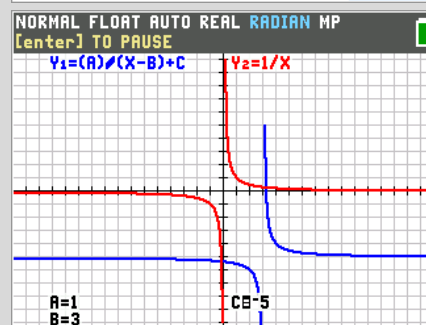
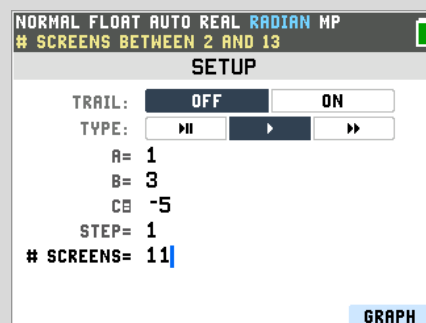
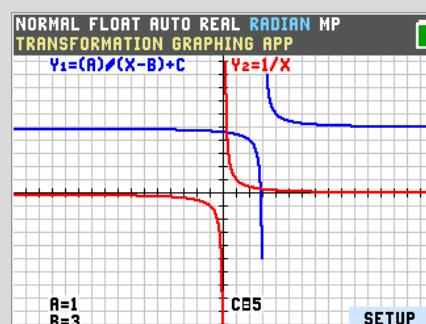
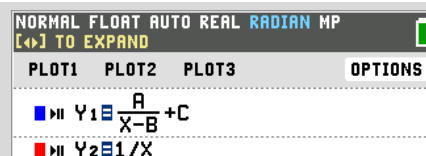
Note the shape and symmetry of the hyperbola and the asymptotic behaviour towards the lines $x = 3$ and $y = 5$.

It may aid the investigation to animate the effect of each parameter. For example, to animate between a set of the graphs for $A = 1$, $B = 3$ and $C = -5$ to 5 :

- Press **SETUP** [f5] to view and edit the **SETUP** options.
- Set the **TYPE** to the middle option ("play").
- Set **A = 1**, **B = 3** and **C = -5**.
- Hover over the "=" to the right of the parameter C and press $\boxed{\text{enter}}$. (this sets b as the parameter to be varied).
- Set **STEP = 1** and set **#SCREENS = 11**.
- Press **GRAPH** [f5] to return to the graph screen and view the animation.

The animation begins automatically, starting with the above values of A , B , and C , then re-graphing for each of the 11 values of C from $C = -5$ to $C = 5$.

Note: Press $\boxed{\text{enter}}$ to pause/continue the animation. To stop the animation, press $\boxed{\text{enter}}$ and then **SETUP** [f5]. Change **TYPE** to the leftmost option ($\triangleright \text{II}$) and then press $\boxed{\text{graph}}$. To quit from the TRANSFORMATION GRAPHING application, press $\boxed{Y=}$ > **OPTIONS** > **Quit App**.



Graphing reciprocal functions using a split screen

Question

Construct a command to graph hyperbolas of the form $Y = \frac{a}{x-h} + k$, permitting the user to enter and store values for the parameters a , h , and k .

Solution

To enter hyperbolas of the above form, using the sample values $a = 1$, $h = 3$, and $k = 5$:

- Press $\boxed{Y=}$ and enter the equation $Y1 = \frac{A}{X-H} + K$.
- Press $\boxed{\text{on}} \boxed{\text{enter}}$ to display the CALCULATOR application.
- Enter the command as follows:

$$1 \rightarrow A : 3 \rightarrow H : 5 \rightarrow K : " \frac{A}{X-H} + K " \rightarrow Y1$$

Note: The arrow symbol ($\rightarrow$) above is entered via the $\boxed{\text{sto} \rightarrow}$ key. The double-quotes are accessed via the $\boxed{\alpha}$ key.

- Press $\boxed{\text{graph}}$ to view the hyperbola for $a = 1$, $h = 3$, $k = 5$.

Note the shape and symmetry of the hyperbola and the asymptotic behaviour towards the lines $x = 3$ and $y = 5$.

To view a different hyperbola:

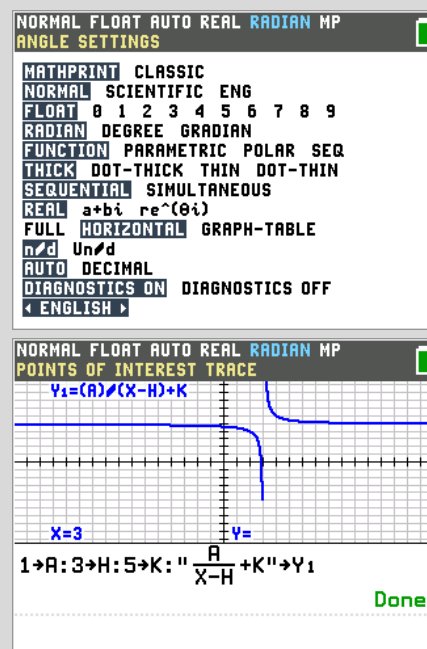
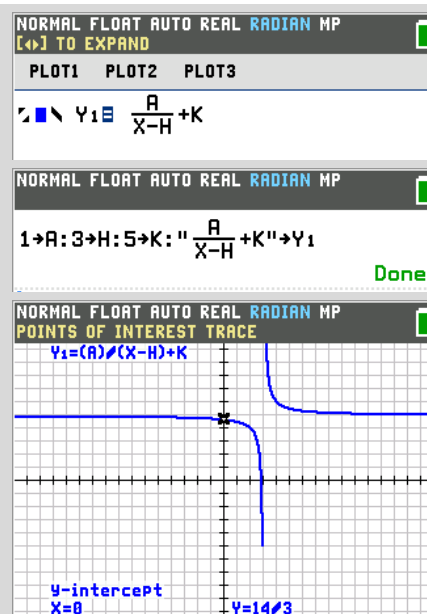
- Press $\boxed{\text{on}} \boxed{\text{enter}}$ to display the CALCULATOR screen,
- Edit the command to change any of the values for parameters a , h , and/or k .
- Press $\boxed{\text{graph}}$ to view the new hyperbola.

It is also possible to split the screen into two parts, so that both the graph and the current values of the a , h , and k can be displayed.

To split the screen, press the following keys:

$\boxed{\text{mode}} > \text{HORIZONTAL} \boxed{\text{enter}}$ (as shown right).

Note: To return to the full screen display, press the following keys: $\boxed{\text{mode}} > \text{FULL}$ and then press $\boxed{\text{enter}}$.



3.4. Topic 4: Trigonometric functions

3.4.1. Circular measure and radian measure

Comparing radian and degrees angle measurement

Question

Create a spreadsheet to display equivalent angle measurements in degrees and radians for angles 0° to 180° at 10-degree intervals.

Solution

- Press $\text{on}\square\text{enter}$ to select the CALCULATOR application.
- Press $\text{2nd}\text{catalog} > \text{ClrAllLists}$ to clear all the lists.
- Press $\text{stat} > \text{Edit ...}$ to view the blank lists.

To enter a sequence of angles 0° to 180° at 10-degree intervals:

- Move to L1 column, and press $\square$ to highlight the column header L1 and press enter .
- Press $\text{2nd}\text{list} > \text{OPS} > \text{seq}$ (and enter the values shown in the screen right for the sequence).
- Press **PASTE** [f2] (but don't press enter yet).
This will paste the command **seq(X,X,0,180,10)** as a column formula for L1. Before pressing enter , add double quotes to the start and finish of the formula as shown right – the double quotes will make the formula editable at a later stage if needed.
- Now press enter to execute the formula and display the sequence in L1.

NORMAL FLOAT AUTO REAL RADI AN MP

Seq

Expr:X
Variable:X
start:0
end:180
step:10

PASTE

L1="seq(X,X,0,180,10)"

NORMAL FLOAT AUTO REAL RADI AN MP

L1	L2	L3	L4	L5
0				
10				
20				
30				
40				
50				
60				
70				
80				
90				
100				

L1="seq(X,X,0,180,10)"

NORMAL FLOAT AUTO REAL RADI AN MP

L1	L2	L3	L4	L5
0	0			
10	$\frac{1}{18}$			
20	$\frac{1}{9}$			
30	$\frac{1}{6}$			
40	$\frac{2}{9}$			

L2="L1/180 F D"

To enter fractions of 180° (or π) that each angle represents:

- Move to the L2 column and press $\square$ to highlight the column header L2 and press enter .
- Complete the equation **L2="L1/180 F D"**.
(for L1, use $\text{2nd}\text{L1}$, and for the fraction conversion command, use $\text{math} > \text{FRAC} > \text{F D}$ or $\text{alpha}\text{f1}$).

The fraction of π each angle represents will now appear in L2.

To calculate the number of radians that each angle represents:

- Move to the L3 column and press $\square$ to highlight the column header for L3 and press enter .
- Complete the equation **L3="L2 $\times$ π "** (for L2, use $\text{2nd}\text{L2}$, and for π , use $\text{2nd}\text{[}\pi\text{]}$).

The equivalent radian measures will now appear in L3.

NORMAL FLOAT AUTO REAL RADI AN MP

L1	L2	L3	L4	L5
0	0	0		
10	$\frac{1}{18}$	0.1745		
20	$\frac{1}{9}$	0.3491		
30	$\frac{1}{6}$	0.5236		
40	$\frac{2}{9}$	0.6981		

L3="L2 $\times$ π "

Converting angle measurements between degrees and radians

Question

(a) Convert $60^{\circ}45'15''$ to radians.

(b) Convert 1.5 radians to degrees.

Solution

(a) Assuming the calculator is in DEGREE mode (use `[mode]` to change the setting if needed), to convert the angle $60^{\circ}45'15''$ to radians:

- Press `[on]``[enter]` to select the CALCULATOR application.
- Enter the angle $60^{\circ}45'15''$, pressing `[2nd]``[angle]` $\gg$ **DEGREE** to select the degrees, minutes, seconds symbols as required.
- Press the `[enter]` key converts the DMS angle into decimal degrees.
- If required, the angle in decimal degrees can be converted to DMS by pressing `[2nd]``[angle]` $\gg$ **DEGREE** $\gg$ $\blacktriangleright^{\circ}$

To convert the DMS or decimal degrees angle to radians:

- Press `[2nd]``[Ans]` to paste the DMS or decimal degrees angle to the editing line.
- Press `[2nd]``[angle]`  $\gg$  **RADIAN**  $\gg$   $\blacktriangleright^r$  and then press `[enter]`.

Note: If the `[mode]` setting is set to RADIAN, then if the angle $60^{\circ}45'15''$ is entered, it will be converted automatically to the equivalent value in radians.

(b) Assuming the calculator is in RADIAN mode (use `[mode]` to change the setting if needed), to convert the angle 1.5^r from radians to decimal degrees:

- Enter the radian value 1.5, then press `[2nd]``[angle]` $\gg$ **RADIAN** $\gg$ $\blacktriangleright^r$.
- Press `[2nd]``[angle]`  $\gg$  **DEGREE**  $\gg$   $\blacktriangleright^{\circ}$  and then press `[enter]`.

To convert the angle 1.5^r from radians to DMS:

- Enter the radian value 1.5, then press `[2nd]``[angle]` $\gg$ **RADIAN** $\gg$ $\blacktriangleright^r$.
- Press `[2nd]``[angle]`  $\gg$  **DEGREE**  $\gg$   $\blacktriangleright^{\circ}$  and then press `[enter]`.

Note: If the calculator is in RADIAN mode, it is not necessary to append the 'r' symbol (see screen right). If the calculator is in DEGREE mode, you can force the 1.5 to be read as a radian value by appending the 'r' symbol.

NORMAL FLOAT AUTO REAL DEGREE MP	
DEGREE	RADIAN GRADIAN R $\blacktriangleright$ P
1: $^{\circ}$	[$\leftrightarrow$]
2: '	
3: "	
4: $\blacktriangleright$ DMS	
5: $\blacktriangleright^{\circ}$	

NORMAL FLOAT AUTO REAL DEGREE MP	
60 $^{\circ}$ 45'15"	60.75416667
Ans $\blacktriangleright$ DMS	60 $^{\circ}$ 45'15"
Ans $\blacktriangleright^{\circ}$	60.75416667 $^{\circ}$

NORMAL FLOAT AUTO REAL DEGREE MP	
DEGREE	RADIAN GRADIAN R $\blacktriangleright$ P
1: r	[$\leftrightarrow$]
2: $\blacktriangleright^r$	

NORMAL FLOAT AUTO REAL DEGREE MP	
60 $^{\circ}$ 45'15"	60.75416667
60.75416667 $\blacktriangleright^r$	1.060360243 ^{r}

NORMAL FLOAT AUTO REAL RADIAN MP	
1.5 $\blacktriangleright^{\circ}$	85.94366927 $^{\circ}$
1.5 $\blacktriangleright$ DMS	85 $^{\circ}$ 56'37.209"
1.5 $\blacktriangleright^{\circ}$	85.94366927 $^{\circ}$
1.5 $\blacktriangleright$ DMS	85 $^{\circ}$ 56'37.209"

3.4.2. Introduction to trigonometric functions

Investigating periodicity of trigonometric functions

Question

- (a) Create a listing of the sine and cosine of all angles that are multiples of $\pi/6$ (from 0 to 4π).
 (b) Create plots of the sine and cosine of all angles that are multiples of $\pi/6$ (from 0 to 4π).

Solution

(a) To generate a sequence of angles 0° to $4\pi^\circ$ at intervals of $\pi/6$:

- Press **on** **enter** to select the CALCULATOR application.
- Press **2nd** **[list]** > **OPS** > **seq** (and enter the values shown in the screen right for the sequence).
- Press **PASTE** **[f2]** (but don't press **enter** yet).
This will paste the command **seq($\pi/6 \times N, N, 0, 24, 1$)**.
- Press **sto** then **2nd** **[list]** > **OPS** > **L**.
- Complete the command
seq($\pi/6 \times N, N, 0, 24, 1$) $\Rightarrow$ LANGLE.
- Now press **enter** to execute the formula and display the sequence of angles (in approximate radian form).

*Note: The small 'L' symbol (found via **2nd** **[list]** > **OPS** > **L**) permits the user to define a list variable with more than one letter. It can also be found via **2nd** **[list]** > **OPS** > **[alpha-lock]** > **L**)*

To calculate the sines and cosines of this sequence of angles from 0° to $4\pi^\circ$ at intervals of $\pi/6$:

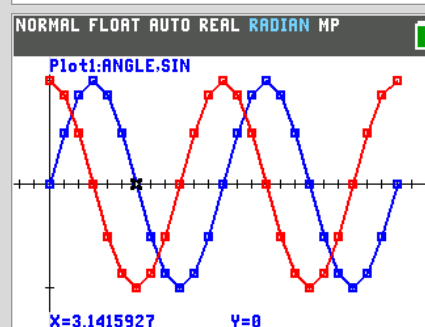
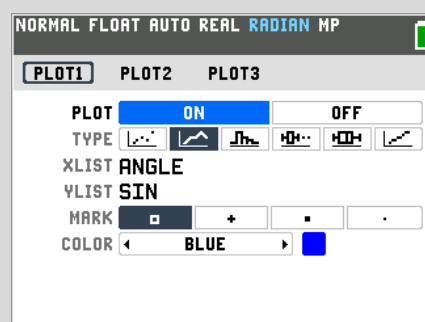
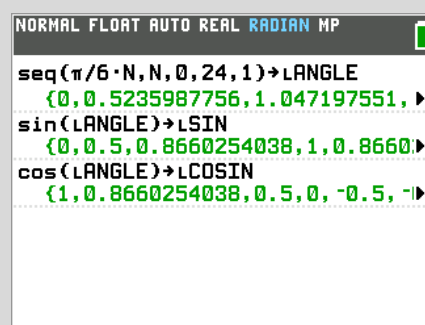
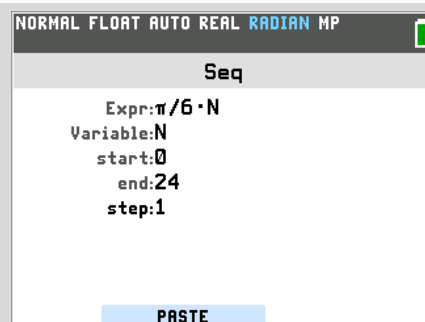
- Enter the command **sin(L**ANGLE) $\rightarrow$ **L**SIN.
- Enter the command **cos(L**ANGLE) $\rightarrow$ **L**COSIN.

*Note: The multiple letter variable names are more easily entered using 'alpha-lock', which is found via **2nd** **[A-lock]**.*

(b) To create plots of the *sine* and *cosine* of angles that are multiples of $\pi/6$ (from 0 to 4π):

- For a plot of the sine values, press **2nd** **[plot]** and select **PLOT1**, and then edit the following settings.
 - Set **PLOT** = **ON**.
 - For **TYPE**, select the **XY Plot**.
 - For **XLIST**, press **2nd** **[list]** > **NAMES** > **ANGLE**.
 - For **YLIST**, press **2nd** **[list]** > **NAMES** > **SIN**.
 - For **MARK**, select the **hollow square mark**.
 - For **COLOR**, select **BLUE**.
- Press **zoom** > **Zoom Stat** to view the plot of the sine graph in a suitable window.
- Repeat the above steps to define **PLOT2** for the plot of the cosine values.

Use the **trace** key to explore the sine and cosine values for angles from 0 to 4π .



Graphing trigonometric functions

Question

Explore the graphs of the following equations: $y = \sin(x)$, $y = \cos(x)$, $y = \tan(x)$ from $x = 0$ to 2π .

Solution

To plot the graph of $y = \sin(x)$ in a suitable window:

- Check calculator is in RADIAN mode (use `mode` to change if required).
- Press `y=` and enter the equation **Y1 = sin(X)**.
- Press `zoom` > **Zoom Trig**.

The graph of $y = \sin(x)$ will appear in a window where $X_{\min} = -2\pi$ and $X_{\max} = 2\pi$, with tick marks on the x -axis at multiples of $\pi/2$. To view the graph for only $x = 0$ to $x = 2\pi$, and to set the tick marks on the x -axis to multiples of $\pi/6$, adjust the viewing window as follows:

- Press `window` and set the following values:
 $X_{\min} = 0$ $X_{\max} = 2\pi$ $X_{\text{scl}} = \pi/6$
 $Y_{\min} = -2$ $Y_{\max} = 2$ $Y_{\text{scl}} = 1$
- Press `graph` to view the graph in the new window.

Note: To view the grid, press `2nd``[format]` and select **LINE**.

To add plots of the graphs of $y = \cos(x)$ and $y = \tan(x)$:

- Press `y=` and enter the other two equations:
Y2 = cos(X) and **Y3 = tan(X)**.
- Press `graph` to view all three graphs from $x = 0$ to 2π .
- Press `trace` to observe and compare the three function values for various values of x .

To view a table of values for all three functions at once:

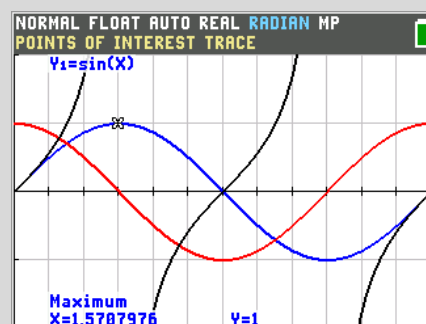
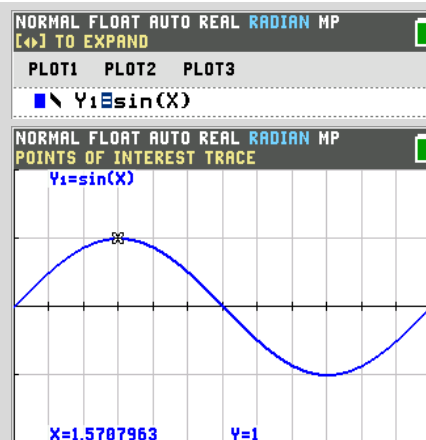
- Press `2nd``[table]`.
- Press `+` to enter a new table increment (ΔTbl) for the independent variable x .
- Set $\Delta Tbl = \pi/6$ and press `enter`.

Scroll through the table to observe and compare trigonometric values from $x = 0$ to 2π .

To view the graphs alongside a table of function values:

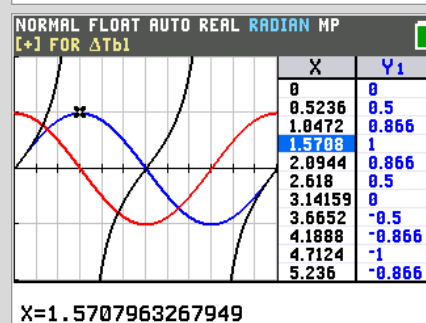
- Press `mode` and select **GRAPH-TABLE**.
- Press `graph` to view all three graphs from $x = 0$ to 2π , alongside a table of the $y = \sin(x)$ values.
- Press `2nd``[table]` to access the table.
- Use the cursor keys to move around the table, and to select the tabled values for $y = \cos(x)$ or $y = \tan(x)$.
- Press `+` to enter a new table increment (ΔTbl) for the independent variable x .

Note: One advantage of the **GRAPH-TABLE** split screen is that tracing the graph also traces through the table at equivalent points. It also works in reverse, that is, tracing through the table values also traces through the equivalent points on the graph.



X	Y1	Y2	Y3
0	0	1	0
0.5236	0.5	0.866	0.5774
1.0472	0.866	0.5	1.7321
1.5708	1	0	ERROR
2.0944	0.866	-0.5	-1.7321
2.618	0.5	-0.866	-0.577
3.14159	0	-1	0
3.6652	-0.5	-0.866	0.5774
4.1888	-0.866	-0.5	1.7321
4.7124	-1	0	ERROR
5.236	-0.866	0.5	-1.7321

$\Delta Tbl = \pi/6$



3.5. Topic 5: Probability

3.5.1. Language of events and sets

Using random number generators for simulation of data

Question

- Generate a random number between 0 and 1.
- Generate 5 random integers between 12 and 20 inclusive (with & without repetitions).
- Simulate rolling a die 100 times, and display the result as a histogram.

Solution

- To generate a random number between 0 and 1:
 - Press **[math]** > **PROB** > **rand** and then press **[enter]**.

The calculator will display a random number in decimal form that is between 0 and 1. Continue to press **[enter]** to repeat the command and produce more random numbers.

- To generate 5 random integers between 12 & 20 (inclusive):
 - Press **[math]** > **PROB** > **randInt**(.
 - In the screen that follows, set the following values:
 - lower = 12** **upper = 20** **n = 5**.
 - Press **PASTE** [f2].
 - Complete by entering **randInt(12,20,5)**.

The calculator will generate a list of randomly selected integers within this range. Note that repetitions are possible. To generate 5 random integers between 12 and 20 (inclusive), but with no repetitions:

- Press **[math]** > **PROB** > **randIntNoRep**(.
- Complete by entering **randIntNoRep(12,20,5)**.

- To generate & store 100 random integers between 1 and 6:
 - Press **[math]** > **PROB** > **randInt**(.
 - In the screen that follows, set the following values:
 - lower = 1** **upper = 6** **n = 100**.
 - Press **PASTE** [f2].
 - Complete by entering **randInt(1,6,100) → L1**.
(press **[sto→]** **[2nd]** **[L1]** to store the set in L1).

To display the results as a histogram:

- Press **[2nd]** **[plot]** and select **PLOT1**, and then edit the following settings.
 - Set **PLOT = ON**.
 - For **TYPE**, select the **Histogram**.
 - For **XLIST**, press **[2nd]** **[L1]**.
- Press **[window]** and enter the following values:

Xmin = 0	Xmax = 8	Xscl = 1
Ymin = -5	Ymax = 25	Yscl = 5
- Press **[graph]** to view and trace the histogram.

Generate another 100 integers and observe the histogram.

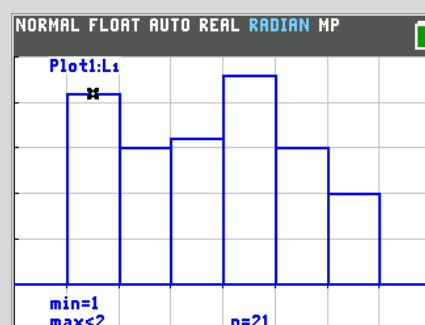
NORMAL FLOAT AUTO REAL RADIAN MP

rand	0.9435974025
rand	0.908318861
rand	0.1466878292

Note: The random number generator can be 'seeded' by storing a unique value into the rand function, so that each student has a different set of random integers. Ask them to enter the last 4 digits of their student ID (e.g. 4378 → rand).

randInt(12,20,5)
{15,18,12,15,20}
randInt(12,20,5)
{12,20,12,12,16}
randIntNoRep(12,20,5)
{19,20,14,13,12}
randIntNoRep(12,20,5)
{18,12,15,14,20}

randInt(1,6,100)→L1
{1,6,3,1,3,4,1,2,5,3,6,2,3,1}



Note: The value of Xscl determines the width of each column.

Exploring sample space through random numbers

Question

If two regular dice are rolled and their numbers added, design a simulation of 500 trials that shows all possible number sums and approximations of their associated probabilities.

Solution

To simulate results from 500 trials of rolling 2 regular dice:

- Press $\boxed{\text{on}}\boxed{\text{enter}}$ to select the CALCULATOR application.
- Press $\boxed{2\text{nd}}\boxed{\text{catalog}} > \text{ClrAllLists}$ to clear all the lists.
- Press $\boxed{\text{math}} > \text{PROB} > \text{randInt}(\text{.})$.
- In the screen that follows, set the following values:
 - lower = 1 upper = 6 n = 500.
- Press **PASTE** [f2] .
- Complete by entering **"randInt(1,6,500)" → L1** (press $\boxed{\text{sto}}\boxed{\rightarrow}\boxed{2\text{nd}}\boxed{[L1]}$ to store 500 results in L1) .
- Press $\boxed{\uparrow}\boxed{\text{enter}}$ to copy and paste the previous entry, then edit it to generate and store another 500 results in L2 (i.e. **"randInt(1,6,500)" → L2**).

To calculate the die sum for each simulated roll of 2 dice, and store each die sum in L3:

- Enter the formula **"L1+L2" → L3** .

Note: The double quotes outside the stored commands above mean that they are editable – that is, the commands can be re-executed to generate another set of 500 trials.

To display the results as a histogram:

- Press $\boxed{2\text{nd}}\boxed{\text{plot}}\boxed{\text{.}}$ and select **PLOT1** and then edit the following settings.
 - Set PLOT = **ON**.
 - For TYPE, select the **Histogram**.
 - For XLIST, press $\boxed{2\text{nd}}\boxed{[L3]}$.
- Press $\boxed{\text{window}}$ and enter the following values:
Xscl = 1 Yscl = 10 (ignore the other parameters).
- Press $\boxed{\text{zoom}} > \text{Zoom Stat}$ to view and trace the histogram.

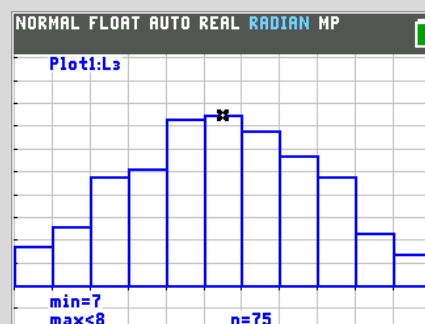
The columns of the histogram will for a two dice sum of between 2 to 12 (inclusive) and the frequency of each can be read by hovering the cursor over each. For example, in the histogram shown right, the dice sum 7 occurred 75 times in that simulation of 500 trials. This corresponds to a relative frequency of $75/500 = 0.15$ (15%).

Note: These relative frequencies provide approximations of theoretical probability. E.g. in this example the modal score is 7 with a relative frequency of $75/500$ or 15%. Compare this to the theoretical probability $1/6 = 16.7\%$. Note also that the simulation can be repeated by pressing $\boxed{\text{zoom}} > \text{Zoom Stat}$. Doing this will also automatically update the window in which the histogram is displayed.

```
NORMAL FLOAT AUTO REAL RADIAN MP
Done
ClrAllLists
Done
"randInt(1,6,500)"→L1
  randInt(1,6,500)
"randInt(1,6,500)"→L2
  randInt(1,6,500)
"L1+L2"→L3
  L1+L2
```

L1	L2	L3	L4	L5
6	3	9		
6	3	9		
1	6	7		
2	4	6		
2	6	8		
2	6	8		
2	4	6		
5	6	11		
1	5	6		
5	1	6		
3	1	4		

L3="L1+L2"



Unit 2: Calculus and further functions

4.1. Topic 1: Exponential functions

4.1.1. Indices and index laws

Using scientific notation

Most modern calculators (including the *TI-84 Evo*) represent numbers using “floating point”. This calculator uses a maximum of 10 digits to represent a decimal number, or 10 digits for the mantissa and 2 digits for the exponent if the number is expressed using scientific notation.

Question

Complete the following calculations and interpret the results obtained.

(a) 123456×54321 . (b) $(1.2345 \times 10^{14}) \times (6 \times 10^{-23})$.

Solution

Note: Unless otherwise stated, the screens show results when the **DECIMAL SETTINGS** is set to **FLOAT**, meaning it will display a maximum of 10 digits. Press **[mode]** to check this setting. The screen right shows how the **FLOAT** setting displays results as the magnitude of the number increases. If the number is larger than can be displayed using 10 digits, the result is displayed using scientific notation, with a maximum of 10 digits for the mantissa, and a maximum of 2 digits for the exponent.

(a) To complete the calculation for question (a) above:

- Press **[on]****[enter]** to select the CALCULATOR application.
- Enter $123\,456 \times 54\,321$.

The result of this product is a 10-digit number and can be displayed in exact form.

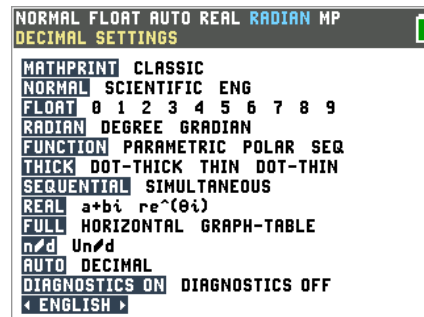
Note: If the product $123\,456 \times 654\,321$ is calculated, the result cannot be displayed in exact form (the exact answer is the 11-digit number 80 779 853 376), and so is displayed using scientific notation with 10 digits in the mantissa and 2 digits in the exponent (as shown right).

(b) Numbers can be added in scientific notation directly by using the **[2nd]****[EE]** key as shown right. To complete the calculation for question (b) above:

- Press **[on]****[enter]** to select the CALCULATOR application.
- Enter 1.2345 **[2nd]****[EE]** 14×6 **[2nd]****[EE]** -23 .

This calculation gives the result in scientific notation, expressing it approximately as 7.407×10^{-9} .

The other calculations show that the result may be displayed as a decimal if the value of the negative exponent is small (as shown here for an exponent of -2 or -3).



HISTORY	
Ans • 197.3	0.95
Ans • 197.3	187.435
Ans • 197.3	36980.9255
Ans • 197.3	7296336.601
Ans • 197.3	1439567211
Ans • 197.3	2.840266108E11

NORMAL FLOAT AUTO REAL RADIAN MP	
123456 • 54321	6706253376
123456 • 654321	8.077985338E10

NORMAL FLOAT AUTO REAL RADIAN MP	
1.2345E14 • 6E-23	7.407E-9
1.2345E14 • 6E-16	0.07407
1.2345E14 • 6E-17	0.007407
1.2345E14 • 6E-18	7.407E-4

4.1.2. Introduction to exponential functions

Transforming exponential functions

Let $f(x) = r^{(x-h)} + k$, $r > 0$. What is the effect on the graph of varying the parameter:

(a) r

(b) h

(c) k ?

(a) To explore the effects of varying the value of the parameter r , let $h = 0$ and $k = 0$.

- Press $\boxed{y=}$ and enter the equation $Y1 = \{1/4, 1/2, 2, 4\}^x$.
- Press $\boxed{\text{window}}$ and enter the following values:
 $X_{\min} = -4$ $X_{\max} = 4$ $X_{\text{scl}} = 1$
 $Y_{\min} = -4$ $Y_{\max} = 20$ $Y_{\text{scl}} = 4$
- Press $\boxed{\text{graph}}$ to display the graphs.

This will generate 4 different exponential function graphs

with the values $r = \frac{1}{4}, \frac{1}{2}, 2$ and 4 .

- Press $\boxed{\text{trace}}$ and the cursor keys to observe how the value of the y -coordinate changes as the x -coordinate changes.

Answer: All graphs of $f(x) = r^x$ (for $r > 0$) will:

- have a horizontal asymptote at $y = 0$.
- include the point $(0, 1)$.
- if $0 < r < 1$, the value of y decreases as x increases.
- if $r > 1$, the value of y increases as x increases.

(b) To explore the effects of varying the value of the parameter h , let $r = 2$ and $k = 0$.

- Press $\boxed{y=}$ and enter the equation $Y1 = 2^{x-(0,2,4)}$.
- Press $\boxed{\text{window}}$ and enter the following values:
 $X_{\min} = -6$ $X_{\max} = 10$ $X_{\text{scl}} = 1$
 $Y_{\min} = -2$ $Y_{\max} = 20$ $Y_{\text{scl}} = 2$

Answer: All graphs of $f(x) = 2^{x-h}$ will:

- have a horizontal asymptote at $y = 0$.
- pass through the point $(h, 1)$.
- translate the point (x, y) on $y = 2^x$ to $(x + h, y)$.

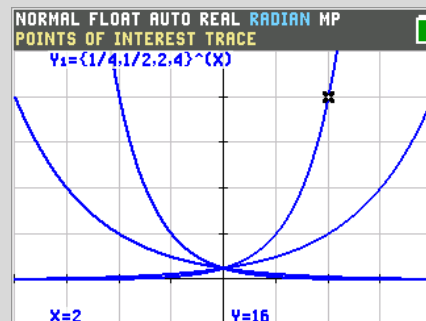
(c) To explore the effects of varying the value of the parameter k , let $r = 2$ and $h = 0$.

- Press $\boxed{y=}$ and enter the equation $Y1 = 2^x + \{-4, 0, 4\}$.
- Press $\boxed{\text{zoom}} \boxed{6}$ to view the three graphs in the standard viewing window.

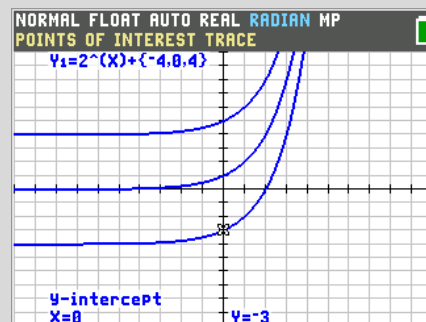
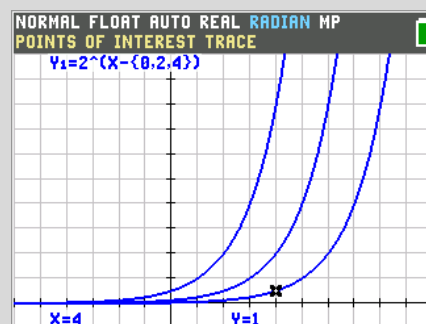
Answer: All graphs of $f(x) = 2^x + k$ will:

- all graphs have a horizontal asymptote at $y = k$.
- all graphs have a y -intercept at $y = 1 + k$.
- translate the point (x, y) on $y = 2^x$ to $(x, y + k)$.

Note: You could also investigate this problem using the TRANSFORMATION GRAPHING application, and the equation $Y1 = A^{X-B} + C$.



Note: The base of the exponential function, r , is often called the 'multiplier'. For instance, a base of $r = 2$ means that as the value of x increases by one unit, the value of y increases by a factor of 2. That is, the y -value is multiplied by 2.



Modelling problems with exponential functions

Question

A student deposits \$1500 into an account with interest rate of 6% per annum, compounded monthly.

- Construct an exponential relationship between the account balance B and t , the number of years over which the money is invested.
- Use your answer to part (a) to find the account balance after each of the first 5 years (to the nearest dollar).
- Find to the nearest month, the time that it would take for the account balance to double.
- Display your answer to part (c) graphically.

Solution

(a) Since compounding occurs monthly, a suitable function model can be defined as:

$$B(t) = 1500 \times \left(1 + \frac{6/12}{100}\right)^{12t} = 1500(1.005)^{12t}.$$

(b) To define this function and find annual account balances for each of the first 5 years:

- Press $\boxed{\text{on}}\boxed{\text{enter}}$ to select the CALCULATOR application.
- Enter $"1500 \times (1.005)^{12X}" \rightarrow Y1$.
- Enter $Y1(\{1,2,3,4,5\})$.
- Enter $\text{round}(\text{Ans},0)$.

Answer: The balances after each of the first 5 years are \$1593, \$1691, \$1795, \$1906, and \$2023.

(c) The time required to double account balance to \$3000 can be found using the numerical solve command as follows:

- Press $\boxed{\text{math}} > \text{MATH} > \text{Numeric Solver ...}$
- For E1, enter $Y1$.
- For E2, enter 3000 .
- Press $\text{NEXT} \boxed{\text{f5}}$ and $\text{SOLVE} \boxed{\text{f5}}$ to find a solution to the equation $Y1 = 3000$ for X .
- Convert the decimal part to months as shown right.

Answer: It will take 11 years and 7 months for the loan balance to double in value.

(d) On a **Graphs** page, and enter the constant function:

- Press $\boxed{\text{y=}}$, add the equation $Y2 = 3000$
- Press $\boxed{\text{window}}$ and enter the following values:
 $X_{\min} = -1$ $X_{\max} = 15$ $X_{\text{scl}} = 1$
 $Y_{\min} = -1000$ $Y_{\max} = 5000$ $Y_{\text{scl}} = 1000$

To show the point of intersection:

- Press $\boxed{\text{2nd}}\boxed{\text{calc}}$ and select **intersect**.
- For **Guess?**, move the cursor close to intersection point, and then press $\boxed{\text{enter}}$.

NORMAL FLOAT AUTO REAL RADIAN MP

"1500(1 + 6/12/100)^12X" → Y1

Y1({1,2,3,4,5})

{1592.516718, 1690.739664, 1795.1906, 2023}

round(Ans,0)

{1593, 1691, 1795, 1906, 2023}

Note: The function label $Y1$ can be accessed by the key sequence $\boxed{\text{vars}} > \text{Y-VARS} > \text{Function} > Y1$ or $\boxed{\alpha}\boxed{\text{f4}}$. The round command is found via $\boxed{\text{math}} > \text{MATH} > \text{round}$. When selecting the command, press $\boxed{\text{v}}$ to view its syntax.

NORMAL FLOAT AUTO REAL RADIAN MP

SOLUTION IS MARKED

Y1=3000

X= 11.581310134224

bounds= {-1E99, 1E99}

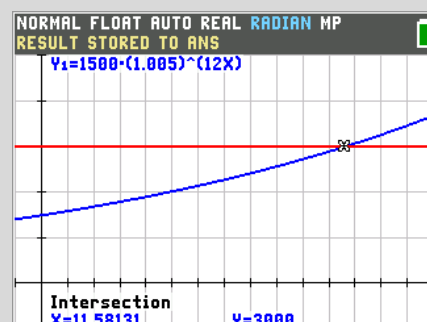
E1-E2= 0

X

Ans-11 11.58131013

Ans-12 0.5813101342

Ans-12 6.975721611



4.2. Topic 2: Logarithms and logarithmic functions

4.2.1. Logarithms and logarithmic laws

Visualising a graph of logarithmic values

The link between a number and its logarithm to a particular base can be visualised by generating a set of numbers and their corresponding log values.

Question

- Generate a small set of numbers which are powers of 2, and then calculate the logarithm of each of these numbers using a base of 2.
- Construct a scatter plot of these numbers, using the powers of 2 as the independent variable, and the logarithm values as the dependent variable.
- Graph the equation $y = \log_2(x)$ on the same axes as the scatter plot.

Solution

- To enter a sequence of powers of 2:
 - Press $\boxed{\text{on}}\boxed{\text{enter}}$ to select the CALCULATOR application.
 - Press $\boxed{2\text{nd}}\boxed{\text{list}} > \text{OPS} > \text{seq(}$ (and enter the values shown in the screen right for the sequence).
 - Press **PASTE** $\boxed{\text{f2}}$ (but don't press $\boxed{\text{enter}}$ yet).
 - This will paste the command $\text{seq}(2^N, N, -2, 3, 1)$.
 - Press $\boxed{\text{sto}}\boxed{\rightarrow}$ then $\boxed{2\text{nd}}\boxed{\text{list}} > \text{OPS} > \text{L}$.
 - Complete the command $\text{seq}(2^N, N, -2, 3, 1) \rightarrow \text{LPOW2}$.
 - Now press $\boxed{\text{enter}}$ to execute the formula and display the powers of 2 for the integers -2 to 3 .

Note: The small 'L' symbol (found via $\boxed{2\text{nd}}\boxed{\text{list}} > \text{OPS} > \text{L}$) permits defining a list variable with more than one letter.

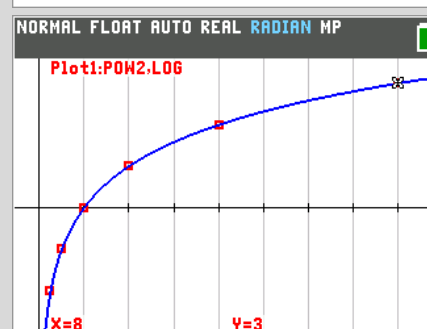
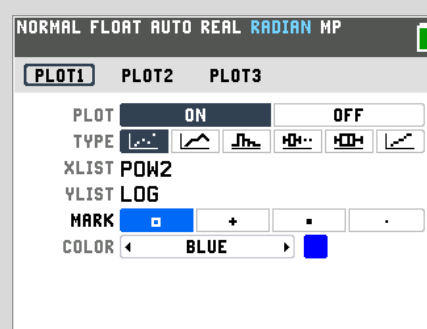
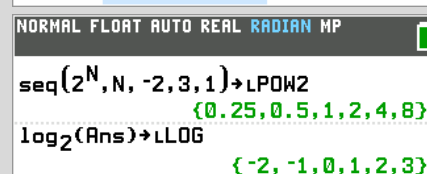
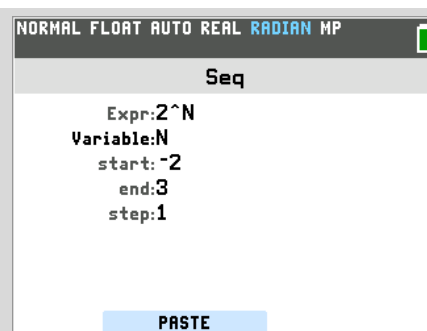
To calculate the logarithms (using base of 2) of this sequence of numbers, press $\boxed{\log}$ and enter $\log_2(\text{Ans}) \rightarrow \text{LLOG}$.

- To create a plot of the logarithms values for each of the numbers:

- Press $\boxed{2\text{nd}}\boxed{\text{plot}}\boxed{\cdot}$ and select **PLOT1**, and then edit the following settings.
 - Set **PLOT** = **ON**.
 - For **TYPE**, select **Scatter Plot**.
 - For **XLIST**, press $\boxed{2\text{nd}}\boxed{\text{list}} > \text{NAMES} > \text{POW2}$.
 - For **YLIST**, press $\boxed{2\text{nd}}\boxed{\text{list}} > \text{NAMES} > \text{LOG}$.
 - For **MARK**, select the **hollow square mark**.
 - For **COLOR**, select **RED**.
- Press $\boxed{\text{zoom}} > \text{Zoom Stat}$ to view the scatterplot.

- To overlay the graph of $y = \log_2(x)$ on the scatter plot:

- Press $\boxed{\text{y=}}$, add the equation **Y1 = $\log_2(X)$**
- Press $\boxed{\text{graph}}$ to view this graph as well as the scatter plot.
- Use the $\boxed{\text{trace}}$ key to explore the value of the logarithm for various values of x .



4.2.2. Logarithmic functions

Transforming logarithmic functions

Let $f(x) = \log_a(x-h) + k$, $a > 1$. What is the effect on the function graph of varying the parameter:

(a) a

(b) h

(c) k

(a) To explore the effects of varying the value of the parameter a , let $h = 0$ and $k = 0$.

- Press $\boxed{Y=}$ and enter the equation $Y1 = \log_{\{2,3,4,5\}}(X)$.
- Press $\boxed{\text{window}}$ and enter the following values:
 $X_{\min} = -1$ $X_{\max} = 10$ $X_{\text{scl}} = 1$
 $Y_{\min} = -5$ $Y_{\max} = 5$ $Y_{\text{scl}} = 1$
- Press $\boxed{\text{graph}}$ to display the graphs.

This will generate 4 different logarithmic function graphs with base values $a = 2, 3, 4$, and 5 .

Answer: With further experimentation, note that:

- all of the graphs have a vertical asymptote at $x = 0$.
- all of the graphs pass through the point $(1,0)$.
- for higher a values, the gradient of the graph decreases more rapidly for $x > 0$.

(b) To explore the effects of varying the value of the parameter h , let $a = 2$, $h = 3$ and $k = 0$.

- Press $\boxed{Y=}$ and enter the equations $Y1 = \log_2(X)$ and $Y2 = \log_2(X-3)$.
- Press $\boxed{\text{graph}}$ to display the graphs.

Notice that for all points on the graph of $Y1 = \log_2(X)$, there is a transformed point on the graph of $Y2 = \log_2(X-3)$ which is 3 units to the right.

Answer: By varying the value of h , it can be observed that:

- all the graphs have a vertical asymptote at $x = h$.
- all the graphs have an x -intercept at $y = 1 + h$.
- the general point (x, y) is translated to $(x + h, y)$.

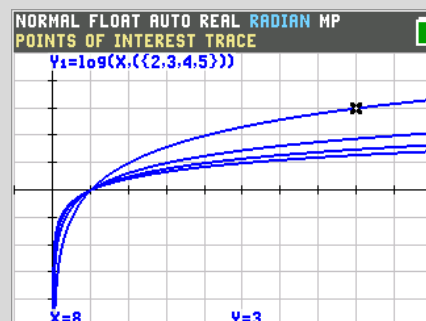
(c) To explore the effects of varying the value of the parameter k , let $a = 2$, $h = 0$ and $k = -3$.

- Press $\boxed{Y=}$ and enter the equations $Y1 = \log_2(X)$ and $Y2 = \log_2(X) - 3$.
- Press $\boxed{\text{graph}}$ to display the graphs.

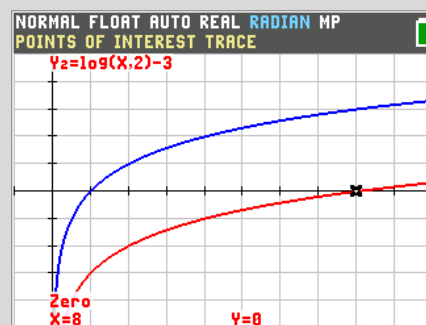
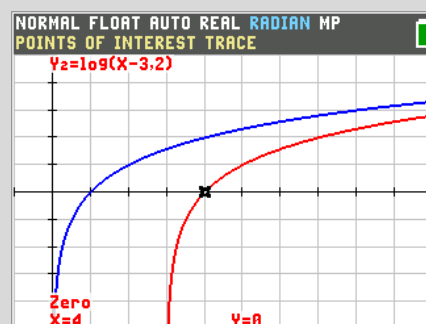
Notice that for all points on the graph of $Y1 = \log_2(X)$, there is a transformed point on the graph of $Y2 = \log_2(X) - 3$ which is 3 units down.

Answer: By varying the value of k , it can be observed that:

- all the graphs have a vertical asymptote at $x = 0$.
- As k increases, the x -intercept approaches 0.
- the general point (x, y) is translated to $(x, y + k)$.



Note: On the $\boxed{Y=}$ screen, equations are in MathPrint format e.g. $Y_1 = \log_{\{2,3,4,5\}}(X)$.
 On the $\boxed{\text{graph}}$ screen, equations are in command line format e.g. $Y_1 = \log(X, \{2,3,4,5\})$



Note: You could also investigate this problem using the TRANSFORMATION GRAPHING application, and $Y1 = \log_A(X - B) + C$.

Modelling problems with logarithmic functions

Question

The intensity of a sound depends on the energy of the sound wave. It is measured as power per unit area, usually in watts/m^2 . An alternative measure is the loudness level, measured in decibels (dB).

The loudness L dB is related to the intensity I watts/m^2 by the formula $L = 10 \log_{10} \left(\frac{I}{10^{-12}} \right)$.

- The threshold for human hearing (i.e. the intensity of a barely audible sound) serves as a base measure for decibels. The intensity is 10^{-12} watts/m^2 . What is the corresponding loudness in decibels?
- Normal conversation takes place at an intensity of about 10^{-6} watts/m^2 . Traffic on a busy street might be 10 times the sound intensity of a normal conversation. Show that this does not mean that the loudness level is 10 times greater and hence explain how the loudness level changes when the sound intensity doubles.
- The human eardrum is in danger of rupturing at a loudness level of 160 dB. What is the intensity in watts/m^2 at which the human eardrum is in danger of rupturing?
- Explore ways to visually represent the relationship between loudness level and sound intensity.

Solution

Define the function for loudness as follows:

- Press $\boxed{\text{on}} \boxed{\text{enter}}$ to select the CALCULATOR application.
- Enter the following command to store as Y1:

$$"10 \times \log_{10} \left(\frac{X}{10^{-12}} \right)" \rightarrow Y1.$$

- To find the loudness level for an intensity of 10^{-12} :

- Enter $Y1(10^{-12})$.

- To find the loudness level for intensities of 10^{-6} and 10^{-5} :

- Enter $Y1(\{10^{-6}, 10^{-5}\})$.

Answer: If the sound intensity is multiplied by a factor of 10, the loudness level is increased by $70 - 60 = 10$ dB.

To find how much the loudness level would increase if the sound intensity is doubled:

- Enter $Y1(2 \times 10^{-6}) - Y1(10^{-6})$.

Answer: If the sound intensity is doubled, the loudness level would increase by about 3 dB. This is true for doubling all sound intensities, which is verified in the screen right, and can be proven algebraically.

- To find the intensity for a loudness level of 160 dB, solve the equation $Y1 = 160$ for X using the numerical solve command as follows:

- Press $\boxed{\text{math}} > \text{MATH} > \text{Numeric Solver ...}$
- For E1, enter $Y1$.
- For E2, enter 160.
- Press **NEXT** $\boxed{\text{f5}}$ and **SOLVE** $\boxed{\text{f5}}$ to find a solution to the equation $L = 160$ for I .

Answer: The sound intensity is 10 000 watts/m^2 (or 10^4 watts/m^2).

NORMAL FLOAT AUTO REAL RADIAN MP

"10 * log10(X / 10^-12)" → Y1

Done

Y1(10^-12)

0

Y1({10^-6, 10^-5})

{60, 70}

NORMAL FLOAT AUTO REAL RADIAN MP

Y1(2 * 10^-6) - Y1(10^-6)

3.010299957

Y1(2 * 10^-5) - Y1(10^-5)

3.010299957

Y1(2 * 10^-4) - Y1(10^-4)

3.010299957

Y1(2 * 10^-3) - Y1(10^-3)

3.010299957

NORMAL FLOAT AUTO REAL RADIAN MP

SOLUTION IS MARKED ■

Y1=160

■ X= 10000.000000003

bounds= {-1E99, 1E99}

■ E1-E2= 0

SOLVE

... continued

Solution (continued)

(d) To display the graph of Y_1 :

- Press $\boxed{Y=}$ and enter the equation $Y_1 = \log_{\{2,3,4,5\}}(X)$.
- Press $\boxed{\text{window}}$ and enter the following values:
 $X_{\min} = -1000$ $X_{\max} = 10000$ $X_{\text{scl}} = 1000$
 $Y_{\min} = -50$ $Y_{\max} = 200$ $Y_{\text{scl}} = 50$
- Press $\boxed{\text{graph}}$ to display the graphs.

Note that this view obscures the graph for lower values of I .

As it is known the values of I change more rapidly than those of L , try plotting the $\log_{10}(I)$ values against the loudness level. From above, a reasonable range for human hearing is from $I = 10^{-12}$ (threshold of hearing) to $I = 10^4$ (ear bleeds). The associated values of L are from $L = 0$ dB to $L = 160$ dB.

Generate these sequences as follows:

- Press $\boxed{\text{on}} \boxed{\text{enter}}$ to select the CALCULATOR application.
- Press $\boxed{2\text{nd}} \boxed{\text{list}} > \text{OPS} > \text{seq}(\text{ and enter the values shown in the screen right for the sequence.$
- Press **PASTE** $\boxed{\text{f2}}$ (but don't press $\boxed{\text{enter}}$ yet).
This will paste the command $\text{seq}(N, N, -12, 4, 1)$.
- Press $\boxed{\text{sto}} \rightarrow$ then $\boxed{2\text{nd}} \boxed{\text{list}} > \text{OPS} > L$.
- Complete the command
 $\text{seq}(N, N, -12, 4, 1) \rightarrow L\text{LGINT}$.
- Now press $\boxed{\text{enter}}$ display and store a sequence of the logarithms of the sound intensity from -12 to 4 .

Repeat the above to generate and store a sequence of the corresponding loudness levels from 0 to 16 using the following command: $\text{seq}(10N, N, 0, 16, 1) \rightarrow L\text{LOUD}$

To create a plot of *loudness level* vs *log (sound intensity)*:

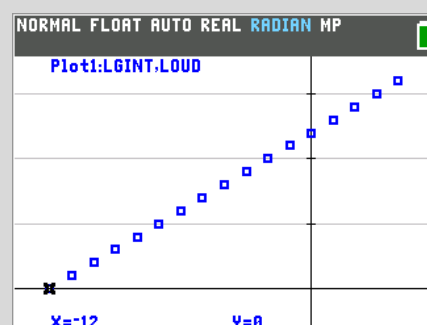
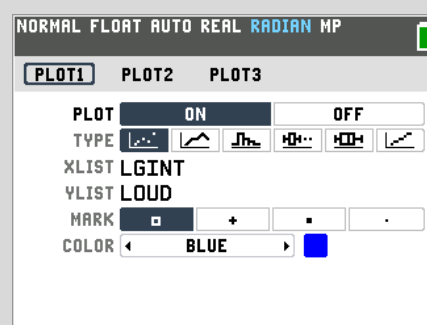
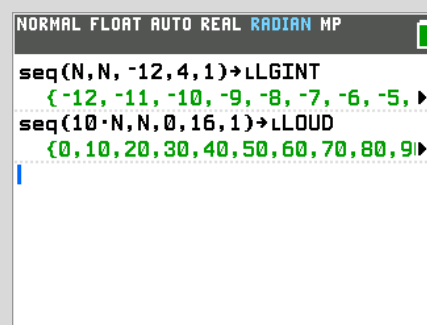
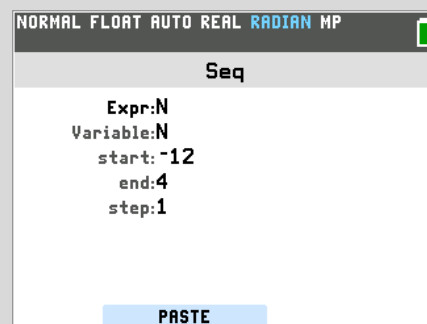
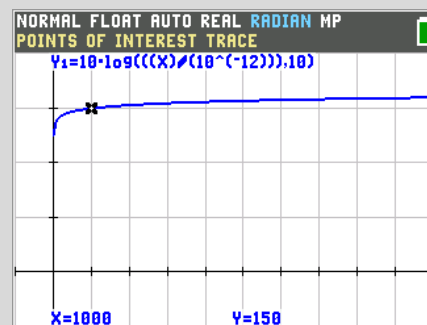
- Press $\boxed{2\text{nd}} \boxed{\text{plot}} \rightarrow$ and select **PLOT1**, and then edit the following settings.
 - Set **PLOT** = **ON**
 - For **TYPE**, select the **Scatter Plot**
 - For **XLIST**, press $\boxed{2\text{nd}} \boxed{\text{list}} > \text{NAMES} > \text{POW2}$
 - For **YLIST**, press $\boxed{2\text{nd}} \boxed{\text{list}} > \text{NAMES} > \text{LOG}$
 - For **MARK**, select the **hollow square mark**
 - For **COLOR**, select **BLUE**
- Press $\boxed{\text{zoom}} > \text{Zoom Stat}$ to view the scatterplot.

The plot shows a perfect linear relationship between the loudness level and the logarithm (base 10) of sound intensity.

The equation of the line relating loudness level and the logarithm (base 10) of sound intensity can be calculated by pressing $\boxed{\text{stat}} > \text{CALC} > \text{LinReg}(ax+b)$.

So according to the regression analysis (and log laws),

$$\text{loudlevel} = 10 \times \log_{10}(\text{intensity}) + 120.$$



4.3. Topic 3: Introduction to differential calculus

4.3.1. Rates of change and the concept of derivatives

Comparing average rates of change with instantaneous rate of change

The volume of water V remaining in an 8000L tank t minutes after a tap is opened can be modelled by the function $V(t) = (20 - t)^3$, $0 \leq t \leq 20$.

- Find the average rate of change in volume during (i) first 10 minutes; (ii) second 10 minutes
- Estimate the rate of change in the volume at $t = 10$ minutes.
- Construct graphs of the above function and the rates of change in the volume requested.

(a) Calculate average rate of change values as follows:

- Press $\boxed{\text{on}} \boxed{\text{enter}}$ to select the CALCULATOR application.
- Enter the function " $(20 - X)^3$ " $\rightarrow Y1$.
- Enter the expression $\frac{Y1(10) - Y1(0)}{10 - 0}$.
- Enter the expression $\frac{Y1(20) - Y1(10)}{20 - 10}$.

Answer: (i) -700 L/min; (ii) -100 L/min.

(b) To estimate the instantaneous rate of change at $t = 10$ min:

- Enter the expression $0.1 \rightarrow H: \frac{Y1(10+H) - Y1(10)}{(10+H) - 10}$.

Answer: By experimenting with very small values of h , it can be estimated that at $t = 10$ minutes, the water in the tank is emptying at a rate of 300 L/min.

(c) Line segments can be constructed to represent the rates calculated using the point at $(10, 1000)$, and drawn on the graph of $V(t)$ over the relevant domain:

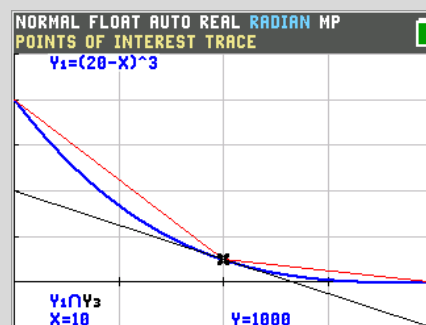
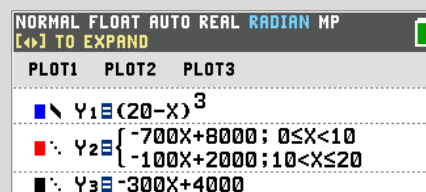
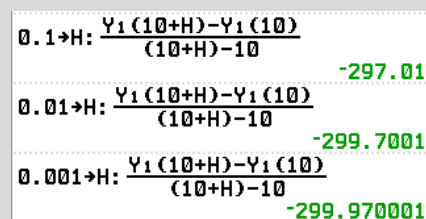
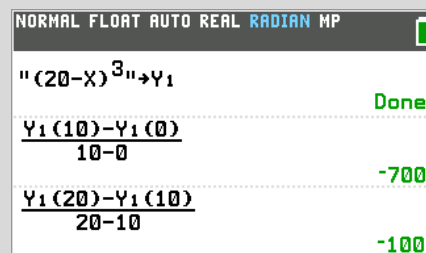
- For first 10 mins: $y = -700t + 8000$.
- For second 10 mins: $y = -100t + 2000$.
- At 10 minutes: $y = -300t + 4000$.

To display the graph of $V(t)$, line segments and the tangent:

- Press $\boxed{y=}$ and enter the equations as follows:

$$Y2 = \begin{cases} -700X + 8000 & 0 \leq X < 10 \\ -100X + 2000 & 10 < X \leq 20 \end{cases}$$
- $Y3 = -300X + 4000$.
- Press $\boxed{\text{window}}$ and enter the following values:
 $X_{\min} = 0$ $X_{\max} = 20$ $X_{\text{scl}} = 5$
 $Y_{\min} = -2000$ $Y_{\max} = 10000$ $Y_{\text{scl}} = 2000$
- Press $\boxed{\text{graph}}$ to display the graph of $V(t)$ and a representation of the relevant rates.

Note: The piecewise function in $Y2$ can be entered by first pressing $\boxed{\text{math}} > \text{MATH} > \text{piecewise}$. The inequality symbols can be accessed via pressing $\boxed{2\text{nd}} \boxed{\text{math}} > \text{TEST}$.



Note: In the screen above, the graphs of $Y2$ and $Y3$ are thinner than $Y1$. The colour/thickness of any function graph can be altered via $\boxed{y=}$. Move the cursor to the left if the Y , then press $\boxed{\text{enter}}$ to access/edit the graph colour and thickness.

4.4. Topic 4: Applications of differential calculus

4.4.1. Graphical applications of derivatives

Drawing a tangent line on a graph

Question

Construct a tangent on the curve of $f(x) = 0.1x(x+4)(x-4)$ at $x = 3$.

Solution

To plot the graph of $f(x)$:

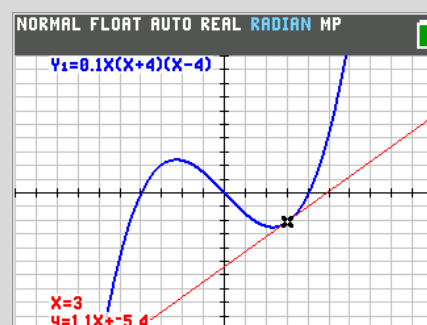
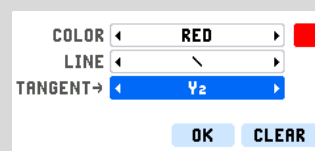
- Press $\boxed{y=}$ and enter the equation $Y1=0.1X(X-4)(X+4)$.
- Press $\boxed{\text{zoom}}$ > **Zoom** $[-10,10]$.

To construct a tangent on the graph of this function at $x = 3$:

- Press $\boxed{2nd}$ > $\boxed{\text{draw}}$ >> **Tangent**
- Press **MENU** $\boxed{f5}$ and use the following tangent settings:
 - Set Color = **RED**.
 - Set LINE = **THIN**.
 - Set TANGENT equation to be stored in **Y2**.
 - Press **OK**.
- To draw the tangent to the curve at $x = 3$, press $\boxed{3}$ $\boxed{\text{enter}}$.

Answer: The tangent line to the curve at $x = 3$ has the equation $y = 1.1x - 5.4$.

Note: Press $\boxed{2nd}$ $\boxed{\text{draw}}$ >> **ClrDraw** to remove the tangent line.



Locating stationary points of a function graphically

Question

Find graphically the coordinates of the local minimum on the curve of $f(x) = 0.1x(x+4)(x-4)$.

Solution

To plot the graph of $f(x)$:

- Press $\boxed{y=}$ and enter the equation $Y1=0.1X(X-4)(X+4)$
- Press $\boxed{\text{zoom}}$ > **Zoom** $[-10,10]$.

To locate the local minimum with the **Trace** feature:

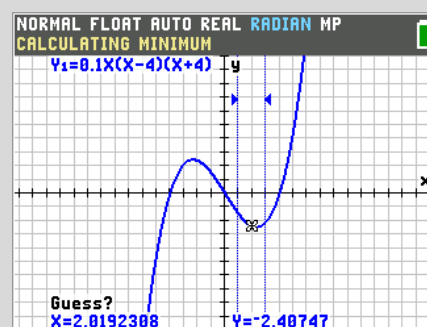
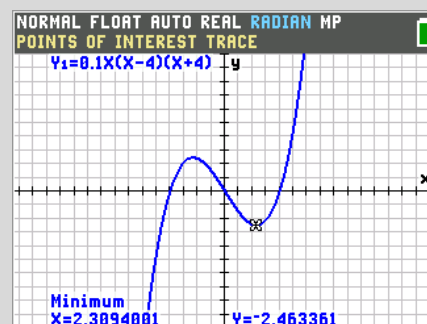
- Press $\boxed{\text{trace}}$ and then press $\boxed{2}$ $\boxed{\text{enter}}$ to locate the cursor on the graph at $x = 2$.
- Press $\blacktriangleright$ until the cursor locates at the local minimum value.

Answer: The local minimum is approximately $(2.31, -2.46)$.

To locate the local minimum with the **calc** feature:

- Press $\boxed{2nd}$ $\boxed{\text{calc}}$ > **minimum**.
- When prompted for a “**Left Bound?**”, position the cursor to the left of the local minimum and press $\boxed{\text{enter}}$.
- When prompted for a “**Right Bound?**”, position the cursor to the right of the local minimum and press $\boxed{\text{enter}}$.
- When prompted for a “**Guess?**”, position the between the **Left Bound** and **Right Bound** you entered and press $\boxed{\text{enter}}$.

Note: It is also possible to enter a numeric value for the left bound, right bound or guess by typing the number directly.



Locating stationary points of a polynomial function numerically

Question

Find numerically the coordinates of any stationary points on the curve of $f(x) = 0.1x(x + 4)(x - 4)$.

Solution

Firstly, display the graph of $f(x)$ and approximate locations of any stationary points:

- Press $\boxed{Y=}$ and enter the equation $Y1=0.1X(X-4)(X+4)$.
- Press $\boxed{\text{zoom}}$ > **Zoom** $[-10,10]$.

From the graph, and a knowledge of cubic functions, we can see that there will be 2 stationary points – a local maximum near $x = -2$ and a local minimum near $x = 2$.

Using a numeric method to find the x -coordinates of stationary points on the graph, find points where the derivative equals zero:

- Press $\boxed{\text{math}}$ > **MATH** > **Numeric Solver ...**
- For E1, press $\boxed{\text{math}}$ > **MATH** > **nDeriv(** and then $\boxed{\alpha}$ $\boxed{f4}$ to enter **Y1**. Then complete the expression for E1 as shown right.
- For E2, enter **0**.
- Press **NEXT** $\boxed{f5}$ and **SOLVE** $\boxed{f5}$ to find a solution to the equation $E1 = E2$ for X .

This locates one stationary point (the local minimum) at approximately $x = 2.3094$.

To locate the x -coordinate of the other stationary point (the local maximum):

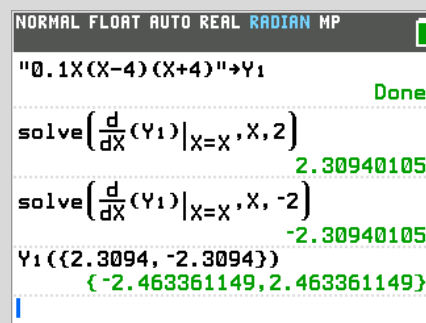
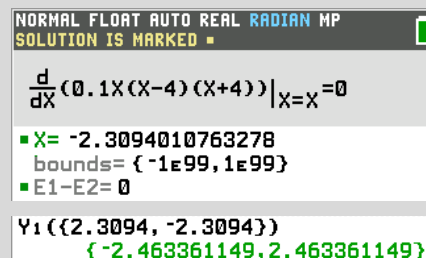
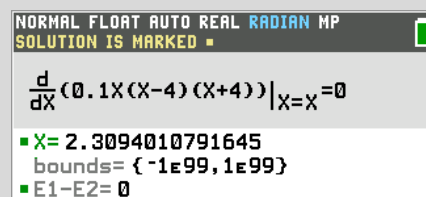
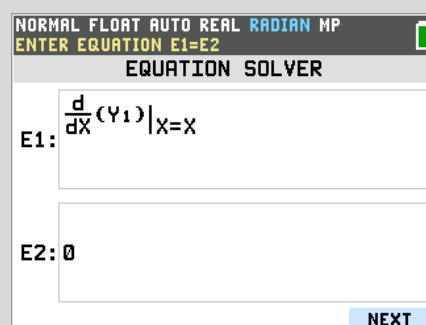
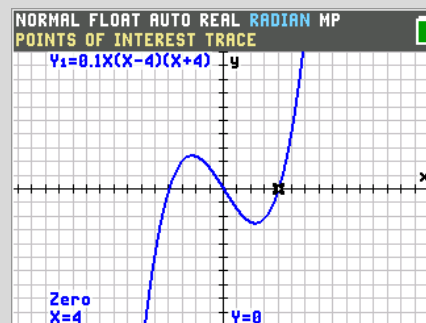
- Move the cursor to the line displaying the current solution for x , and change this to a negative number (for example $x = -2$).
- Press **SOLVE** $\boxed{f5}$ to find the other solution at approximately $x = -2.3094$.

To find the y -coordinates of the 2 stationary points:

- Press $\boxed{\text{on}}\boxed{\text{enter}}$ to select the **CALCULATOR** application.
- Press $\boxed{\alpha}$ $\boxed{f4}$ to enter **Y1**.
- Complete the expression **Y1({2.3094,-2.3094})**

Answer: The coordinates of the stationary points are approximately $(-2.3094, 2.4633)$ and $(2.3094, -2.4633)$.

Note: It is also possible to solve the above equation using the **solve** command (found via $\boxed{2nd}\boxed{\text{catalog}}\boxed{S}$). Assuming that the original function is stored in **Y1**, the command **solve(expression, variable, guess)** can be used. In this command, 'expression' means the LHS of the equation to be solved assuming the RHS equals zero. Here, we are solving for $dy/dx = 0$, so the 'expression' is dy/dx . In the example shown right, the 'guess' has been set to $x = 2$ and $x = -2$. The **POLYNOMIAL ROOT FINDER** could also be used here.



Building a tangent and normal calculator (TI-BASIC)

Question

Construct an executable program in **TI-Basic** that will find the equation of a tangent and normal to a function curve at a given x -value. Test the program by finding the equation of the tangent and normal of $y = x^2 - 8$ at $x = 2$.

Solution

To construct a program in **TI-Basic**:

- Press **[prgm]** > **TI-Basic** > **NEW** > **Create New**.
- Enter the program title **TANGNORM**.
- Enter the commands exactly as shown right.

Key sequences for entering selected program commands:

- For **ClrHome**, enter **[prgm]** > **I/O** > **ClrHome**.
- For **Y1**, enter **[alpha]****[f5]****[1]**.
- For **Prompt**, enter **[prgm]** > **I/O** > **Prompt**.
- For **Diso**, enter **[prgm]** > **I/O** > **Disp**.
- For **nDeriv**, enter **[math]** > **MATH** > **nDeriv**.
- For **Pause**, enter **[prgm]** > **CTL** > **Pause**.

To execute the program **TANGNORM** and test it by finding the equation of the tangent and normal of $y = x^2 - 8$ at $x = 2$, complete the following steps:

- Execute the program by pressing
 - **[alpha]****[f5]** (if inside the program code) or
 - **[prgm]** > **TI-Basic** > **EXEC** > **TANGNORM** (if outside the program).
- For **Y1**, enter **"X² - 8"** (include double quotes as shown).
- For **X**, enter **2**.

The program will calculate and display the gradient and y-intercept of the tangent at $x = 2$. Press **[enter]** and the program will calculate and display the

Answer: The equation of the tangent and normal to the graph of $y = x^2 - 8$ when $x = 2$ are $y = 4x - 12$ and $y = -0.25x - 3.5$.

To view the graph of $y = x^2 - 8$ and the tangent and normal lines at $x = 2$:

- Press **[y=]** and enter **Y2 = MX+C** and **Y3 = NX+D**.
- Press **[zoom]** > **Zoom [-10,10]**.
- Press **[zoom]** > **Zoom Square** (to use equal scales for the axes, so that the tangent & normal look perpendicular).

Note: The fourth last line of the **TANGNORM** program uses the command **PAUSE**. This pauses the program, so that you can record the tangent equation. To un-pause and continue the program, press **[enter]**. This will then display the equation of the normal.

```

NORMAL FLOAT AUTO REAL RADIAN MP
EDIT MENU: [alpha] [f5]

PROGRAM: TANGNORM

:ClrHome
:Prompt Y1,X
:nDeriv(Y1,X,X)→M
:-1/M→N
:Y1(X)-MX→C
:Y1(X)-NX→D
:Disp "TANG:Y=MX+C"
:Disp "M=",M
:Disp "C=",C
:Pause
:Disp "NORM:Y=NX+D"
:Disp "N=",N
:Disp "D=",D

```

```

NORMAL FLOAT AUTO REAL RADIAN MP

Y1?"X2-8"
X?2
TANG:Y=MX+C
M=
C=
NORM:Y=NX+D
N=
D=

```

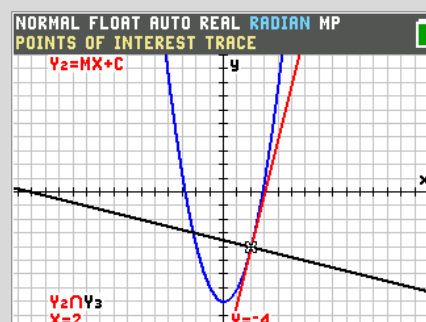
4
-12
-0.25
-3.5
Done

```

NORMAL FLOAT AUTO REAL RADIAN MP

PLOT1 PLOT2 PLOT3
Y1=X2-8
Y2=MX+C
Y3=NX+D

```



Locating global maxima/minima of a function

Question

Find numerically the minimum and maximum values of $f(x) = 0.1x(x-4)(x+4)$ in the domain $x \in [-9, 4.5]$.

Solution

To define a function with the restricted domain:

- Press $\boxed{Y=}$ and enter the equation and domain as follows:
Y1=0.1X(X-4)(X+4)/(X≥-9 and X≤4.5).
- Press $\boxed{\text{zoom}}$ > **Zoom [-10,10].**
- Press $\boxed{\text{zoom}}$ > **Zoom Fit.**

Notes: (1) To enter the domain, press $\boxed{2\text{nd}}\boxed{\text{math}}$ < **TEST** to access the inequality symbols, or $\boxed{2\text{nd}}\boxed{\text{math}}$ < **LOGIC** to enter the 'and' logical operator. Alternatively, you can define Y1 as a piecewise function with one part (via $\boxed{\text{math}}$ > **MATH** > **piecewise**).

(2) **Zoom Fit** adjusts Ymin and Ymax so that all defined points for x values between Xmin and Xmax are visible in the viewing window.

It looks as if global minimum and local minimum are not the same, but it appears as though the local maximum at around $x = -2.3$ is also the global maximum within the domain.

To numerically calculate the x value at which the global minimum occurs in the domain given:

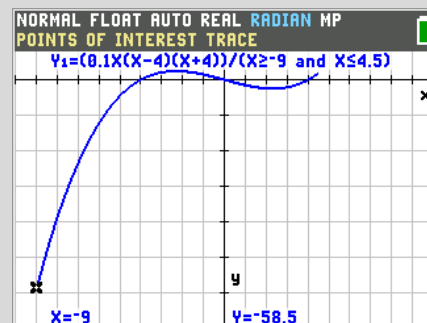
- Press $\boxed{\text{on}}\boxed{\text{enter}}$ to select the CALCULATOR application
- Enter $\boxed{\text{math}}$ > **MATH** > **fMin(**
- Complete the expression:
fMin(Y1, X, -9,4.5).
- Press $\boxed{\alpha}$ > $\boxed{\text{f4}}$ > **Y1.**
- Complete the expression **Y1(Ans).**

Answer: The global minimum in the given domain occurs at the point with coordinates $(-9, -58.5)$.

To numerically calculate the x value at which the global maximum occurs in the domain given:

- Enter $\boxed{\text{math}}$ > **MATH** > **fMax(**
- Complete the expression:
fMax(Y1, X, -9,4.5).
- Press $\boxed{\alpha}$ > $\boxed{\text{f4}}$ > **Y1.**
- Complete the expression **Y1(Ans).**

Answer: The global minimum in the given domain occurs at the point with approximate coordinates $(-2.31, 2.46)$.



NORMAL FLOAT AUTO REAL RADIAN HP	
POINTS OF INTEREST TRACE	
Y1=(0.1X(X-4)(X+4)/(X≥-9 and X≤4.5)	
fMin(Y1,X,-9,4.5)	-8.999992744
Y1(Ans)	-58.49983529
fMax(Y1,X,-9,4.5)	-2.309401157
Y1(Ans)	2.463361149

Note: The commands **fMin** and **fMax** use a numeric routine to locate the x values associated with the minimum or maximum function value across a specified domain. Such commands will sometimes give an answer very close to, but not the same as, the correct answer. Keep this in mind when using these commands.

Unit 3: Further calculus and introduction to statistics

5.1. Topic 1: Differentiation of exponential & logarithmic functions

5.1.1. Calculus of exponential functions

Exploring Euler's number

Question

- (a) Find the value of a for which the value of $\lim_{h \rightarrow 0} \frac{a^h - 1}{h} = 1$.
- (b) Explore the relationship between the graph of the function $f(x) = a^x$ and its derivative graph.

Solution

(a) To find the value of a for which $\lim_{h \rightarrow 0} \frac{a^h - 1}{h} = 1$:

- Press $\boxed{\text{on}} \boxed{\text{enter}}$ to select the CALCULATOR application.
- Enter the following to start testing values of h and a :

$$0.001 \rightarrow H : 3 \rightarrow A : \frac{A^H - 1}{H}$$

Experiment with various values of a until the expression generates a value close to 1 (to do this, press $\boxed{\text{on}} \boxed{\text{enter}}$ to copy and paste the previous entry, then edit the value of a). This highlights the limiting value of 1 as the value of a approaches Euler's number e . Then try smaller values of h until the expression generates a value even closer to 1. The calculator may re-express the values of h in scientific notation.

(b) To explore the relationship between the graph of the function $f(x) = a^x$ and its derivative graph, start with $a = 5$:

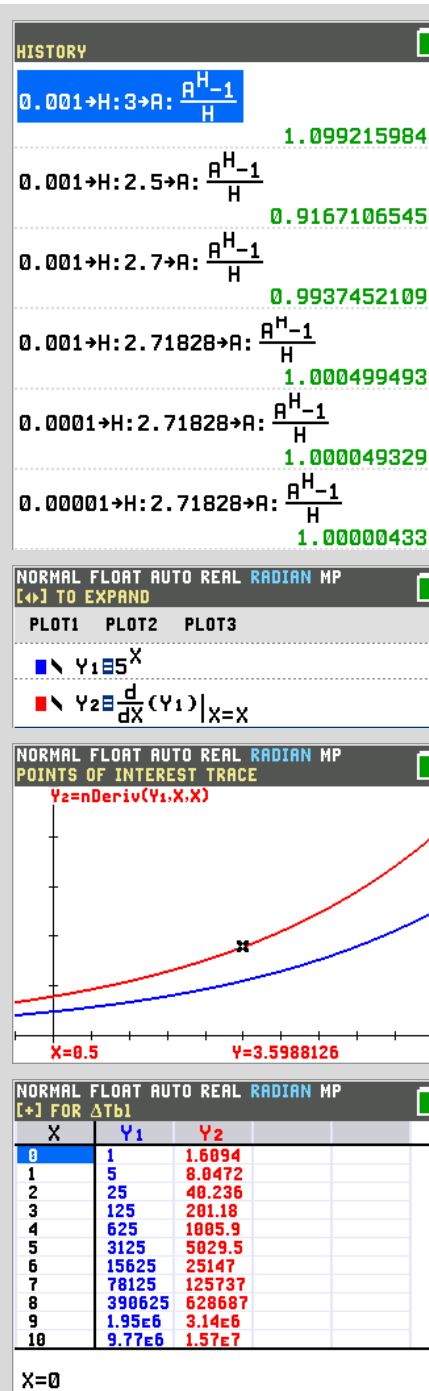
- Press $\boxed{y=}$ and enter the equation $Y1 = 5^x$.
- Move to $Y2$ line, then press $\boxed{\text{math}} > \text{MATH} > \text{nDeriv}(\cdot)$.
- Press $\boxed{\alpha} > \boxed{\text{f4}} > Y1$ to enter $Y1$.
- Then complete the derivative template for $Y2$ as shown.
- Press $\boxed{\text{window}}$ and enter the following values:
 $X_{\min} = -0.1$ $X_{\max} = 1$ $X_{\text{scl}} = 0.1$
 $Y_{\min} = -1$ $Y_{\max} = 10$ $Y_{\text{scl}} = 2$
- Press $\boxed{\text{graph}}$ and then $\boxed{\text{trace}}$.

Now both the graphs of the function $f(x) = 5^x$ and its derivative graph are displayed in a suitable window. Notice that the graph of the derivative appears to be above the graph of $f(x) = 5^x$ for the visible domain.

- Press $\boxed{2\text{nd}} \boxed{\text{table}}$ to view a table comparing the function value with the derivative value.

Try various values of a until the function graph and its derivative graph appear to coincide. What is this value?

Also, for this a value, what is the value of $f'(0)$?



Solving exponential equations

Question

Find a few different ways to calculate the solution to the equation $2^x = 7$.

Solution

On a **Calculator** page, enter the commands as follows:

Method 1: Press $\boxed{2\text{nd}}\boxed{[\text{catalog}]}\boxed{[S]}$ and locate the solve command, and then enter **solve**($2^X-7,X,2.5$).

This will calculate the decimal approximation to the solution of the given equation. Note that the original equation $2^x = 7$ is entered as finding 'zeros' to $2^x - 7$

(that is, solutions for x to the equation $2^x - 7 = 0$).

Method 2: Press $\boxed{\log}$ to display the logarithm template, then enter 2 as the base and 7 as the number.

Method 3: The solution can also be expressed using the

'change of base' law $\left(\log_a b = \frac{\log_c b}{\log_c a}, c > 1 \right)$.

In the screen right, Euler's number e is chosen as the base.

Method 4: The **Numeric Solver** can also be used in the following manner:

- Press $\boxed{\text{math}} > \mathbf{MATH} > \mathbf{Numeric Solver} \dots$
- For E1, enter 2^X .
- For E2, enter 7.
- Press **NEXT** $\boxed{[f5]}$ and **SOLVE** $\boxed{[f5]}$ to find a solution to the equation $E1 = E2$ for X .

Note that **Method 4** gives a decimal approximation with 14 significant digits.

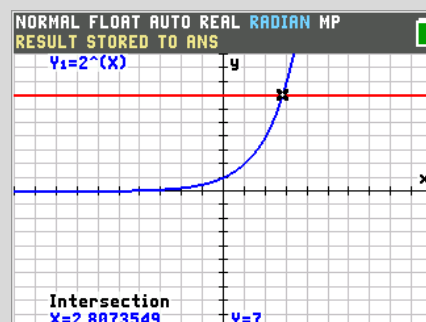
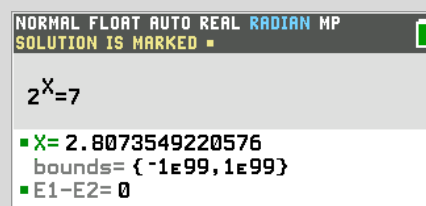
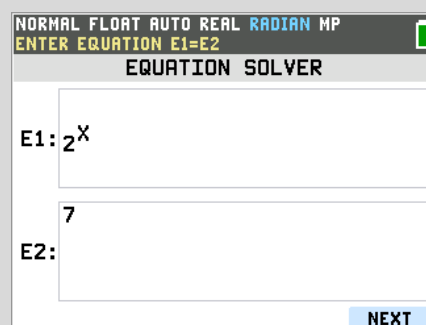
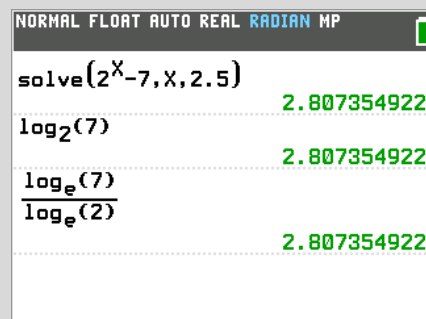
Note: There is at most one solution to the equation $a^x = b$, ($a \in \mathbb{R}^+, b \in \mathbb{R}$) so **nsolve** will find this if it exists.

Method 5: Use function graphs to find the intersection between two equations $y = 2^x$ and $y = 7$, or find the x -intercept of the equation $y = 2^x - 7$.

As an example, to find the intersection point between the graph of $y = 2^x$, and $y = 7$;

- Press $\boxed{y=}$
- Enter the equation **Y1** = 2^X .
- Enter the equation **Y1** = 7.
- Press $\boxed{2\text{nd}}\boxed{[\text{calc}]} > \mathbf{intersect}$.
- For **Guess**, enter 2.8.

Answer: All the above methods confirm that the solution to the given equation is approximately $x = 2.8073549$.



5.1.2. Calculus of logarithmic functions

Analysing the inverse relationship between exponential and logarithmic functions

Question

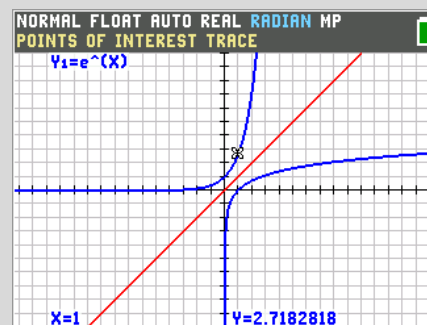
Draw the graph of $f(x) = e^x$ and then draw its inverse.

Solution

To graph $f(x) = e^x$ and its inverse:

- Press $\boxed{y=}$ and enter the equation $Y1 = e^X$.
- Enter the equation $Y2 = X$.
- Press $\boxed{\text{zoom}} > \text{Zoom} [-10,10]$.
- Press $\boxed{\text{zoom}} > \text{Zoom Square}$ to redraw with equal scale on both axes.
- Press $\boxed{2\text{nd}}\boxed{\text{draw}} > \text{DrawInv}$.
- Complete the command **DrawInv** $Y1$ ($Y1$ can be entered via $\boxed{\alpha}\boxed{\text{f4}} > Y1$).

Both the graphs of the function $f(x) = e^x$ and its inverse are displayed. The graph of the $y = x$ is added as well, as it illustrates that inverse graphs are reflections in this line.



*Note: The **DrawInv** command will display the inverse graphs, even if the inverse graph is not a function.*

Modelling with exponential and logarithmic functions

Question

The population of a particular species of fish in a lake is declining due to a chemical spill, which can be modelled by the logarithmic function $P(t) = a \log_e t + b$, $1 \leq t \leq c$.

Two weeks after the date of the chemical spill, the population was 11,500 and after 4 weeks the population was 8,500.

- Find the value of a and the value of b .
- Find the value of c (correct to one decimal place), which is the time at which the fish population is completely gone.
- Sketch the graph of this model in a suitable window.

Solution

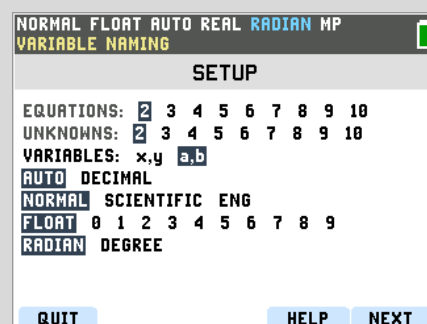
- We know $P(2) = 11500$ and $P(4) = 8500$ so:

$$P(2) = 11500 = a \ln(2) + b \Leftrightarrow \ln(2)a + b = 11500$$

$$P(4) = 8500 = a \ln(4) + b \Leftrightarrow \ln(4)a + b = 8500$$

Solve this pair of linear equations simultaneously as follows:

- Press $\boxed{\text{on}}\boxed{\text{2nd}}\boxed{\text{7}}$ and select the SYSTEM SOLVER application.
- On the setup page:
 - Set EQUATIONS = 2.
 - Set UNKNOWN = 2.
 - Set VARIABLES = **a,b**
- Press **NEXT** $\boxed{\text{f5}}$.



... continued

Solution (continued)

- In the template created, enter the two equations from the information provided.

$$\ln(2)a + b = 11500$$

$$\ln(4)a + b = 8500$$

- Press **SOLVE** [f5].

The result displays the solution for a and b respectively.

Answer: $a = -4328.085$, $b = 14\,500$.

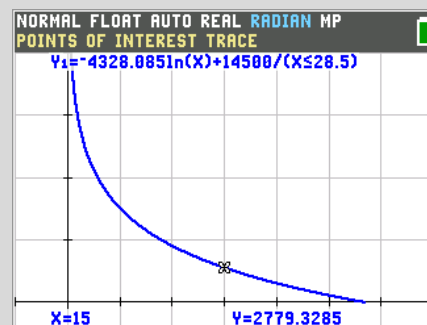
(b) To find the value of c , find the time when the fish population is predicted to have a value of zero, as follows:

- Press [math] > \mathbf{MATH} > \mathbf{Numeric Solver} \dots
- For E1, enter $-4328.085 \ln(X) + 14500$.
- For E2, enter 0.
- Press **NEXT** [f5] and **SOLVE** [f5] to find a solution to the equation $E1 = E2$ for X .

Answer: The model predicts the fish will be gone after approximately 28.5 weeks.

(c) Construct a graph of the population model as follows:

- Press [y=] and enter the equation $\mathbf{Y1=-4328.085 \ln(X)+14500/(X \leq 28.5)}$
- Move to **Y2** line, then press [math] > \mathbf{MATH} > \mathbf{nDeriv(}
- Press [alpha] > [f4] > **Y1** to enter **Y1**.
- Then complete the derivative template for **Y2** as shown.
- Press [window] and enter the following values:
 $X_{\min} = -5$ $X_{\max} = 35$ $X_{\text{scl}} = 5$
 $Y_{\min} = -2000$ $Y_{\max} = 20000$ $Y_{\text{scl}} = 5000$
- Press [graph] and then [trace].



Note: The “ $\leq$ ” symbol is found via the keys [2nd][=≤≠>].

Question

The shape of a cable hanging between two poles is modelled by $y(x) = 4(e^{\frac{x}{5}} + e^{-\frac{x}{5}})$ where x metres is the horizontal displacement from the lowest point of the cable and $y(x)$ metres is the height of the cable above the ground.

- Find the minimum height (above the ground) of the cable.
- Use calculus to find the exact gradient of the cable when $x = \ln_e 32$.
- The cable hangs between two poles, attached to the top of each pole at a height of 11 metres. How far apart are the poles, correct to the nearest centimetre?
- Plot the graph of $y(x)$ in a suitable window with the domain restricted by the distance between the two poles.

Solution

First enter the equation for $y(x)$ as follows:

- Press $\boxed{Y=}$ and enter the equation $Y1 = 4(e^{X/5} + e^{-X/5})$.
- (a) For the minimum height (occurs when $y'(x) = 0$):
- Press $\boxed{\text{on}} \boxed{\text{enter}}$ to select the CALCULATOR application.
- Press $\boxed{\text{math}} > \text{MATH} > \text{Numeric Solver ...}$
- For E1, press $\boxed{\text{math}} > \text{MATH} > \text{nDeriv(}$ and then complete the command $\frac{d}{dX}(Y1) \Big|_{X=X}$ as shown.
- For E2, enter 0.
- Press **NEXT** $\boxed{[f5]}$ and **SOLVE** $\boxed{[f5]}$ to find a solution to the equation $E1 = E2$ for X .

Answer: The model predicts minimum height is 8 metres.

- To find the gradient when $x = \ln(32)$:

- press $\boxed{\text{math}} > \text{MATH} > \text{nDeriv(}$ and then complete the command $\frac{d}{dX}(Y1) \Big|_{X=\ln(32)}$ as shown.

Answer: The gradient of the cable is 1.2.

- To find the horizontal distance between the poles:

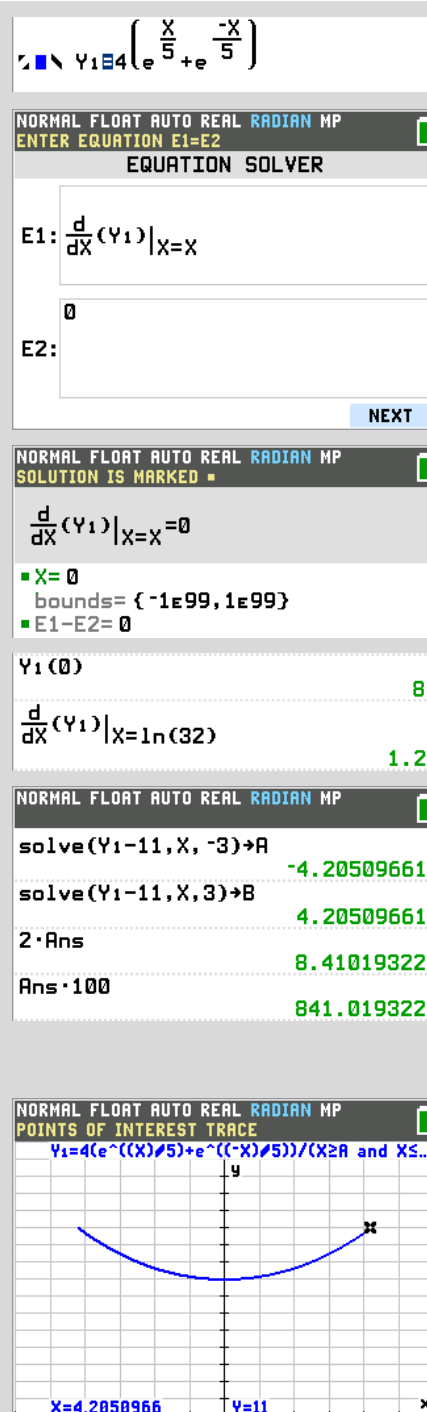
- Press $\boxed{2\text{nd}} \boxed{\text{catalog}} \boxed{[s]}$ and locate the solve command, and then, using a guess of $x = 3$, enter **solve(Y1-11, X, 3)**.
- repeat the above solve command with a guess of $x = -3$ **solve(Y1-11, X, -3)**.
- Multiply the answer by 2 (for the distance between poles).
- Multiple that answer by 100 (to convert to centimetres).

Answer: The distance between poles is approximately 841cm.

- Construct a graph of $y(x)$ as follows:

- Press $\boxed{Y=}$ and modify the equation as follows:
 $Y1 = 4(e^{X/5} + e^{-X/5}) / (X \geq A \text{ and } X \leq B)$.
- Press $\boxed{\text{window}}$ and enter the following values:
 $X_{\min} = -6$ $X_{\max} = 6$ $X_{\text{scl}} = 1$
 $Y_{\min} = -1$ $Y_{\max} = 15$ $Y_{\text{scl}} = 2$
- Press $\boxed{\text{graph}}$ and then $\boxed{\text{trace}}$.

Note: Inequality & “and” symbols are found via $\boxed{2\text{nd}} \boxed{[= < >]}$.



5.2. Topic 2: Differentiation of trigonometric functions & differentiation rules

5.2.1. Calculus of trigonometric functions

Modelling with trigonometric functions

Question

The concentration, C units, of a certain drug in the bloodstream t hours after taking the drug can be modelled by $C(t) = 0.4 - 0.4 \cos\left(\frac{\pi t}{4}\right)$, valid until all the drug is absorbed from the bloodstream.

- Find the time, in hours, after taking the drug at which C is at its maximum, and the maximum value of C .
- The drug is ineffective when $C < 0.4$. After the drug is taken, for how long is it effective?
- Let a be the time (in hours) between the first and second dose of the drug. Create graphs to explore the impact of various values of a on the effectiveness of the drug over the dosage time.

Solution

(a) The period and range of C are 8 hours and $[0, 0.8]$ respectively. To plot a graph of C in a suitable window (make sure calculator is RADIAN mode):

- Press $\boxed{Y=}$ and enter the equation as follows:

$$Y1 = 0.4 - 0.4 \cos\left(\frac{\pi X}{4}\right) / (X \geq 0 \text{ and } X \leq 8).$$
- Press $\boxed{\text{window}}$ and enter the following values:
 $X_{\min} = -1$ $X_{\max} = 10$ $X_{\text{scl}} = 1$
 $Y_{\min} = -0.1$ $Y_{\max} = 1.1$ $Y_{\text{scl}} = 0.2$
- Press $\boxed{\text{graph}}$ and then $\boxed{\text{trace}}$ to the local maximum.

Answer: The maximum value of C is 0.8, at $t = 4$ hours.

(b) To find the time when the drug is effective:

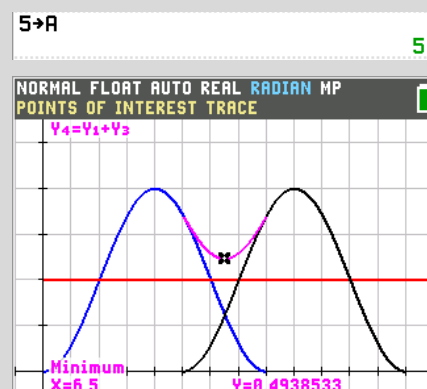
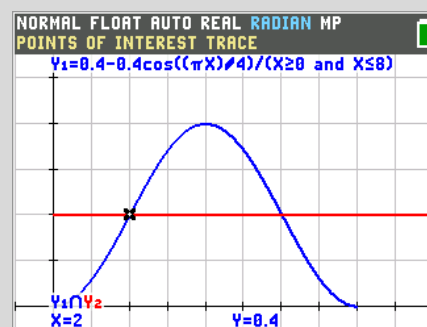
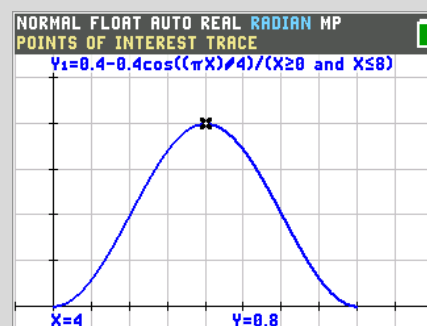
- Press $\boxed{Y=}$ and enter the $Y2 = 0.4 / (X \geq 0)$.
- Press $\boxed{\text{graph}}$ and then $\boxed{\text{trace}}$.
- Move the cursor to locate the points of intersection (displays the text $Y1 \cap Y2$ at such points).

Answer: The drug is effective for 4 hours (from $t = 2$ to $t = 6$).

(c) To explore the impact of varying a (time between doses):

- Enter $Y3 = 0.4 - 0.4 \cos\left(\frac{\pi(X - A)}{4}\right) / (X \geq A \text{ and } X \leq A + 8)$
- Enter $Y4 = Y1 + Y3$.

The function $Y4$ represents the combined effect of the two doses from $t = a$ to $t = 8$ hours. Change the value of a to view its effect on the effectiveness over the dosage time (for example, enter $5 \rightarrow A$ via the CALCULATOR application). As an example, in the screen right, if $a = 5$, the drug remains effective between $t = 2$ and $t = 11$ (a total of 9 hours).



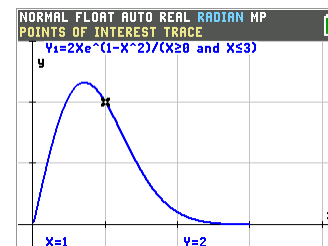
5.2.2. Differentiation rules

Analysing advanced functions

Question

The graph of the function $f(x) = 2xe^{1-x^2}$, where $0 \leq x \leq 3$, is shown right.

- Find the gradient of the tangent to $f(x)$ at $x = 1$.
- Find the obtuse angle that the tangent to $f(x)$ at $x = 1$ makes with the positive direction of the horizontal axis (to nearest degree).
- Find the value of p for which the tangent to $f(x)$ at $x = 1$ and the tangent to $f(x)$ at $x = p$ are perpendicular to each other (to 3 d.p.).
- Find the coordinates of the point where the tangents to the graph of $f(x)$ at $x = 1$ and $x = p$ intersect when they are perpendicular. Give your answer correct to two decimal places.



Solution

- Press $\boxed{\text{on}} \boxed{\text{enter}}$ to select the CALCULATOR application.
- Enter " $2Xe^{1-X^2} / (X \geq 0 \text{ and } X \leq 3)$ " $\rightarrow Y_1$.
- (a) For the gradient of the tangent to $f(x)$ at $x = 1$:
 - Press $\boxed{\text{math}} > \text{MATH} > \text{nDeriv(}$, and then complete the template as follows: $\frac{d}{dx}(Y_1)|_{X=1}$.

*Note: the **nDeriv** command can also be accessed via the shortcut $\boxed{\alpha} \boxed{f2}$.*

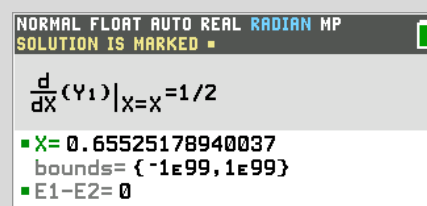
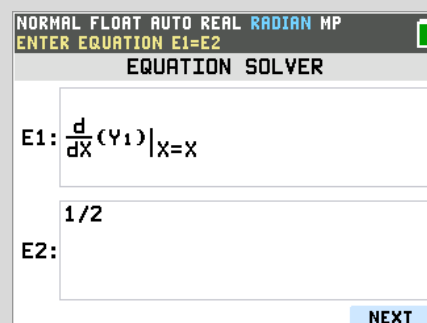
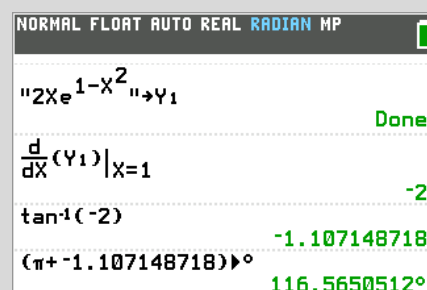
Answer: The gradient of the tangent at $x = 1$ is -2 .

- For the obtuse angle, use $m = \tan^{-1}(\theta)$ as follows:
 - Enter $\tan^{-1}(-2)$.
 - To find the obtuse angle, type $(\pi + \tan^{-1}(-2))$ but don't press $\boxed{\text{enter}}$.
 - Press $\boxed{2\text{nd}} \boxed{\text{angle}} \boxed{5}$ and then press $\boxed{\text{enter}}$ to convert the radian answer to decimal degrees.

Answer: The angle that the tangent at $x = 1$ makes with the positive direction of the horizontal axis is approximately 117 degrees.

- To find the value of p for which tangents at $x = 1$ and $x = p$ are perpendicular, use the fact that $m_2 = \frac{-1}{m_1}$ as follows:

- Press $\boxed{\text{math}} > \text{MATH} > \text{Numeric Solver ...}$
- For E1, press $\boxed{\text{math}} > \text{MATH} > \text{nDeriv(}$, and then complete the template as follows: $\frac{d}{dx}(Y_1)|_{X=X}$.
- For E2, enter $1/2$.
- For X (initial guess), enter 0 .
- Press **NEXT** $\boxed{f5}$ and **SOLVE** $\boxed{f5}$ to find a solution to the equation $E1 = E2$ for X.



... continued

Solution (continued)

(d) The equations of the tangents at $x = 1$ and $x = p$ can be found by hand and rearranged to be solved simultaneously:

$$y = -2x + 4 \Leftrightarrow 2x + y = 4$$

$$y = 0.5x + 1.99 \Leftrightarrow -0.5x + y = 1.99$$

Solve this pair of linear equations simultaneously as follows:

- Press **on** and select the SYSTEM SOLVER application.
- On the setup page:
 - Set EQUATIONS = 2.
 - Set UNKNOWN = 2.
 - Set VARIABLES = x,y.
- Press **NEXT** [f5].
- Enter the tangent equations in the form given above.
- Press **SOLVE** [f5].

The result displays the solution for x and y respectively.

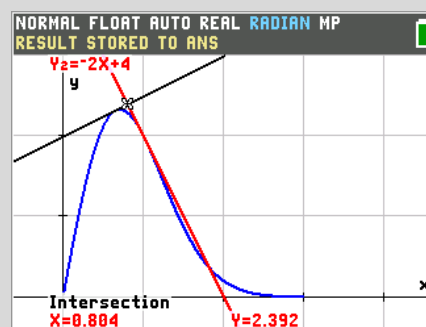
Answer: $x = 0.804$, $y = 2.392$. Round the answers to show the point of intersection of the tangents is at (0.80, 2.39).

NORMAL FLOAT AUTO REAL RADIAN MP			
SYSTEM SOLVER APP			
SYSTEM OF EQUATIONS			
2 x +	1 y =	4	
-0.5 x +	1 y =	1.99	

NORMAL FLOAT DEC REAL RADIAN MP			
SYSTEM SOLVER APP			
SOLUTION			
x =	0.804		
y =	2.392		

Note: Alternatively, the tangent lines and their equations can be found via a graphs page, and the point of intersection can be found using **2nd**[calc] (see right). Use **Zoom Square** to make the scales equal to highlight that the two tangents are perpendicular to each other.

NORMAL FLOAT AUTO REAL RADIAN MP			
[←] TO EXPAND			
PLOT1	PLOT2	PLOT3	
Y1=	2X	1-X ² / (X≥0 and X≤3)	
Y2=	-2X+4		
Y3=	0.5X+1.99		
Y4=			
Y5=			
Y6=			
Y7=			



5.3. Topic 3: Further applications of differentiation

5.3.1. The second derivative and applications of differentiation

Analysing the graphs of the first and second derivative

Question

Construct a graphing template to view the graph of $f(x) = 0.1(x-1)(x+2)(x-4)$ alongside the graphs of its first and second derivatives.

Solution

To enter equations for $f(x)$, as well as its first and second derivative:

- Press $\boxed{Y=}$ and enter the equation as follows:
 $Y1 = 0.1(X-1)(X+2)(X-4)$.
- Move to the line for $Y2$, and press $\boxed{\text{math}} > \text{MATH} > \text{nDeriv(}$, and then complete the template as follows:
 $Y2 = \frac{d}{dx}(Y1)|_{X=X}$.
- Move to the line for $Y3$, and press $\boxed{\text{math}} > \text{MATH} > \text{nDeriv(}$, and then complete the template as follows:
 $Y3 = \frac{d}{dx}(Y2)|_{X=X}$.
- Press $\boxed{\text{zoom}} > \text{Zoom} [-10,10]$.

The graphs of $y = f(x)$ and its associated first and second derivative graphs are now displayed in the standard viewing window.

To improve the viewing window:

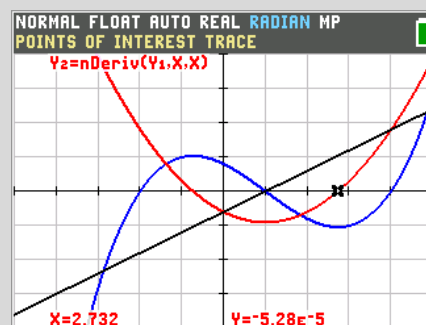
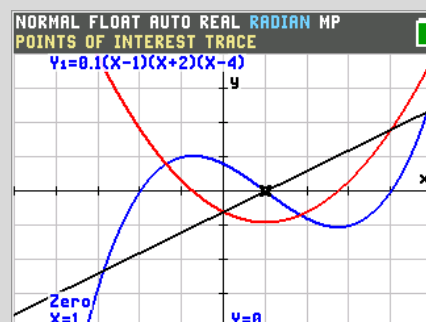
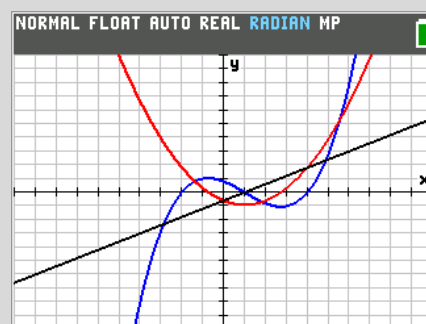
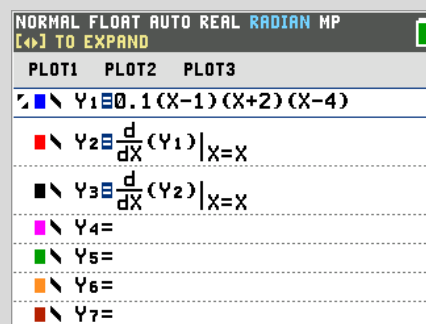
- Press $\boxed{\text{window}}$ and enter the following values:
 $X_{\min} = -5$ $X_{\max} = 5$ $X_{\text{scl}} = 1$
 $Y_{\min} = -4$ $Y_{\max} = 4$ $Y_{\text{scl}} = 1$

To trace the values of the function, first and second derivatives for a given x -value:

- Press $\boxed{\text{trace}}$.
- Use the up or down arrow key to jump to the three different graphs.
- Use the left or right arrow key to view these values for different values for x (or enter a new x value directly).

Explore how the values of the first and second derivatives are linked to function values for particular x -values.

Note: To display the grid lines, press $\boxed{2\text{nd}}\boxed{\text{format}}$ and set **GRID** to **LINE**.



5.4. Topic 4: Introduction to integration

5.4.1. Antidifferentiation

Solving kinematics problems with integrals

An ant is travelling along a straight path from point A . The ant's velocity (cm/min) is modelled by $v(t) = 2\sqrt{t}$, where t is time (in minutes) for $0 \leq t \leq 15$.

A spider passes point A 10 minutes after the ant and follows the ant's path. The spider moves with a constant acceleration of 2 cm/min^2 and catches up to the ant at $t = 15$ minutes.

Determine the spider's velocity at point A .

Define the velocity of the ant's as follows:

- Press $\boxed{\text{on}}\boxed{\wedge}\boxed{\text{enter}}$ to select the CALCULATOR application.
- Enter $"2\sqrt{X}" \rightarrow Y1$.

For the total displacement of the ant from A after 15 minutes:

- Press $\boxed{\text{math}} > \text{MATH} > \text{fnInt}($.
- Complete the definite integral as follows: $\int_0^{15} Y1 \, dX$.

Define the velocity of the spider as the antiderivative of the spider's acceleration 2 cm/min^2 , which is $2t + c$.

- Enter $"2X + C" \rightarrow Y2$.

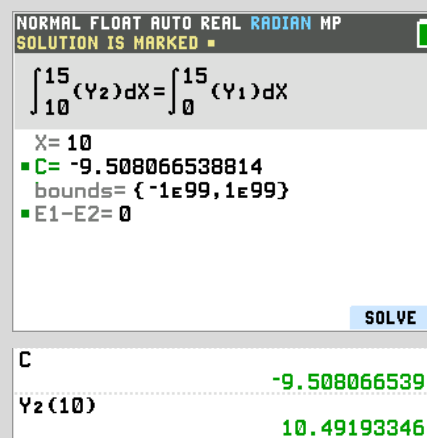
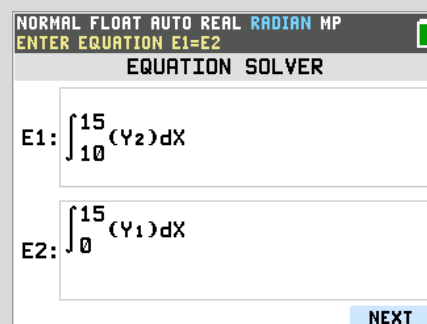
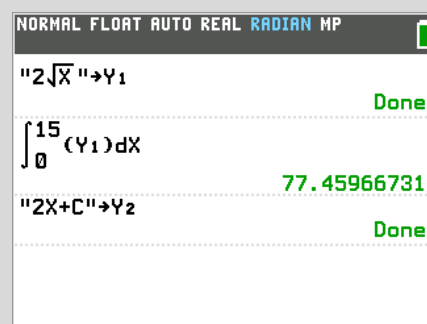
The total displacement of the spider from $t = 10$ to 15 minutes equals the total displacement of the ant in $t = 0$ to 15 minutes, so an equation can be solved for c as follows:

- Press $\boxed{\text{math}} > \text{MATH} > \text{Numeric Solver ...}$
- For E1, press $\boxed{\text{math}} > \text{MATH} > \text{nDeriv}($ and then complete the command $\int_{10}^{15} Y2 \, dX$ as shown.
- For E2, enter $\int_0^{15} Y1 \, dX$.
- Press **NEXT** [f5] and position the cursor on the line for C.
- Press **SOLVE** [f5] to find a solution to the equation $E1 = E2$ for c .

Now the spider velocity at A (at $t = 10$) can be calculated:

- Press $\boxed{2\text{nd}}\boxed{\text{quit}}$ to display the CALCULATOR application.
- Press $\boxed{C}\boxed{\text{enter}}$ to show that the previous solution has been stored as c .
- Enter **Y2(10)**.

Answer: The spider's velocity at A was approximately 10.5 cm/min.



5.5. Topic 5: Discrete random variables

5.5.1. General discrete random variables

Finding the mean and variance of a discrete random variable

Question

A discrete random variable X has the following probability distribution:

x	0	10	20	30	40
$\Pr(X = x)$	0.05	0.20	0.35	0.25	0.15

- (a) Calculate the value of $E(X)$ and $\text{Var}(X)$, and the bounds within which 95% of the values of X are expected to lie.
- (b) Create a graph of the distribution of X .

Solution

(a) Enter the values of X and the probabilities, and calculate statistics as follows:

- Press $\text{on}(\square)\text{enter}$ to select the CALCULATOR application.
- Enter $\{0,10,20,30,40\} \rightarrow \text{LX}$.
- Enter $\{0.05,0.2,0.35,0.25,0.15\} \rightarrow \text{LP}$.

Note: The small 'L' symbol (via $\text{2nd}[\text{list}] > \text{OPS} > \text{L}$) permits the user to name a list variable with more than one letter.

- Enter $\text{sum}(\text{LP})$ to check the values in prob sum to 1.
- To define $E(X)$, enter $\text{sum}(\text{LX} \cdot \text{LP}) \rightarrow \text{E}$.
- To define $\text{Var}(X)$, enter $\text{sum}(\text{LX}^2 \cdot \text{LP}) \rightarrow \text{V}$.
- To estimate the bounds for 95% of the values of X , enter the command $\{E - 2\sqrt{V}, E + 2\sqrt{V}\}$.

Note: The answers can be rounded to two decimal places by typing $\text{round}(\text{ans},2)$.

Answers: $E(X) = 22.5$; $\text{Var}(X) = 118.75$
Estimate of 95% bounds are $\{0.71, 44.29\}$.

Note: The value of $E(X)$ and $\text{Var}(X)$ can also be found by pressing $\text{stat} > \text{CALC} > \text{1-Var Stats}$, and then set $\text{LIST} = \text{LX}$ and $\text{FREQLIST} = \text{LP}$. See results screen right.

(b) To create a plot of the distribution of X :

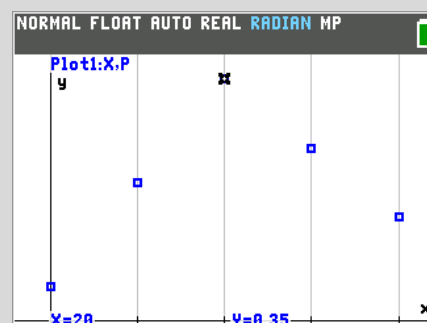
- Press $\text{2nd}[\text{plot}]$ and select **PLOT1**, and then edit the following settings.
 - Set **PLOT** = **ON**.
 - For **TYPE**, select the **Scatter Plot**.
 - For **XLIST**, press $\text{2nd}[\text{list}] > \text{NAMES} > \text{X}$.
 - For **YLIST**, press $\text{2nd}[\text{list}] > \text{NAMES} > \text{P}$.
 - For **MARK**, select the **hollow square mark**.
 - For **COLOR**, select **BLUE**.
- Press $\text{zoom} > \text{Zoom Stat}$ to view the scatterplot.

A plot of the distribution of X will be displayed.

HISTORY	
$\{0,10,20,30,40\} \rightarrow \text{LX}$	$\{0,10,20,30,40\}$
$\{0.05,0.20,0.35,0.25,0.15\} \rightarrow \text{LP}$	$\{0.05,0.2,0.35,0.25,0.15\}$
$\text{sum}(\text{LP})$	1
$\text{sum}(\text{LX} \cdot \text{LP}) \rightarrow \text{E}$	22.5
$\text{sum}(\text{LX}^2 \cdot \text{LP}) \rightarrow \text{V}$	118.75

NORMAL FLOAT AUTO REAL RADIAN MP	
$\{E - 2\sqrt{V}, E + 2\sqrt{V}\}$	$\{0.7055052823, 44.29449472\}$
$\text{round}(\text{Ans},2)$	$\{0.71, 44.29\}$

NORMAL FLOAT AUTO REAL RADIAN MP	
1-Var Stats	
$\bar{x} =$	22.5
$\Sigma x =$	22.5
$\Sigma x^2 =$	625
$Sx =$	
$\sigma x =$	10.89724736
$n =$	1
$\text{minX} =$	0
$Q_1 =$	15



5.5.2. Bernoulli distributions

Finding $E(X)$ from $SD(X)$

Question

Let X be a Bernoulli random variable. If $SD(X) = 2/5$, find the possible values of $E(X)$.

Solution

$SD(X) = \sqrt{p(1-p)} = 2/5$, so $\text{Var}(X) = p(1-p) = (2/5)^2$.

To solve for p numerically with an initial estimate of $p = 0$:
(returns only single result):

- Enter **solve(P(1-P) - (2/5)², 0)**.

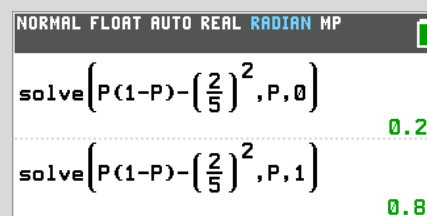
To solve for p numerically with an initial estimate of $p = 1$:
(returns only single result):

- Enter **solve(P(1-P) - (2/5)², 1)**.

Answer: Since $E(X) = p$, $E(X) = 1/5$ or $4/5$.

Note: The **solve** command can be accessed via **[2nd][catalog]**.

The command is solving the equivalent equation $p(1-p) - (2/5)^2 = 0$.



5.5.3. Binomial distributions

Visualising the distribution of a binomial random variable

Question

Create commands to list and graph the distribution of a binomial random variable.

Solution

Create commands to generate values for a binomial random variable with a given n and p value.

- Press **[on][enter]** to select the CALCULATOR application.

To generate a list of r values, enter the following command:

$$8 \rightarrow N : \frac{1}{4} \rightarrow P : \text{seq}(X, X, 0, N, 1) \rightarrow \text{L R.}$$

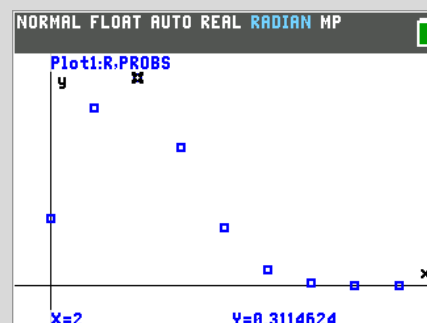
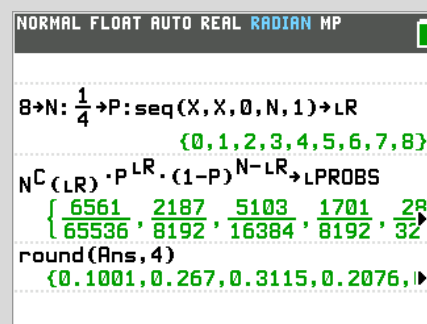
To generate a list of binomial probabilities, enter the following command:

$${}_N C_{L R} \times P^{L R} \times (1-P)^{N-L R} \rightarrow \text{L PROBS.}$$

- Enter the command **round (Ans,4)** round to 4 d.p..

To create a plot of the distribution of X :

- Press **[2nd][plot]** and select **PLOT1**, and then edit the following settings.
 - Set **PLOT = ON**.
 - For **TYPE**, select the **Scatter Plot**.
 - For **XLIST**, press **[2nd][list] > NAMES > R**.
 - For **YLIST**, press **[2nd][list] > NAMES > PROBS**.
 - For **MARK**, select the **hollow square mark**.
 - For **COLOR**, select **BLUE**.
- Press **[zoom] > Zoom Stat** to view a plot of the distribution of X will be displayed.



Solving binomial random variable problems

Question

A salesperson has a 30% probability of making a sale to each customer who enters the store. Each sale is independent of all other sales. Determine the:

- probability that the number of sales exceeds 20 on a day when 40 customers enter the store.
- mean & standard deviation of the number of sales on a day when 40 customers enter the store.
- number of sales expected on 95% of days when 40 customers enter the store.
- minimum number of customers who would have to enter the store to have a 90% chance or more of making at least one sale.

Solution

For parts (a) to (c), X is the number of sales on a day when 40 customers enter the store, so $X \sim Bi(n = 40, p = 0.3)$

(a) $P(X > 20)$ can be found using the following command:

- Press $\boxed{\text{on}} \boxed{\text{enter}}$ to select the CALCULATOR application.
- Press $\boxed{\alpha} \boxed{\text{distr}} > \text{Binomial} > \text{binomcdf}(\cdot)$.
- Complete the command **binomcdf(N,P,21,40)**.

Answer: $P(X > 20) \approx 0.0024$.

(b) The mean number of sales can be found using the formula $E(X) = np$, and the standard deviation can be found using the formula $SD(X) = \sqrt{np(1-p)}$.

- To store $E(X)$ as **E**, enter **NP→E**.
- To store $SD(X)$ as **S**, $\sqrt{NP(1-P)} \rightarrow S$.

Answer: $E(X) = 12$; $SD(X) \approx 2.898$.

(c) Assuming that the distribution of sales is reasonably symmetric about the mean sales (see plot shown on the screen right), it can be estimated that on 95% of days when 40 customers enter the store, the number of sales will be on the interval $[E(X) - 2 \times SD(X), E(X) + 2 \times SD(X)]$.

- To calculate the 95% bounds for X rounded to nearest whole numbers, enter **round({E-2*S,E+2*S},0)**.

Answer: The 95% bounds for sales are {6,18}.

(d) To find the minimum number of customers who would have to enter the store for there to be a 90% chance of at least one sale, let $X \sim Bi(\text{custno}, p = 0.3)$ and $P(X \geq 1) \geq 0.9$.

To find the minimum number of customers (by guess, check and improve):

- Enter **binomcdf(custno,0.3,1,custno)|custno=5**.
- Increment **custno** until the probability exceeds 0.9.

Answer: The minimum number of customers is 7.

HISTORY	
40→N:0.3→P	0.3
binomcdf(N,P,21,40)	0.0024193599
NP→E	12
$\sqrt{NP(1-P)} \rightarrow S$	2.898275349

HISTORY	
40→N:0.3→P	0.3
binomcdf(N,P,21,40)	0.0024193599
NP→E	12
$\sqrt{NP(1-P)} \rightarrow S$	2.898275349

round({E-2*S,E+2*S},0)	{6,18}
------------------------	--------

NORMAL FLOAT AUTO REAL RADIAN MP	
5→N:binomcdf(N,P,1,N)	0.83193
6→N:binomcdf(N,P,1,N)	0.882351
7→N:binomcdf(N,P,1,N)	0.9176457

Unit 4: Further calculus, trigonometry and statistics

6.1. Topic 1: Further integration

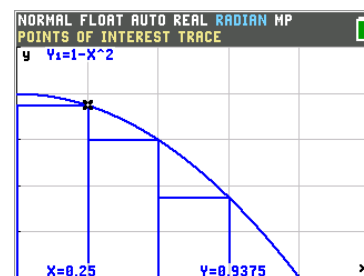
6.1.1. Fundamental theorem of calculus and definite integrals

Using sums of the form $\sum_i f(x_i) \delta x_i$ to estimate the area under the curve $y=f(x)$

Question

Consider the region bounded by the curve $f(x) = 1 - x^2$ and the coordinates axes.

Estimate the area of this region using sums of the form $\sum_i f(x_i) \delta x_i$ where $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$ and $x_4 = 1$.



Solution

Let A denote the area of the region.

- Press $\boxed{\text{on}} \boxed{\text{enter}}$ to select the CALCULATOR application.
- Enter the function " $1 - X^2$ " $\rightarrow$ Y1.

The sum of the rectangles (an estimate for A) is given by:

$$\begin{aligned} A &\approx \sum_{i=1}^4 f(x_i) \delta x_i \\ &= f(0.25) \times 0.25 + f(0.5) \times 0.25 + f(0.75) \times 0.25 + f(1) \times 0.25 \\ &= 0.25(f(0.25) + f(0.5) + f(0.75) + f(1)) \\ &= 0.53125 \end{aligned}$$

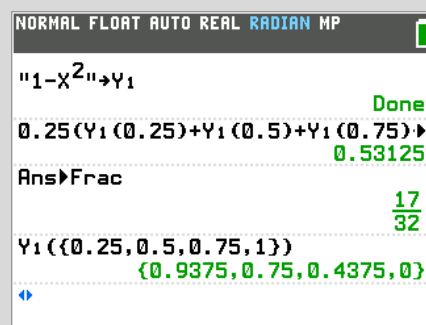
- Enter either the line 2 expression or the line 3 expression from above. The line 3 expression is shown at right.

To represent the estimate as a fraction:

- Press $\boxed{2\text{nd}} \boxed{\text{Ans}} > \boxed{\text{math}} > \text{MATH} > \boxed{\text{Frac}}$ (or press $\boxed{\text{Frac}}$).

As a fraction, an estimate for A using these rectangles is $\frac{17}{32}$ square units.

Note: If intermediate function values are required, enter as shown on the fourth entry line of the screenshot at right.



Estimating areas: Brief background

The area under a curve of a continuous nonnegative function $f(x)$ between $x = a$ and $x = b$ can be approximated with n rectangles and $\delta x = \frac{b-a}{n}$.

The left endpoint approximation, L_n , is given by $L_n = \sum_{i=1}^n f(a + (i-1)\delta x) \delta x$.

The right endpoint approximation, R_n , is given by $R_n = \sum_{i=1}^n f(a + i\delta x) \delta x$.

In the previous example, a right endpoint approximation, R_4 , was used.

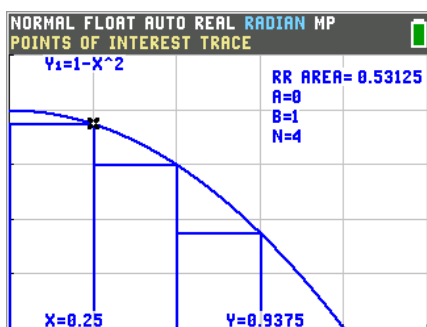
In that example, $R_4 = 0.53125 < A$ as the function f is decreasing on the interval $[0,1]$.

When the number of rectangles (of equal width) is increased and a smaller value for δx (rectangles of smaller width) is used, a better approximation to the area A is obtained.

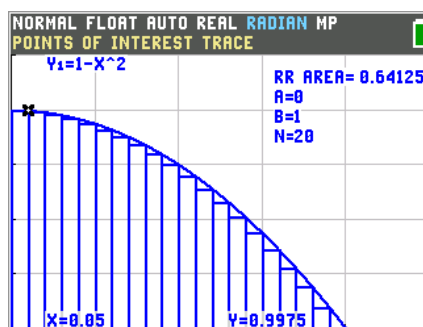
In the next example, the number of rectangles is increased.

The **right-endpoint approximation**, R_n , of the area bounded by the curve $f(x) = 1 - x^2$ and the coordinate axes with n rectangles is $\sum_{i=1}^n \left(1 - \left(\frac{i}{n}\right)^2\right) \left(\frac{1}{n}\right)$.

Here is a representation of R_4 .

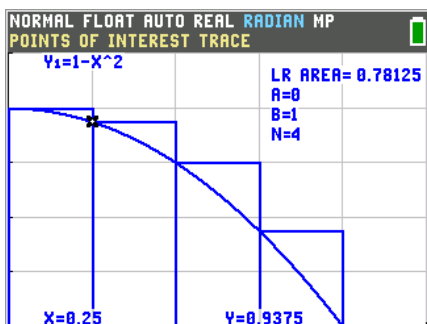


Here is a representation of R_{20} .

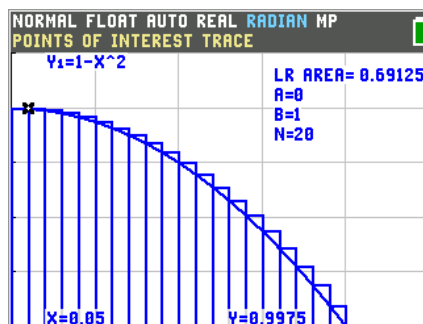


The **left-endpoint approximation**, L_n , of the area bounded by the curve $f(x) = 1 - x^2$ and the coordinate axes with n rectangles is $\sum_{i=1}^n \left(1 - \left(\frac{i-1}{n}\right)^2\right) \left(\frac{1}{n}\right)$.

Here is a representation of L_4 .



Here is a representation of L_{20} .



In this example, $L_4 = 0.78125 > A$ as the function f is decreasing on the interval $[0,1]$.

Recognising the definite integral as a limit of sums of the form $\sum_i f(x_i) \delta x_i$

Question

Consider the region bounded the curve $f(x) = 1 - x^2$ and the coordinates axes.

(a) Estimate the area of this region using sums of the form $\sum_i f(x_i) \delta x_i$ where

(i) $x_1 = 0.1, x_2 = 0.2, \dots, x_9 = 0.9$ and $x_{10} = 1$.

(ii) $x_1 = 0.01, x_2 = 0.02, x_{99} = 0.99$ and $x_{100} = 1$.

(iii) $x_1 = 0.001, x_2 = 0.002, x_{999} = 0.999$ and $x_{1000} = 1$.

(b) Find $\int_0^1 (1 - x^2) dx$ and compare this value with the estimates found in part (a).

Solution

(a) Let A denote the area of the region. $A \approx \sum_{i=1}^n f(x_i) \delta x_i$.

- Press $\boxed{\text{on}} \boxed{\text{enter}}$ to select the CALCULATOR application.
- Enter the function " $1 - X^2$ " $\rightarrow$ Y1.

To calculate this sum:

- Press $\boxed{\text{math}} > \text{MATH} > \text{summation}$.
- Complete the summation template as shown right.

The following estimates are obtained:

(i) With $n = 10$, $R_{10} = 0.615$.

(ii) With $n = 100$, $R_{100} = 0.66165$.

(iii) With $n = 1000$, $R_{1000} = 0.6661665$

*Note: If required, press $\boxed{\text{on}} \boxed{\text{on}}$, select **Settings** and then select **Document Settings**. Change Display Digits to Float.*

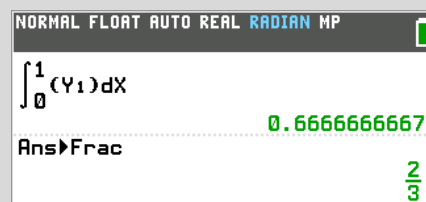
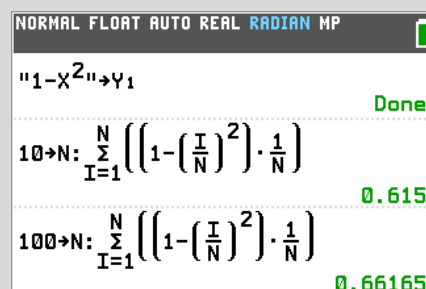
(b) The required region lies entirely above the x -axis and can therefore be calculated using a single definite integral.

- Press $\boxed{\text{on}} \boxed{\text{enter}}$ to select the CALCULATOR application.
- Press $\boxed{\text{math}} < \text{MATH} < \text{fnInt}(\text{.}$
- Complete the integral template as shown right.

Answer: So $\int_0^1 (1 - x^2) dx = \frac{2}{3}$.

From part (a), $\lim_{n \rightarrow \infty} R_n = A = \frac{2}{3}$.

The exact area is equal to the limit as $n \rightarrow \infty$ of R_n (or L_n).



Visualising the area under the curve $y=f(x)$ using right endpoint approximation

Question

Consider the region bounded by the curve $f(x) = 1 - x^2$ and the coordinates axes (as per the previous question). Write a program in TI-Basic to visualise and calculate the area under the curve

using the right endpoint approximation. Test the program for $a = 0$, $b = 1$, and $n = 4$. Then experiment with different n values, and come up with an estimate of the actual area under the curve.

Solution

- Press $\boxed{\text{on}}\boxed{\text{enter}}$ to select the CALCULATOR application.
- Enter the function " $1 - X^2$ " $\rightarrow$ Y1.

To set the viewing window:

- Press $\boxed{\text{window}}$ and enter the following values:
 $X_{\min} = 0$ $X_{\max} = 1.25$ $X_{\text{scl}} = 0.25$
 $Y_{\min} = 0$ $Y_{\max} = 1.25$ $Y_{\text{scl}} = 0.25$
- Press $\boxed{\text{graph}}$ to view the graph.

To construct a program in **TI-Basic** to visualise and calculate the area under the curve using the right endpoint approximation:

- Press $\boxed{\text{prgm}} > \text{TI-Basic} > \text{New}$.
- Enter the program title **RIGHTREC**.
- Enter the commands exactly as shown right.

Key sequences for entering selected program commands:

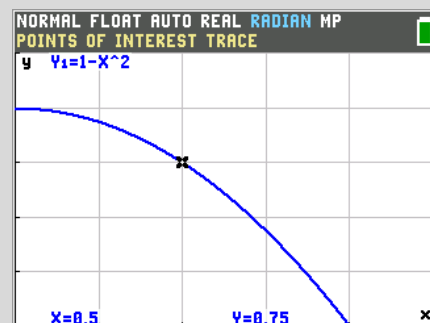
- For **ClrHome**, enter $\boxed{\text{prgm}} > \text{I/O} > \text{ClrHome}$.
- For **ClrDraw**, enter $\boxed{2\text{nd}}\boxed{\text{draw}} > \text{ClrDraw}$.
- For **Prompt**, enter $\boxed{\text{prgm}} > \text{I/O} > \text{Prompt}$.
- For the **For** loop, enter $\boxed{\text{prgm}} > \text{CTL} > \text{For}(\text{.}$
- For line drawing command to constructing the rectangles, enter $\boxed{2\text{nd}}\boxed{\text{draw}} > \text{DRAW} > \text{Line}(\text{.}$
- For **Y1**, enter $\boxed{\text{alpha}}\boxed{\text{f5}}\boxed{1}$.
- For **DispGraph**, press $\boxed{\text{prgm}} > \text{I/O} > \text{DispGraph}$.
- For **Text**, enter $\boxed{2\text{nd}}\boxed{\text{draw}} > \text{DRAW} > \text{Text}(\text{.}$
- For **Pause**, enter $\boxed{\text{prgm}} > \text{CTL} > \text{Pause}$.

To execute the program **RIGHTREC** and test it by drawing rectangles representing the right endpoint approximation for $a = 0$, $b = 1$, and $n = 4$, complete the following steps:

- Execute the program by pressing
 - $\boxed{\text{alpha}}\boxed{\text{f5}}$ (if inside the program code) or
 - $\boxed{\text{prgm}} > \text{TI-Basic} > \text{EXEC} > \text{RIGHTREC}$ (if outside the program)
- For a , enter $A = 0$.
- For b , enter $B = 1$.
- For n , enter $N = 4$.

Answer: For $n = 4$, the area using right endpoint approximation is 0.53125 (as found previously). Larger values of n (e.g. $n = 100$, $n = 1000$) suggest that the area under the curve approaches a value of $2/3$ square units.

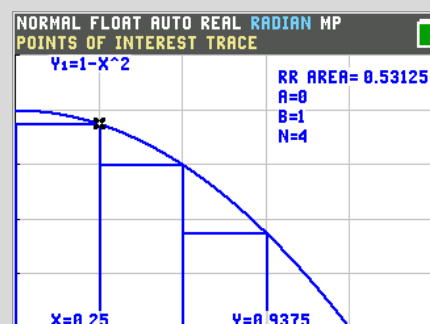
NORMAL FLOAT AUTO REAL RADIAN MP
 "1-X^2"→Y1
 Done



NORMAL FLOAT AUTO REAL RADIAN MP
 EDIT MENU: [alpha] [f5] [F5]
 PROGRAM: RIGHTREC

```

:ClrHome
:ClrDraw
:Disp "RIGHT RECT AREA"
:Prompt A,B,N
:(B-A)/N→H
:A→D
:A+H→E
:0→R
:For(I,1,N,1)
:Line(D,0,D,Y1(E))
:Line(D,Y1(E),E,Y1(E))
:Line(E,Y1(E),E,0)
:R+H·Y1(E)→R
:D+H→D
:E+H→E
:End
:DispGraph
:Text(10,200,"RR AREA=")
:Text(10,265,R)
:Text(25,200,"A=")
:Text(25,215,A)
:Text(40,200,"B=")
:Text(40,215,B)
:Text(55,200,"N=")
:Text(55,215,N)
:Pause
:ClrHome
  
```

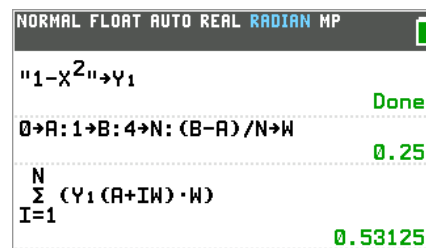


Estimating areas: A general result

The right endpoint approximation, R_n , can be used to approximate an area under a curve as follows:

$$R_n = \sum_{i=1}^n f(a + iw)w \text{ where } w = \frac{b-a}{n}.$$

- Enter the function $"1 - X^2" \rightarrow Y1$.
- Enter values for a, b, n and w as follows: $0 \rightarrow A : 1 \rightarrow B : 4 \rightarrow N : (B - A) / N \rightarrow W$.
- Press $\boxed{\text{math}} > \mathbf{MATH} > \mathbf{\text{summation}}$.
- Complete the summation template as shown right.



This mirrors the method for calculating R_n used in the TI-Basic program in the previous example.

Finding the area under the curve $y = f(x)$ between $x = a$ and $x = b$ if $f(x) > 0$ over this interval

Question

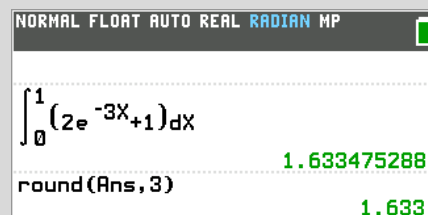
Find the area of the region enclosed by the curve of the equation $y = 2e^{-3x} + 1$, the coordinate axes and the line $x = 1$. Give your answer correct to three decimal places.

Solution

The required region lies entirely above the x -axis and can therefore be calculated using a single definite integral.

- Press $\boxed{\text{on}} \boxed{\text{enter}}$ to select the CALCULATOR application.
- Press $\boxed{\text{math}} < \mathbf{MATH} < \mathbf{fnInt(}$.
- Complete the integral template as shown right.

Answer: $\int_0^1 (2e^{-3x} + 1) dx = 1.633$ (sq. units) (correct to 3 d.p.).



Note: The answer can be rounded to three decimal places by typing **round(ans,3)**. The round command can be found via $\boxed{\text{math}} > \mathbf{NUM} > \mathbf{round(}$.

Alternatively the area can be found graphically:

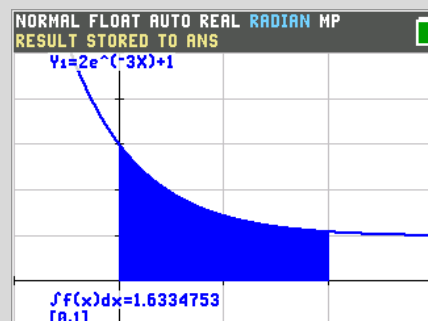
- Press $\boxed{y=}$ and enter $Y1 = 2e^{-3x} + 1$.

To set the viewing window:

- Press $\boxed{\text{window}}$ and enter the following values:
 $X_{\min} = -0.5$ $X_{\max} = 1.5$ $X_{\text{scl}} = 0.5$
 $Y_{\min} = -1$ $Y_{\max} = 5$ $Y_{\text{scl}} = 1$
- Press $\boxed{\text{graph}}$, then to select the integral tool, press $\boxed{2\text{nd}} \boxed{\text{calc}} \boxed{7}$.
- Type 0 for the **Lower Limit** and press $\boxed{\text{enter}}$.
- Type 1 for the **Upper Limit** and press $\boxed{\text{enter}}$.

The required region is shaded and the area displayed.

Answer: The area is 1.633 (sq. units) (correct to 3 d.p.).



6.1.2. Applications of integration

Calculating the area enclosed by a curve and the x-axis over a given domain

Question

Find the area of the region enclosed by the curve $y = \cos(2x)$, the x-axis and the lines

$x = 0$ and $x = \frac{2\pi}{3}$. Give your answer correct to two decimal places.

Solution

The area is made up of one region above the x-axis and one region below the x-axis. The absolute value function can be used to reflect in the x-axis any points that with a negative y-coordinate.

Graphical method

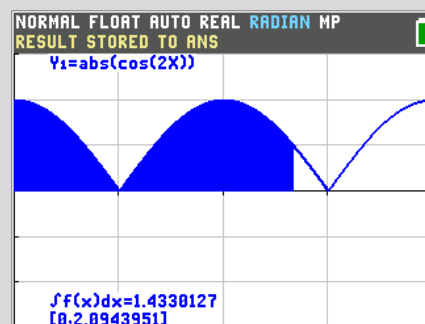
- Press $\boxed{Y=}$ and position the cursor on Y1 line then press $\boxed{\text{math}} > \text{NUM} > \text{abs}(\text{)}$.
- Complete the absolute value template to enter the function $Y1 = |\cos(2X)|$.



Note: A shortcut to the absolute value template is $\boxed{\alpha}[\text{f2}]$.

To set the viewing window:

- Press $\boxed{\text{window}}$ and enter the following values:
Xmin = 0 Xmax = π Xscl = $\pi/4$
Ymin = -1.5 Ymax = 1.5 Yscl = 0.5
- Press $\boxed{\text{graph}}$ to view the curve.
- Press $\boxed{2\text{nd}}[\text{calc}][7]$ to select the integral tool.
- Type 0 for the **Lower Limit** and press $\boxed{\text{enter}}$.
- Type $\frac{2\pi}{3}$ for the **Upper Limit** and press $\boxed{\text{enter}}$.



The required region is shaded and the area displayed.

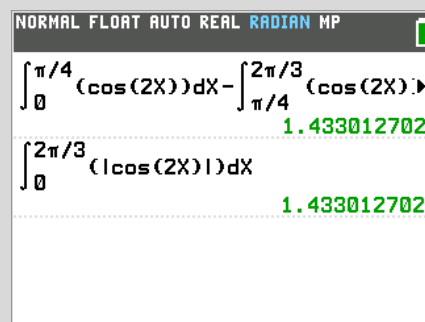
Answer: The area is 1.433 (sq. units) (correct to 3 d.p.).

Numeric methods

- Press $\boxed{\text{on}}[\text{enter}]$ to select the CALCULATOR application.
- Press $\boxed{\text{math}} > \text{MATH} > \text{fnInt}(\text{)}$.
- Complete the integral template as shown below and right.

Either: $\int_0^{\pi/4} (\cos(2x)) dx - \int_{\pi/4}^{2\pi/3} (\cos(2x)) dx = 1.43$ (to 2 d.p.).

Or: $\int_0^{2\pi/3} (|\cos(2x)|) dx = 1.43$ (correct to 2 decimal places).



Note: A shortcut to $\text{fnInt}()$ is via $\boxed{\alpha}[\text{f2}]$.

Calculating the area between curves

Question

Find the area of the region enclosed by the line $y = 2x$ and the curve $y = \frac{1}{2}x^2$ (to 2 d.p.)

Solution

First view the graphs of these equations.

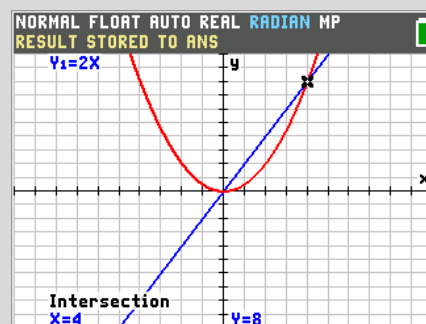
- Press $\boxed{y=}$ and enter the following equations:
 - $Y1 = 2X$
 - $Y2 = \frac{1}{2}X^2$
- Press $\boxed{\text{zoom}} > \text{ZOOM} > \text{Zoom} [-10,10]$.
- Press $\boxed{2\text{nd}}\boxed{\text{calc}} > \text{intersect}$ and follow the prompts.

The two graphs intersect at (0,0) and (4,8).

- Press $\boxed{\text{on}}\boxed{\text{enter}}$ to select the CALCULATOR application.
- Press $\boxed{\text{math}} > \text{MATH} > \text{fnInt}()$.
- Complete the integral template as shown right.

Answer: $\int_0^4 \left(2x - \frac{1}{2}x^2 \right) dx = 5.33$ (correct to 2 decimal places).

Note: The x-coordinates of the points of intersection can also be found with the POLYNOMIAL ROOT FINDER application, by solving the polynomial equation $\frac{1}{2}x^2 - 2x = 0$ for x . The screens shown right show how this might be done.



Calculator screen showing the integral calculation:

$$\int_0^4 (2x - x^2/2) dx$$

The result is displayed as 5.333333333.

Calculator screen showing the Polynomial Root Finder application:

POLYNOMIAL - ORDER 2

$\frac{1}{2}x^2 - 2x + 0 = 0$

Press $[+]$ or $[-]$

Buttons: QUIT, SETUP, RESET, LOAD, SOLVE

Calculator screen showing the Polynomial Root Finder application results:

POLYNOMIAL ROOT FINDER APP

$\frac{1}{2}x^2 - 2x + 0 = 0$

$x1 = 0$

$x2 = 4$

Buttons: QUIT, SETUP, COEFF, STORE, $\leftrightarrow$

Using the trapezoidal rule to approximate an area

Question

Use the trapezoidal rule with four strips to find an approximate value for $\int_1^2 \log_{10}(x) dx$.

Solution

Let A be the area calculated using the trapezoid rule approximation as follows:

$$A = \int_a^b f(x) dx \approx \frac{w}{2} [f(x_0) + 2(f(x_1) + f(x_2) + \dots + f(x_{n-1})) + f(x_n)] \text{ where } w = \frac{b-a}{n}.$$

$$\text{Here, } \int_1^2 \log_{10}(x) dx \approx \frac{2-1}{2 \times 4} [f(1) + 2(f(1.25) + f(1.5) + f(1.75)) + f(2)] \approx 1.667.$$

The **TI-Basic** program uses an alternative form for the trapezoidal rule.

$$A = \int_a^b f(x) dx \approx w \left[\frac{f(x_0) + f(x_n)}{2} + \sum_{i=1}^{n-1} f(x_i) \right].$$

To construct a program in **TI-Basic** to visualise and calculate the area under the curve using the right endpoint approximation:

- Press **[prgm]** > **TI-Basic** > **New**.
- Enter the program title **TRAPRULE**.
- Enter the commands exactly as shown right.

Key sequences for entering selected program commands:

- For **ClrHome**, enter **[prgm]** > **I/O** > **ClrHome**.
- For **ClrDraw**, enter **[2nd]****[draw]** > **ClrDraw**.
- For **Input**, enter **[prgm]** > **CTL** > **Input**.
- For **Prompt**, enter **[prgm]** > **I/O** > **Prompt**.
- For **Y1**, enter **[alpha]****[f5]** > **Y1**.
- For **Disp**, press **[prgm]** > **I/O** > **Disp**.

Now execute the program **TRAPRULE** and use the trapezoidal rule with four strips to find an approximate value

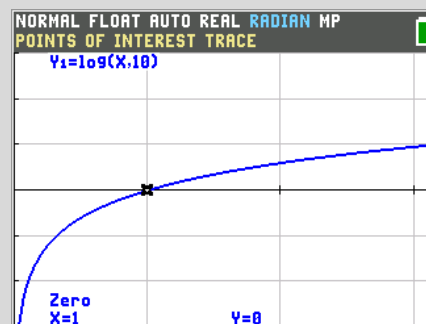
for $\int_1^2 \log_{10}(x) dx$:

- Execute the program by pressing
 - **[alpha]****[f5]** (if inside the program code) or
 - **[prgm]** > **TI-Basic** > **EXEC** > **TRAPRULE** (if outside the program) .
- For equation, enter **"log(X,10)"** .
- For a , enter $A = 1$.
- For b , enter $B = 1$.
- For n , enter $N = 4$.

Answer: Area = 0.1667 (correct to 3 decimal places)

Note: As this curve is concave down, the trapezoidal rule underestimates the integral.

$\int_1^2 \log_{10}(x) dx = 0.167$, correct to 3 decimal places.



```

NORMAL FLOAT AUTO REAL RADIAN MP
EDIT MENU: [alpha] [f5]
PROGRAM: TRAPRULE

:ClrHome
:Disp "TRAP RULE APPROX"
:Input "EQN=",Y1
:Prompt A,B,N
:(B-A)/N→W
:W•((Y1(A)+Y1(B))/2+Σ(Y1(A+IW),
I,1,N-1)→T
:Disp "TRAP AREA="
:Disp T
  
```

```

NORMAL FLOAT AUTO REAL RADIAN MP

TRAP RULE APPROX
EQN="log(X,10)"
A?1
B?2
N?4
TRAP AREA=
0.1666385796
Done
  
```

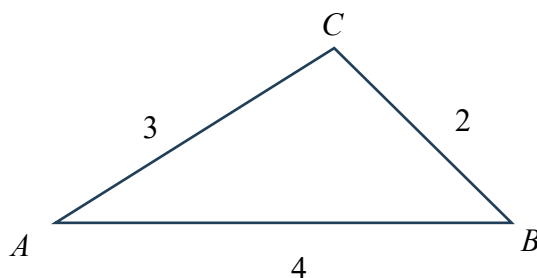
6.2. Topic 2: Trigonometry

6.2.1. Cosine and sine rules

Using the cosine rule

Question

For triangle ABC , find the size of the largest angle. Give your answer correct to the nearest minute.



Solution

The largest angle is always opposite the longest side so the angle at C is required.

Use the cosine rule as three side lengths are known.

Substituting into the cosine rule gives

$$4^2 = 2^2 + 3^2 - 2(2)(3)\cos C.$$

To solve for C (ensuring that calculator is in DEG mode):

- Press $\boxed{\text{math}}$ > **MATH** > **Numeric Solver ...**
- For E1, enter 4^2 .
- For E2, enter $2^2 + 3^2 - 2(2)(3)\cos(X)$.
- Press **NEXT** $\boxed{\text{f5}}$ and **SOLVE** $\boxed{\text{f5}}$ to find a solution to the equation $E1 = E2$ for X .

This gives the answer $X = 104.4775 \dots$ degrees. To convert this answer to degrees, minutes and seconds:

- Press $\boxed{\text{on}} \boxed{\text{2nd}}$ $\boxed{\text{enter}}$ to select the **CALCULATOR** application.
- Press $\boxed{\text{X,T,}\theta,\text{n}}$ and then press $\boxed{\text{enter}}$.
- Enter $\boxed{\text{2nd}}$ $\boxed{\text{angle}}$ > **DMS** to convert the previous answer to the required form.

Answer: Solving $4^2 = 2^2 + 3^2 - 2(2)(3)\cos C$ for C with $0^\circ < C < 180^\circ$ gives $C = 104^\circ 29'$, correct to the nearest minute.

Note: To change between **RAD** and **DEG** modes, press $\boxed{\text{mode}}$ and select the desired angle setting.

NORMAL FLOAT AUTO REAL DEGREE MP
ENTER EQUATION E1=E2

EQUATION SOLVER

E1: 4^2

E2: $2^2 + 3^2 - 2(2)(3)\cos(X)$

NEXT

NORMAL FLOAT AUTO REAL DEGREE MP
SOLUTION IS MARKED

$4^2 = 2^2 + 3^2 - 2(2)(3)\cos(X)$

X = 104.47751218593
bounds = {-1e99, 1e99}
E1-E2 = 0

SOLVE

NORMAL FLOAT AUTO REAL DEGREE MP

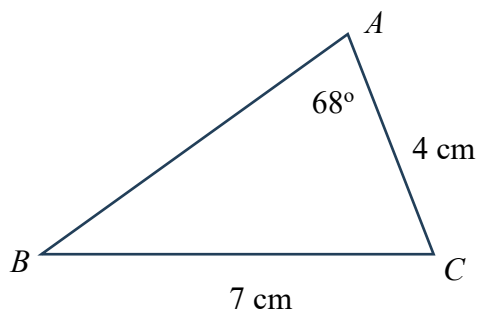
X 104.4775122

Ans $\rightarrow$ DMS 104°28'39.044"

Finding the area of a triangle

Question

Find the area of triangle ABC . Give your answer in cm^2 correct to one decimal place.



Solution

As the angle between the two side lengths needs to be calculated, use the sine rule to find angle B and hence angle C .

To solve for B (ensuring that calculator is in DEG mode):

- Press $\boxed{\text{math}} > \text{MATH} > \text{Numeric Solver ...}$
- For E1, enter $7/\sin(68)$.
- For E2, enter $4/\sin(X)$.
- Press **NEXT** $\boxed{\text{f5}}$ and set a starting value $X = 1$ (or any small positive value)
- Press **SOLVE** $\boxed{\text{f5}}$ to find a solution to the equation $E1 = E2$ for X .

Answer: Solving $\frac{7}{\sin(68^\circ)} = \frac{4}{\sin(B)}$ for B with

$0^\circ < B < 180^\circ$ gives $B = 31.993\dots^\circ$.

Note: $B \neq 148.006\dots^\circ$ as a triangle's angle sum is 180° .

Using $A + B + C = 180^\circ$,

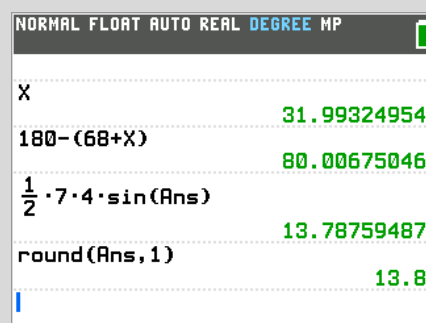
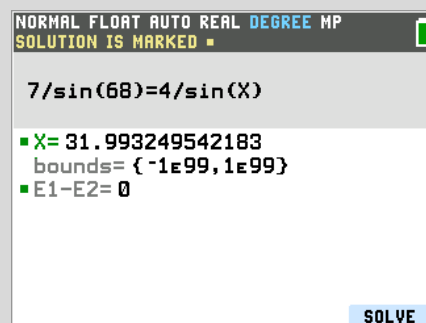
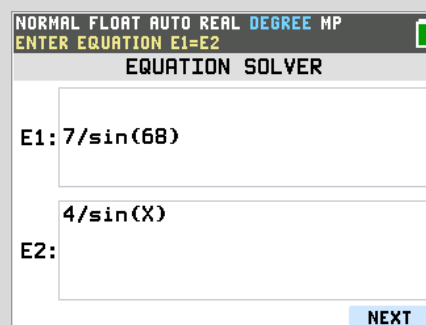
- Press $\boxed{\text{on}} \boxed{\text{enter}}$ to select the CALCULATOR application.
- Press $\boxed{\text{X,T,}\theta,\text{n}}$ and then press $\boxed{\text{enter}}$.
- Enter $C = 180^\circ - (68^\circ + 31.993\dots^\circ) = 80.006\dots^\circ$

Substituting into $A = \frac{1}{2}ab \sin C$ gives:

- Enter $A = \frac{1}{2} \times 7 \times 4 \sin(\text{Ans}) = 13.787\dots$

The area of the triangle is 13.8 cm^2 , correct to 1 decimal place.

Note: The answer can be rounded to one decimal place by typing **round(ans,1)**. The round command can be accessed via the key sequence $\boxed{\text{math}} > \text{NUM} > \text{round}(\text{.})$.



6.3. Topic 3: Continuous random variables & normal distribution

6.3.1. General continuous random variables

Performing calculations involving probability density functions

Question

A continuous random variable X has probability density function f defined by

$$f(x) = \begin{cases} kx(2-x), & 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

where k is a real number.

- Find the value of k .
- Plot the curve $y = f(x)$ and verify graphically that $\int_0^2 f(x) dx = 1$.
- Find $P(1.2 \leq X \leq 2)$.
- Find $E(X)$.
- Find $\text{Var}(X)$.

Solution

The probability distribution of a continuous random variable X is a function $f(x)$, called a probability density function, such that $f(x) \geq 0$ for all real x and $\int f(x) dx = 1$.

- Press $\boxed{y=}$ and position the cursor in the Y1 line.
- Press $\boxed{\text{math}} > \mathbf{MATH} > \mathbf{piecwise(}$ (and set **PIECES** = 2).
- Complete the piecewise template as shown right.

Note: For inequality symbols and “or”, press $\boxed{2\text{nd}} \boxed{[\leq \neq >]}$.

As probability is given by the area under the curve $y = f(x)$ and the sum of probabilities is 1, then the total area under the curve $y = f(x)$ must be 1.

To solve for k in the equation $\int_0^2 f(x) dx = 1$:

- Press $\boxed{\text{math}} > \mathbf{MATH} > \mathbf{Numeric Solver ...}$
- For E1, press $\boxed{\text{math}} > \mathbf{MATH} > \mathbf{fnInt(}$ (and then complete the expression $\int_0^2 Y1 dX$.
- For E2, enter 1.
- Press **NEXT** $\boxed{[f5]}$ and set a starting value **K** = 1 (or any small positive value).
- Press **SOLVE** $\boxed{[f5]}$ to find a solution to the equation $E1 = E2$ for k .

Answer: Solving $\int_0^2 f(x) dx = 1$ for k gives $k = \frac{3}{4} = 0.75$.

... continued

NORMAL FLOAT AUTO REAL RADIAN MP			
[↵] TO EXPAND			
PLOT1	PLOT2	PLOT3	
 Y1=	{ KX(2-X); 0≤X≤2 0; X<0 or X>2		
 Y2=			
 Y3=			
 Y4=			
 Y5=			
 Y6=			
 Y7=			

NORMAL FLOAT AUTO REAL RADIAN MP	
SOLUTION IS MARKED ■	
$\int_0^2 (Y_1) dX = 1$	
■ K =	0.749999999999998
X =	0
bounds =	{ -1E99, 1E99 }
■ E1-E2 =	0
SOLVE	

Solution (continued)

Note: As well as solving for k , the solver stores that value in the letter K (see screen right). This is very useful for use in the following question parts.

(b) To view a graph of this continuous pdf:

First set the viewing window:

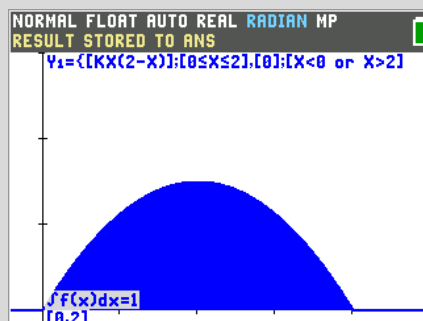
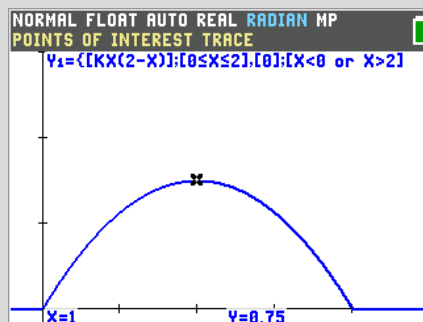
- Press **[window]** and enter the following values:
 $X_{\min} = -0.2$ $X_{\max} = 2.5$ $X_{\text{scl}} = 0.5$
 $Y_{\min} = -0.1$ $Y_{\max} = 1.5$ $Y_{\text{scl}} = 0.5$
- Press **[graph]** to view the curve.

To check graphically that $\int_0^2 f(x) dx = 1$:

- Press **[2nd][calc][7]** to select the integral tool.
- Type **0** for the **Lower Limit** and press **[enter]**.
- Type **2** for the **Upper Limit** and press **[enter]**.

Answer: As expected, the area is 1.

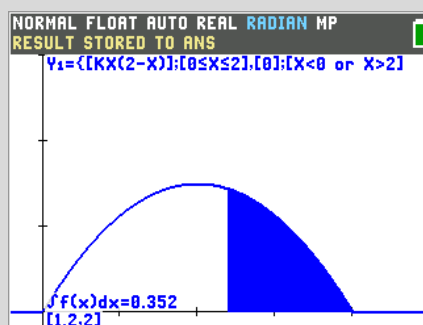
K 0.75



(c) On the same **Graphs** page that was used for part (b):

- Press **[2nd][draw] > ClrDraw** to remove the previous shading and area calculation.
- Press **[2nd][calc][7]** to select the integral tool.
- Type **1.2** for the **Lower Limit** and press **[enter]**.
- Type **2** for the **Upper Limit** and press **[enter]**.

$$\begin{aligned} \text{Answer: } P(1.2 \leq X \leq 2) &= \int_{1.2}^2 f(x) dx \\ &= 0.352 \end{aligned}$$



Alternatively:

- Press **[on] [enter]** to select the **CALCULATOR** application.
- Press **[math] > MATH > fnInt(**.
- Complete the integral template as shown right.

Answer: $P(1.2 \leq X \leq 2) = 0.352$ as before.

NORMAL FLOAT AUTO REAL RADIAN MP
 $\int_{1.2}^2 (Y_1) dX$
 0.352

... continued

Solution (continued)

(d) The expected value is given by $E(X) = \mu = \int_{-\infty}^{\infty} x f(x) dx$.

To calculate the mean:

- Press $\boxed{\text{on}} \boxed{\Delta}$ $\boxed{\text{enter}}$ to select the CALCULATOR application.
- Press $\boxed{\text{math}}$ > **MATH** > **fnInt**(.
- Complete the integral template as shown right.

$$\int_0^2 (x \cdot y_1) dx$$

1

$$\begin{aligned} E(X) &= \int_0^2 x \cdot f(x) dx \\ &= 1 \end{aligned}$$

The curve $y = f(x)$ for $0 \leq x \leq 2$ is part of a parabola which is symmetrical about $x = 1$.

For a symmetrical pdf, the mean is the x -value corresponding to the line of symmetry.

(e) The variance is given by

$$\text{Var}(X) = \sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx.$$

To calculate the variance:

- Press $\boxed{\text{on}} \boxed{\Delta}$ $\boxed{\text{enter}}$ to select the CALCULATOR application.
- Press $\boxed{\text{math}}$ > **MATH** > **fnInt**(.
- Complete the integral template as shown right.

Either using $\text{Var}(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$:

$$\int_0^2 ((x-1)^2 \cdot y_1) dx$$

0.2

$$\begin{aligned} \text{Var}(X) &= \int_0^2 (x-1)^2 f(x) dx \\ &= 0.2 \left(= \frac{1}{5} \right). \end{aligned}$$

Or using $\text{Var}(X) = \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2$:

$$\int_0^2 (x^2 \cdot y_1) dx - 1^2$$

0.2

$$\begin{aligned} \text{Var}(X) &= \int_0^2 x^2 f(x) dx - 1^2 \\ &= 0.2 \left(= \frac{1}{5} \right). \end{aligned}$$

Performing calculations involving cumulative density functions

Question

The lifetime, X , in tens of hours, of a battery has a cumulative distribution function, $F(x)$, given by

$$F(x) = \begin{cases} 0, & x < 1 \\ \frac{4}{9}(x^2 + 2x - 3), & 1 \leq x \leq 1.5 \\ 1, & x > 1.5 \end{cases}$$

- Plot the curve $y = F(x)$.
- Find $P(X \geq 1.2)$. Give your answer correct to three decimal places.
- Find the median of X . Give your answer correct to two decimal places.

A torch runs on four of these batteries, all of which must be working.

Four new batteries are put into the torch.

- Find the probability that the torch will still be working after 12 hours of use. Give your answer correct to three decimal places.

Solution

The cumulative probability distribution function (cdf) of a continuous random variable X is a function $F(x)$ such that

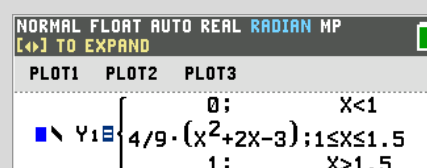
$$F(x) = P(X \leq x) \text{ for all real } x.$$

A cdf describes the probabilities of a random variable having values less than or equal to x .

It is a cumulative function because it sums the total probabilities up to that point. Its output always ranges between 0 and 1.

- First construct the piecewise function as follows:

- Press $\boxed{y=}$ and position the cursor in the Y1 line.
- Press $\boxed{\text{math}} > \text{MATH} > \text{piecewise}(\text{.}$
- Set **PIECES** = 3.
- Complete the piecewise template as shown right.



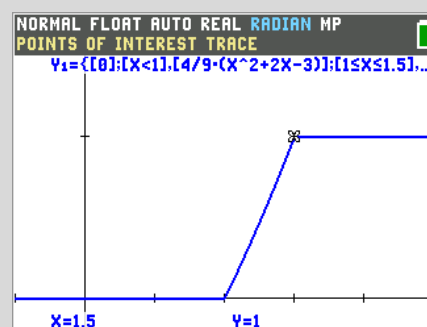
Note: For inequality symbols and “or”, press $\boxed{2\text{nd}} \boxed{[\leq \neq >]}$.

To view a graph of $F(x)$:

First set the viewing window:

- Press $\boxed{\text{window}}$ and enter the following values:
 $X_{\min} = -0.5$ $X_{\max} = 2.5$ $X_{\text{scl}} = 0.5$
 $Y_{\min} = -0.2$ $Y_{\max} = 1.5$ $Y_{\text{scl}} = 1$
- Press $\boxed{\text{graph}}$ to view the curve.

The required graph is displayed in a suitable window.



... continued

Solution (continued)

(b) To calculate $P(X \geq 1.2)$

- Press $\boxed{\text{on}}\boxed{\text{enter}}$ to select the CALCULATOR application.
- Enter $1 - Y1(1.2)$ as shown

Answer: $P(X \geq 1.2) = 1 - F(1.2)$

$$= 0.627 \quad (\text{correct to 3 d.p.})$$

Note: To obtain this probability as a fraction, press $\boxed{\frac{\Box}{\Box}}$.

Alternatively, using a pdf:

The pdf, $f(x)$, can be found from the cdf, $F(x)$, by using

$$f(x) = \frac{d}{dx}(F(x)).$$

- Press $\boxed{\text{math}} > \mathbf{MATH} > \mathbf{fnInt}(\cdot)$.
- Position the cursor inside the brackets, and then press $\boxed{\text{math}} > \mathbf{MATH} > \mathbf{nDeriv}(\cdot)$.
- Complete the integral and derivative templates as shown.

Answer: $\int_{1.2}^{1.5} \frac{8}{9}(x+1)dx = 0.627$, correct to 3 d.p.

(c) To find the median, plot the horizontal line $Y2 = 0.5$ and find the point of intersection of the line and the cdf curve:

- Press $\boxed{Y=}$ and position the cursor in the $Y2$ line.
- Enter $Y2 = 0.5$.

To find the point of intersection:

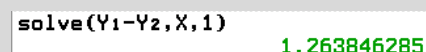
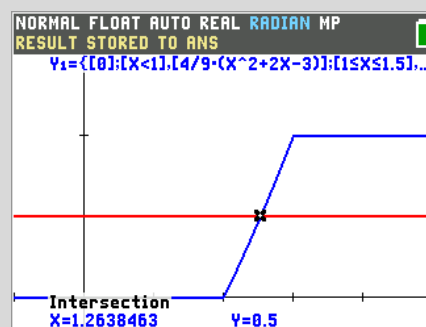
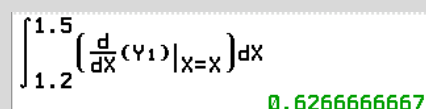
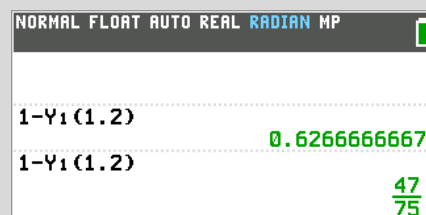
- Press $\boxed{2nd}\boxed{\text{calc}} > \mathbf{intersect}$.
- For Guess, type **1** and $\boxed{\text{enter}}$.

Answer: Let m be the median.

Solving $\frac{4}{9}(m^2 + 2m - 3) = 0.5$ for m with $1 < m < 1.5$ gives $m = 1.26$, correct to 2 decimal places.

Note: The intersection point can also be found in the CALCULATOR application, via the numerical solve command (found via $\boxed{2nd}\boxed{\text{catalog}}$). Recall that to solve $Y1 = Y2$, the **solve** command requires this to be expressed as the LHS of the equation $Y1 - Y2 = 0$.

(d) $(0.6266\dots)^4 = 0.154$ (correct to 3 decimal places).



6.3.2. Normal distributions

Recognising features of the graph of the normal distribution probability density function

If X is a normally distributed random variable with mean μ and standard deviation σ , then the probability density function of X is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$

A random variable Z with the standard normal distribution has probability density function

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}.$$

The standard normal distribution has mean $\mu = 0$ and standard deviation $\sigma = 1$.

The TI-84 Evo has a **Normal Pdf** command which can be used to plot the graph of a normal probability density function. For a normal distribution with mean μ and standard deviation σ , the syntax is **normalpdf**(x, μ, σ).

Question

Use the **normalpdf** command to explore how changing the values of μ and σ affect the behaviour of the normal probability density curve.

Solution

Note: In this example, the parameters used on TI-84 Evo will be denoted by A (for μ) and B (for σ).

First construct the standard normal curve as follows:

- Press **[y=]** and position the cursor in the Y1 line.
- Press **[alpha][distr] > Normal... > normalpdf**(.
 - Set x value = X .
 - Mean = 0 .
 - Standard deviation = 1 .
- Press **PASTE [f3]** to paste **normalpdf(X,0,1)** to the Y1 line.

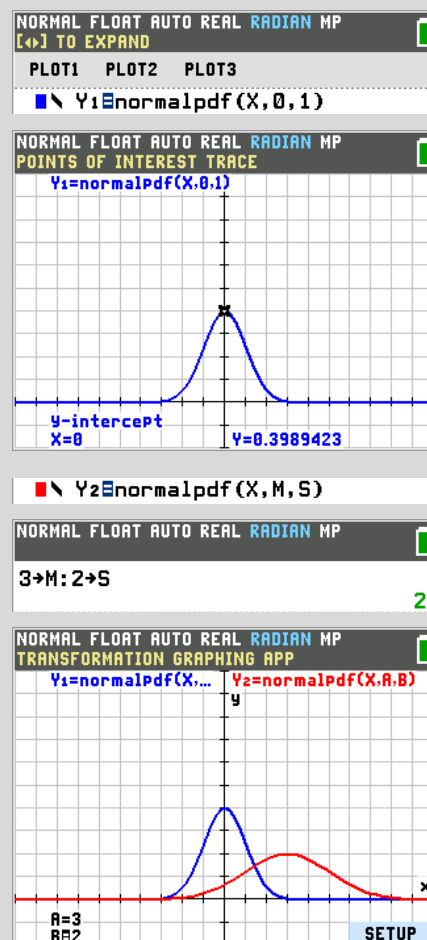
To set the viewing window $[-10,10]$ by $[-0.2,1]$:

- Press **[window]** and enter the following values:

Xmin = -10	Xmax = 10	Xscl = 1
Ymin = -0.2	Ymax = 1	Yscl = 0.1
- Press **[graph]** to view the standard normal curve.

For comparison with the standard normal curve, enter a second function for changing the values of μ and σ as follows:

- Press **[on|off][enter]** and select the TRANSFORMATION application.
- Press **[y=]** and enter **Y2 = normalpdf(X,A,B)**.
- Press **[graph]** to view plots of the two normal curves for comparison purposes.
- Using the sliders for **A** and **B**, set a mean of 3 and a standard deviation of 2 (see result screen at right).



Solution (continued)**Exploration**

Starting with $\mu = 0$ and $\sigma = 1$, vary the value of μ (the value of σ is kept constant).

What do you notice about the behaviour of the curve?

Now repeat for σ (the value of μ is kept constant).

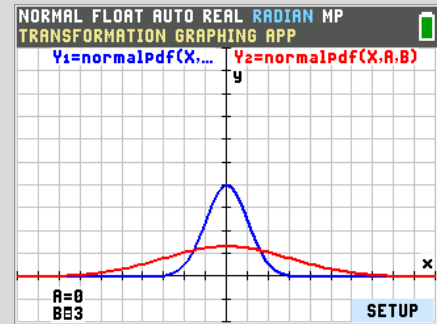
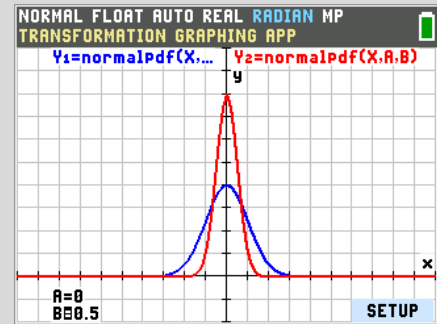
What do you notice about the behaviour of the curve?

Answer:

- Changing μ (keeping σ constant) translates the curve horizontally left and right.
- Increasing σ (keeping μ constant) causes the curve to be stretched horizontally and compressed vertically. In other words, increasing σ produces a flatter and wider bell-shaped curve.
- Decreasing σ (keeping μ constant) causes the curve to be compressed horizontally and stretched vertically. In other words, decreasing σ produces a taller and narrower curve.

Note: It is possible to use sliders and the TRANSFORMATION GRAPHING application to compare two normal curves where neither are the standard normal pdf. That is, use four sliders:

$Y2 = \text{normalpdf}(X, A, B)$ and $Y3 = \text{normalpdf}(X, C, D)$



Calculating probabilities and quantiles associated with a given normal distribution

Question

Given that X has a normal distribution with mean 75 and standard deviation 15, find

- (a) $P(48 \leq X \leq 100)$. (b) $P(X < 60)$. (c) $P(X > 70)$.
 (d) the value of c if $P(X < c) = 0.8$. (e) the value of d if $P(X \geq d) = 0.745$.

In parts (a), (b) and (c), give your answers correct to three decimal places and in parts (d) and (e), give your answers correct to two decimal places.

Solution

The **Normal Cdf** command calculates probabilities of the form $P(a < X < b)$. For parts (a), (b) and (c), press **on** **enter** to select the CALCULATOR application, and then:

- Press **[alpha][distr] > Normal > normalcdf**.
 - Set lower = **48** and upper = **100**.
 - Set mean = **75** and standard deviation = **15**.
- Press **PASTE [f3]** to paste **normcdf(48,100,75,15)**.

Answer: $P(48 \leq X \leq 100) = 0.9163$, correct to 4 d.p.

- For part (b), enter **normCdf(-10000,60,75,15)**.

Answer: $P(X < 60) = 0.1587$, correct to 4 decimal places.

- For part (c), enter **normCdf(70,10000,75,15)**.

Answer: $P(X > 70) = 0.6306$, correct to 4 decimal places.

Notes: (1) Generally, using a lower bound of $-10,000$ and an upper bound of $10,000$ will be sufficient. (2) The answers to parts (a), (b) and (c) can be rounded to four decimal places by typing **round(ans,4)**.

(d) and (e):

The **Inverse Normal** command calculates the value of x for a given probability p where $P(X < x) = p$. The command and its syntax are **invNorm(p, μ, σ)** or simply **invNorm(p)** if $\mu = 0$ and $\sigma = 1$.

For part (d):

- Press **[alpha][distr] > Normal > invNorm**.
 - Set area = **0.8**.
 - Set mean = **75**.
 - Set standard deviation = **15**.
 - tail = **LEFT**.
- Press **PASTE [f3]** to paste **invnorm(0.8,75,15,LEFT)**.

Answer: If $P(X < c) = 0.8$, then $c = 87.62$, correct to 2 d.p.

- For part (e), enter **invnorm(0.745,75,15,RIGHT)**.

Answer: If $P(X \geq d) = 0.745$, $d = 65.12$ correct to 2 d.p.

NORMAL FLOAT AUTO REAL RADIAN MP

Normalcdf

lower:48
upper:100
 μ :75
 σ :15

PASTE

HISTORY

normalcdf(48,100,75,15)
0.9162794042
round(ans,4) 0.9163
normalcdf(-10000,60,75,15)
0.1586552596
round(ans,4) 0.1587
normalcdf(70,10000,75,15)
0.6305585963
round(ans,4) 0.6306

NORMAL FLOAT AUTO REAL RADIAN MP

InvNorm

area:0.8
 μ :75
 σ :15
tail: LEFT CENTER RIGHT

PASTE

invNorm(0.8,75,15,LEFT)
87.6243185
invNorm(0.745,75,15,RIGHT)
65.11743464

Note: It is often helpful to sketch a labelled diagram before attempting a normal probability distribution question.

Calculating the standard deviation of a normal distribution

Question

The random variable X has a normal distribution with mean 32 and standard deviation σ . Given that $P(X > 36.8) = 0.3$ find σ . Give your answer correct to two decimal places.

Solution

Use of Inverse Normal (CALCULATOR application):

- Press **[math]** > **MATH** > **Numeric Solver ...**
- For E1, enter **[alpha][distr]** > **Normal** > **invNorm(**, then enter **invnorm(0.7,32,S,LEFT)**
- For E2, enter **36.8**.
- Press **NEXT** [f5] and set a starting value **S = 1** (or any small positive value)
- Press **SOLVE** [f5] to find a solution to the equation $E1=E2$ for **S**.

So $\sigma = 9.15$, correct to 2 decimal places.

Use of Normal Cdf (CALCULATOR application):

- Press **[math]** > **MATH** > **Numeric Solver ...**
- For E1, enter **[alpha][distr]** > **Normal** > **normalcdf(**, then enter **normalcdf(36.8,10000,32,S)**.
- For E2, enter **0.3**.
- Press **NEXT** [f5] and set a starting value **S = 1** (or any small positive value).
- Press **SOLVE** [f5] to find a solution to the equation $E1=E2$ for **S**.

So $\sigma = 9.15$ as before.

Use of Inverse Normal (GRAPHS application):

- Press **[y=]** and enter **Y1 = invNorm(0.7,32,X,LEFT)** and **Y2 = 36.8**.

To set the viewing window $[-0.5, 15]$ by $[-10, 60]$:

- Press **[window]** and enter the following values:
 $X_{\min} = -0.5$ $X_{\max} = 15$ $X_{\text{scl}} = 1$
 $Y_{\min} = -10$ $Y_{\max} = 60$ $Y_{\text{scl}} = 10$
- Press **[graph]** to view the intersection of the two graphs.

To find the point of intersection:

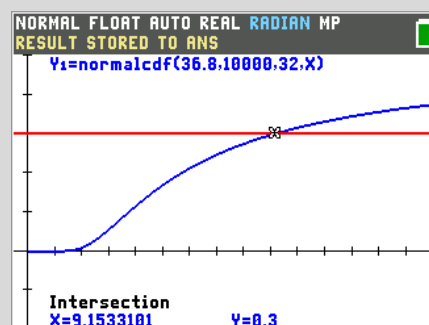
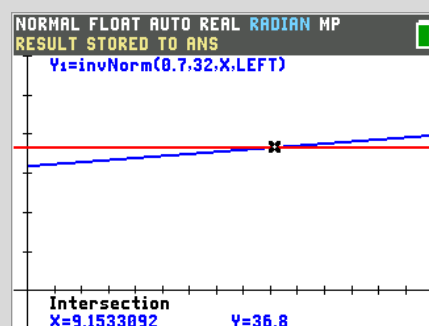
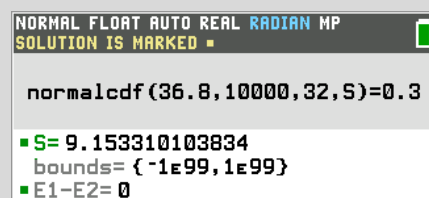
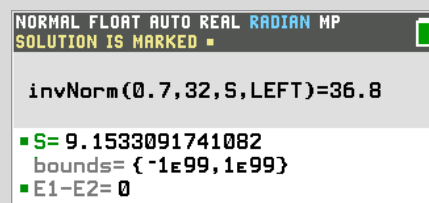
- Press **[2nd][calc]** > **intersect**.
- Click on each line and locate a guess value.

So $\sigma = 9.15$ as before.

Use of Normal Cdf (GRAPHS application):

- Repeat above with **Y3 = normalcdf(36.8,10000,32,X)** and the line **Y4 = 0.3** for $[-0.5, 15]$ by $[-0.2, 0.5]$.
- Set the viewing window and find the point of intersection.

So $\sigma = 9.15$ as before.



Calculating the mean and standard deviation of a normal distribution

Question

The random variable X has a normal distribution with mean μ and standard deviation σ .

Given that $P(X < 30) = 0.4$ and $P(X < 55) = 0.9$ find μ and σ . Give your answer correct to one decimal place.

Solution

Transform from X to Z and form two linear equations:

$$P(Z < z) = 0.4 \Rightarrow \frac{30 - \mu}{\sigma} = \text{invNorm}(0.4) \Leftrightarrow \mu - 0.2533\sigma = 30$$

(1)

$$P(Z < z) = 0.9 \Rightarrow \frac{55 - \mu}{\sigma} = \text{invNorm}(0.9) \Leftrightarrow \mu + 1.2816\sigma = 55$$

(2)

Solve this pair of linear equations simultaneously as follows:

- Press **[on]** and select the SYSTEM SOLVER application.
- On the setup page:
 - Set EQUATIONS = 2.
 - Set UNKNOWNNS = 2.
 - Set VARIABLES = a,b.
- Press **NEXT** [f5].
- In the template created, enter the two equations from the information provided (using $a = \mu$ and $b = \sigma$).

$$a - 0.2533b = 30$$

$$a + 1.2816b = 55$$

- Press **SOLVE** [f5]

The result displays the solution for μ and σ respectively.

Answer: $\mu = 34.1$ and $\sigma = 16.3$.

Use of Inverse Normal on a Graphs page:

- Plot the line $Y1 = \frac{30 - X}{\text{invNorm}(0.4)}$ and the line

$$Y2 = \frac{55 - X}{\text{invNorm}(0.9)}.$$

To set the viewing window $[-0.5, 50]$ by $[-5, 25]$:

- Press **[window]** and enter the following values:

Xmin = -0.5	Xmax = 50	Xscl = 10
Ymin = -5	Ymax = 25	Yscl = 5
- Press **[graph]** to view the intersection of the two graphs.

To find the point of intersection:

- Press **[2nd][calc] > intersect**.
- Click on each line and locate a guess value.

Answer: So $\mu = 34.1$ and $\sigma = 16.3$, correct to 1 d.p.

NORMAL FLOAT AUTO REAL Radian MP

```
invNorm(0.4,0,1,LEFT)
-0.2533471011
invNorm(0.9,0,1,LEFT)
1.281551567
```

NORMAL FLOAT AUTO REAL Radian MP

NUMBER OF EQUATIONS

SETUP

EQUATIONS: 2 3 4 5 6 7 8 9 10

UNKNOWNNS: 2 3 4 5 6 7 8 9 10

VARIABLES: x,y a,b

AUTO DECIMAL

NORMAL SCIENTIFIC ENG

FLOAT 0 1 2 3 4 5 6 7 8 9

Radian DEGREE

QUIT HELP NEXT

NORMAL FLOAT AUTO REAL Radian MP

SYSTEM SOLVER APP

SYSTEM OF EQUATIONS

1 a- 0.2533 b= 30

1 a+ 1.2816 b= 55

NORMAL FLOAT AUTO REAL Radian MP

SYSTEM SOLVER APP

SOLUTION

a= 34.12567594

b= 16.28770604

NORMAL FLOAT AUTO REAL Radian MP

[<=>] TO EXPAND

PLOT1 PLOT2 PLOT3

Y1= (30-X)/invNorm(0.4,0,1,LEFT)

Y2= (55-X)/invNorm(0.9,0,1,LEFT)

NORMAL FLOAT AUTO REAL Radian MP

RESULT STORED TO ANS

Y1=(30-X)/(invNorm(0.4,0,1,LEFT))

Intersection X=34.126447 Y=16.28772

6.4. Topic 4: Sampling and proportions

6.4.1. Random sampling

Sampling from a binomial distribution

Question

Thirty students sit a test consisting of 25 multiple-choice questions. Each question has four possible answers. Each of the 30 students guess the answer to each question.

Random variable X denotes the number of correct answers obtained by a student.

- Use the random binomial (**randBin**) command to simulate this situation.
- Determine $\bar{x}$, the sample mean number of correct answers obtained from your simulation. Give the value for $\bar{x}$ correct to two decimal places. Compare the value for $\bar{x}$ to the theoretical mean number of correct answers (expected value).
- Repeat part (b) for 100 students and then for 1000 students.
- Plot a histogram for the simulation of 1000 students sitting the test.

Solution

- (a) The syntax for the random binomial command is **randBin**($n, p, [\text{\#Simulations}]$).

There are 25 multiple-choice questions so $n = 25$.

Each question has 4 alternatives so the probability of obtaining a correct answer (success), p , is 0.25.

There are 30 'students' so the number of simulations is 30.

To enter the randBin command, press **[math] > PROB > randBin**.

We can simulate and store the results of the above scenario by entering the following command:

randBin(25, 0.25, 30) → L1 as shown right.

The results of the first few 'students' are shown. Here:

Student 1 obtained 5 correct answers.

Student 2 obtained 4 correct answers.

Student 3 obtained 6 correct answers. And so on.

*Note: The results can also be shown via **[stat] > EDIT > Edit**.*

- (b) The sample mean for these 30 students can be calculated as follows:

- Press **[2nd][list] > MATH > mean(**.
- Press **[2nd][list] > NAMES > L1**.
- Press **[enter]** to calculate the sample mean.

Answer: For 30 students (30 simulations), $\bar{x} = 6.53$, correct to 2 decimal places. Compare to $\mu = 25 \times 0.25 = 6.25$.

*Note: If required, the random number generator can be seeded by entering a large number, and storing it as **rand** (the **rand** command can be found using **[math] > PROB > rand**). An example of seeding is shown below.*

NORMAL FLOAT AUTO REAL RADIAN MP					
957846218→rand					
				957846218	
randBin(25, 0.25, 30)→L1					
{5, 4, 6, 9, 4, 8, 3, 5, 3, 8, 7, 6, 8, 1}					

NORMAL FLOAT AUTO REAL RADIAN MP					
L1	L2	L3	L4	L5	
5					
4					
6					
9					
4					
8					
3					
5					
3					
8					
7					

L1(1)=5

mean(L1) 6.533333333

Sampling from a uniform distribution

Question

For brevity, some instructions required here are described in the previous example. Please refer to them as required.

A random variable X has probability density function

$$f(x) = \begin{cases} 0.5, & 2 \leq x \leq 4 \\ 0, & \text{else} \end{cases}.$$

It can be shown that $E(X) = 3$ and $Var(X) = \frac{1}{3} \left(\sigma = \frac{1}{\sqrt{3}} \right)$.

- (a) Generate a random sample of size 100 from this distribution and plot a histogram of the sample.
- (b) Determine $\bar{x}$, the sample mean and s , the sample standard deviation. Give these values for $\bar{x}$ and s correct to two decimal places. Compare the values for $\bar{x}$ and s to the theoretical mean number of correct answers (expected value).
- (c) Repeat parts (a) and (b) for a random sample of size 999.

Solution

(a) The **rand** command has syntax **rand(#Trials)**.

This returns a list containing random values between 0 and 1.

The required formula for this simulation is **2*rand(100)+2**.

To calculate and store these 100 values:

- Press **on** **enter** to select the CALCULATOR application.
- Enter **2*rand(100)+2 → L1**.

The first few values of the random sample are shown.

To construct a histogram to display these values:

To obtain a histogram:

- Press **2nd** **[plot]** and turn on **Plot 1**.
- Select the **Histogram** icon.
- Set **XLIST = L1**.

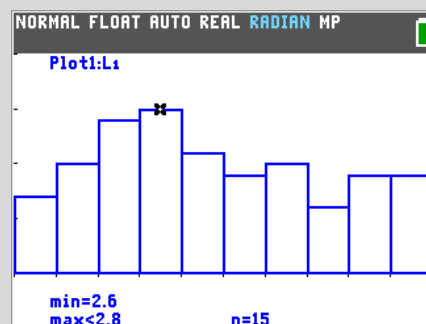
To set an appropriate viewing window for the histogram:

- Press **window** and enter the following values:
 $Xmin = 2$ $Xmax = 4$ $Xscl = 0.2$
 $Ymin = -5$ $Ymax = 20$ $Yscl = 5$
- Press **graph** to view the histogram.

*Note: You may need to clear any other defined functions in the **y=** screen or other plots that may have been set up.*

It is expected that the sample values are reasonably evenly distributed between 2 and 4. That is, it is expected that all the columns in the histogram should be about the same height.

NORMAL FLOAT AUTO REAL RADIAN HP
 2*rand(100)+2→L1
 {2.781768028, 2.651160339, 3.2 ▶



... continued

Solution (continued)**(b)** Considering $\bar{x}$:

- Press $\boxed{2\text{nd}}\boxed{[list]} > \text{MATH} > \text{mean}(\cdot)$.
- Press $\boxed{2\text{nd}}\boxed{[list]} > \text{NAMES} > \text{L1}$.
- Press $\boxed{\text{enter}}$ to calculate the sample mean.

For this random sample, $\bar{x} = 2.95$, correct to 2 decimal places.

Compare to $E(X) = 3$.

Considering s :

- Press $\boxed{2\text{nd}}\boxed{[list]} > \text{MATH} > \text{stdDev}(\cdot)$.
- Press $\boxed{2\text{nd}}\boxed{[list]} > \text{NAMES} > \text{L1}$.
- Press $\boxed{\text{enter}}$ to calculate the sample standard deviation.

For this random sample, $s = 0.55$, correct to 2 decimal places.

Compare to $\sigma = \frac{1}{\sqrt{3}} (= 0.577\dots)$.

(c) To calculate and store 999 values:

- Press $\boxed{\text{on}}\boxed{\square}\boxed{\text{enter}}$ to select the CALCULATOR application.
- Enter $2 \times \text{rand}(999) + 2 \rightarrow \text{L1}$.
- Find the sample mean and standard deviation as before.

For this random sample, $\bar{x} = 3.00$ and $s = 0.58$, correct to 2 decimal places.

Compare to $E(X) = 3$ and $\sigma = \frac{1}{\sqrt{3}} (= 0.577\dots)$.

As the sample size increases, it appears that $\bar{x}$ is getting closer to μ and s is getting closer to σ .

To construct a histogram to display these values:

To obtain a histogram:

- Press $\boxed{2\text{nd}}\boxed{[\text{plot}]} \cdot$ and turn on **Plot 1**.
- Select the **Histogram** icon.
- Set **XLIST** = **L1**.

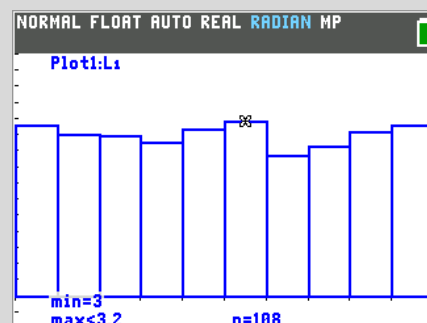
To set an appropriate viewing window for the histogram:

- Press $\boxed{\text{window}}$ and enter the following values:
 $X_{\min} = 2$ $X_{\max} = 4$ $X_{\text{scl}} = 0.2$
 $Y_{\min} = -20$ $Y_{\max} = 150$ $Y_{\text{scl}} = 10$
- Press $\boxed{\text{graph}}$ to view the histogram.

The sample values are reasonably evenly distributed between 2 and 4 (columns in the histogram are about the same height).

```
mean(L1)
2.948949359
stdDev(L1)
0.5485688843
```

```
2*rand(999)+2→L1
{2.703588021,2.424885725,2.27
mean(L1)
2.995457049
stdDev(L1)
0.5826511227
```



Sampling from a normal distribution

Question

Random variable X is normally distributed with mean 100 and standard deviation 15.

- (a) Use the random normal (**randNorm**) command to generate a random sample of size 10.
- (b) Generate a normal probability plot.
- (c) Repeat the above for random samples of size 30 and 100. Describe briefly what happens to the normal probability plot as the sample size increases.
- (d) For the random sample of size 100, plot the histogram and the normal probability density function.

Solution

- (a) The syntax for the random normal command is **randNorm($\mu, \sigma, [\#Trials]$)**.

The initial random sample size is 10.

To calculate and store these 10 values:

- Press **on** **enter** to select the CALCULATOR application.
- Enter **randNorm(100,15,10) → L1**.
- Press **2nd** **[list]** **> MATH > mean(**.
- Press **2nd** **[list]** **> NAMES > L1**.
- Press **enter** to calculate the sample mean.
- Press **2nd** **[list]** **> MATH > stdDev(**.
- Press **2nd** **[list]** **> NAMES > L1**.
- Press **enter** to calculate the sample standard deviation.

For this random sample, $\bar{x} = 96.31$ and $s = 13.21$, correct to 2 decimal places.

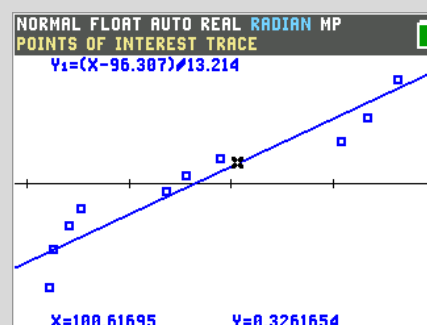
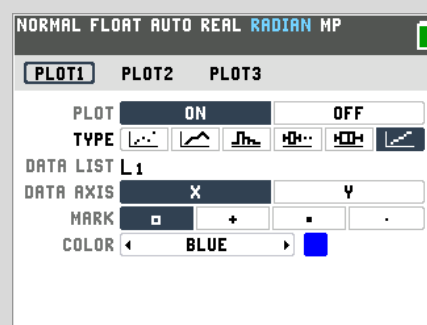
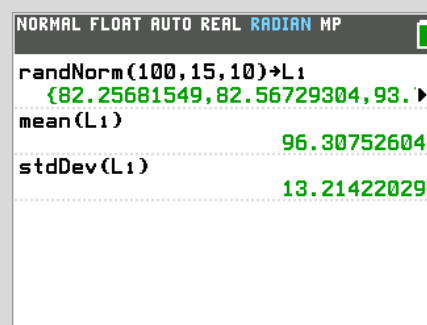
- (b) The normal probability plot helps to assess whether a data set is approximately normally distributed.

Data are plotted against a theoretical normal distribution in such a way that the points should form an approximate straight line. Departures from this straight line indicate departures from normality.

To construct a normal probability plot:

- Press **2nd** **[plot]** and turn on **Plot 1**.
- Select the **Normal probability plot** icon.
- Set **DATA LIST = L1**.
- Set **DATA AXIS = X**.
- Press **zoom** **< Zoom Stat**.

Note that the equation of the straight line fitted to the normal probability plot gives $\bar{x} = 96.31$ and $s = 13.21$, correct to 2 decimal places.



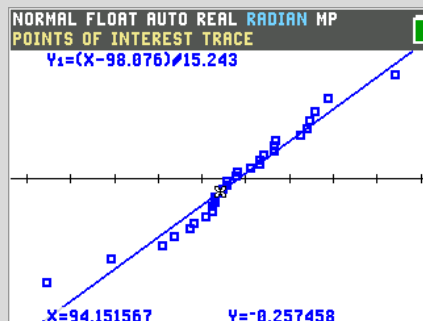
... continued

Solution (continued)

(c) Consider a random sample of size 30 from this distribution. To calculate and store these 30 values, repeat the previous steps (as shown right).

For this random sample of size 30, $\bar{x} = 98.08$ and $s = 15.24$, correct to 2 decimal places. The normal probability plot is shown, along with the fitted line.

```
randNorm(100,15,30)→L1
{94.43902185,114.6978751,133▶
mean(L1)
98.07595579
stdDev(L1)
15.24319533
```

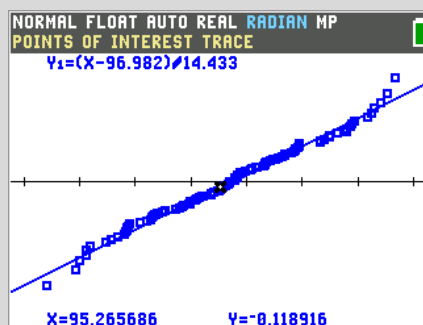


Repeating the above for a random sample of 100 from this distribution.

For this random sample of size 100, $\bar{x} = 96.98$ and $s = 14.43$, correct to 2 decimal places. The normal probability plot is shown, along with the fitted line.

As the sample size increases, it becomes more obvious that the samples are from a normal population.

```
randNorm(100,15,100)→L1
{71.31833851,114.7383191,83.▶
mean(L1)
96.9821636
stdDev(L1)
14.43347593
```

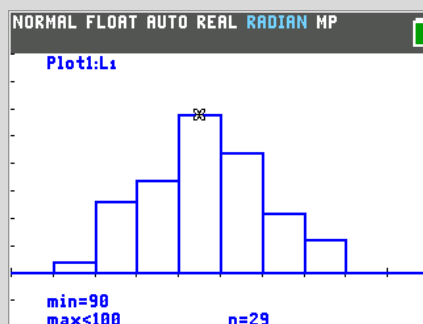


(d) A histogram for the sample can be constructed.

- Press $\boxed{2nd}[\boxed{plot}]$ and turn on **Plot 1**.
- Select the **Histogram** icon.
- Set **XLIST** = **L1**.

To set an appropriate viewing window for the histogram:

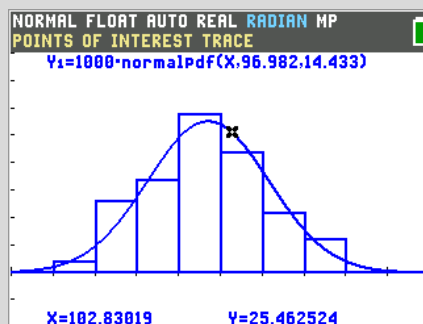
- Press $\boxed{window}$ and enter the following values:
 $Xmin = 50$ $Xmax = 150$ $Xscl = 10$
 $Ymin = -10$ $Ymax = 40$ $Yscl = 5$
- Press $\boxed{graph}$ to view the histogram.



To plot the normal pdf:

- Press $\boxed{y=}$ and enter the equation
 $Y1 = 1000 \text{normcdf}(X, 96.982, 14.433)$.

For the random sample, this offers a visual comparison between the histogram and a plot of the normal pdf.



6.4.2. Sample proportions

Understanding the concept of the sample proportion as a random variable

Question

Assume that 67% of people in Queensland have brown eyes.

- Generate a distribution of sample proportions when 200 samples of size 10 ($n = 10$) are randomly selected from the population. Display the distribution on a histogram. Comment on the distribution generated.
- Repeat part (a) for 200 random samples of size 30 ($n = 30$). Display the distribution on a histogram. What do you notice when the sample size is increased?
- Explore how changing the sample size, population proportion or the number of samples affects the distribution of sample proportions.

Solution

(a) To generate a distribution of sample proportions when 200 samples of size 10:

Enter the formula $\frac{\text{randBin}(10, 0.67, 200)}{10} \rightarrow \text{LPHAT}$.

```
randBin(10,0.67,200) → LPHAT
10.
{0.6,0.9,0.6,0.5,0.4,0.7,0.7}
```

Note: The decimal point after the number 10 in the denominator. The subscript “L” before PHAT is to define PHAT as a list. This subscript “L” character can be entered via $\boxed{2\text{nd}}\boxed{[list]} > \text{OPS} > \text{L}$.

- Press $\boxed{2\text{nd}}\boxed{[list]} > \text{MATH} > \text{mean}(\cdot)$.
- Press $\boxed{2\text{nd}}\boxed{[list]} > \text{NAMES} > \text{LPHAT}$.
- Press $\boxed{\text{enter}}$ to calculate the sample mean.
- Press $\boxed{2\text{nd}}\boxed{[list]} > \text{MATH} > \text{stdDev}(\cdot)$.
- Press $\boxed{2\text{nd}}\boxed{[list]} > \text{NAMES} > \text{LPHAT}$.
- Press $\boxed{\text{enter}}$ to calculate the sample standard deviation.
- Enter the formula $10 \rightarrow N : 0.67 \rightarrow P : \sqrt{\frac{P(1-P)}{N}}$.

```
mean(LPHAT) 0.6755
stdDev(LPHAT) 0.1485262612
10→N:0.67→P: √(P(1-P)/N) 0.1486943173
```

Answer: The mean value of LPHAT is approximately equal to the population parameters p . Note also that the value of

$\sqrt{\frac{P(1-P)}{N}}$ and the sample standard deviation are close.

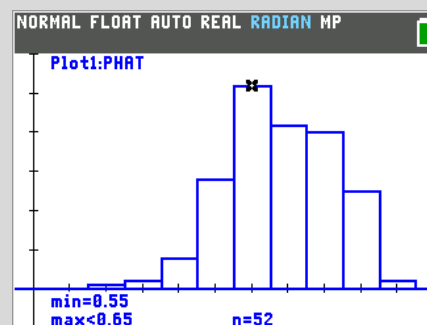
The sample proportion is a random variable whose value varies between samples. A histogram for the sample can be constructed.

- Press $\boxed{2\text{nd}}\boxed{[plot\cdot]}$ and turn on **Plot 1**.
- Select the **Histogram** icon.
- For **XLIST**, press $\boxed{2\text{nd}}\boxed{[list]} > \text{NAMES} > \text{LPHAT}$.

To set an appropriate viewing window for the histogram:

- Press $\boxed{\text{window}}$ and enter the following values:

Xmin = 50	Xmax = 150	Xscl = 10
Ymin = -10	Ymax = 40	Yscl = 5
- Press $\boxed{\text{graph}}$ to view the histogram.



... continued

Solution (continued)

For small sample sizes, a skewed (negative) distribution results because the population proportion used is not 0.5. We would not expect many samples to have no people with brown eyes or even only one or two people with brown eyes.

The distribution is centred roughly around 0.7 (the mode is 0.6) and has values ranging from 0.1 to 1.

For this simulation, 52 samples had $\hat{p} = 0.6$.

To plot a normal pdf using these sample statistics:

- Press $\boxed{Y=}$ and enter the equation
 $Y1 = 20 \times \text{normcdf}(X, 0.6755, 0.1485)$.

The calculated normal pdf based on the data set shows the mean and standard deviation of the sample proportion.

(b) Repeat part **(a)** with a larger sample size of $n = 30$.

Re-enter the formulas used previously, changing any reference to $n = 10$ to $n = 30$. See screen right.

As the sample size increases, the distribution has become more symmetric and more closely centred around 0.7. The distribution is squeezing more closely to $p = 0.7$.

Generate the histogram and superimpose the normal pdf curve over it (as shown on previous page).

Despite variations in the distribution of $\hat{p}$, the distribution tends to behave predictably in terms of shape, centre and spread.

Conclusions: Answers may vary.

As the sample size increases, the distribution becomes more normal.

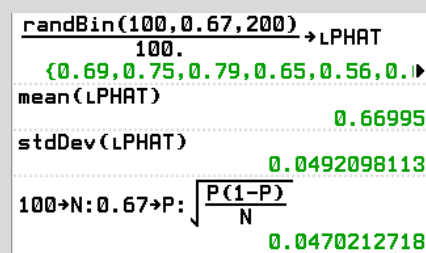
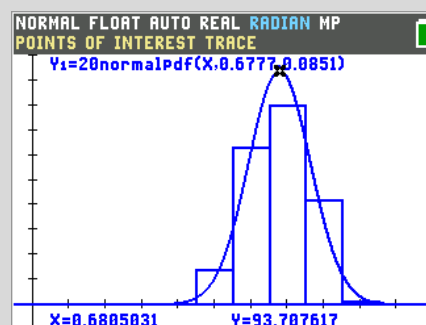
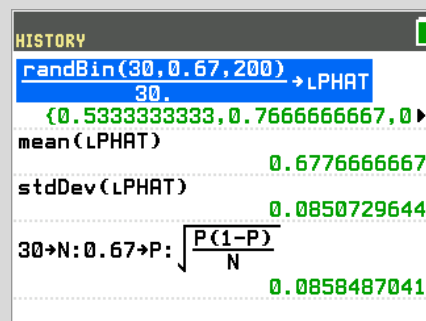
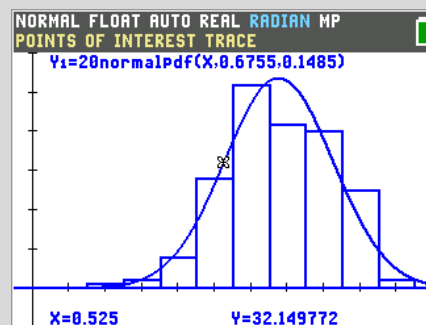
When the sample size is large enough, the distribution of a binomial random variable is well approximated by the normal distribution.

Hence, for large sample sizes, note the approximate normality of the distribution of sample proportions.

For $n = 100$, the sample standard deviation is very close to the population standard deviation and the sample mean is very close to the population mean (see screen right)

When the sample size, n , is large, the sample proportion $\hat{P}$ has an approximately normal distribution with mean $\mu = p$

and standard deviation $\sigma = \sqrt{\frac{p(1-p)}{n}}$.



6.5. Topic 5: Interval estimates for proportions

6.5.1. Confidence intervals for proportions

Understanding and calculating confidence intervals for proportions

Question

A survey found that 350 out of 500 people have brown eyes.

Calculate and compare approximate 90%, 95% and 99% confidence intervals for the proportion of people in Queensland who have brown eyes. Give your answer correct to three decimal places.

Solution

The approximate confidence interval

$$\left(\hat{p} - z \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \right) \text{ is an interval estimate for}$$

p , the population proportion, where z is the appropriate quantile for the standard normal distribution. Use $\hat{p}$ from one sample (survey) to calculate an interval that we are 95% certain contains p (unknown).

To find the 90% confidence interval for p :

- Press **[stat]** > **INTERVALS** > **1-PropZInt...**
- Set **x** = 350.
- Set **n** = 500.
- Set **C-LEVEL**=0.9.
- Press **CALCULATE** [**f3**].

Answer: The 90% approximate confidence interval (0.666, 0.734), correct to 3 decimal places.

To determine the approximate 95% and 99% confidence intervals, repeat the above calculation, changing only the value of C-LEVEL as required:

- Change the assigned confidence level (**cl**) to 0.95 and then to 0.99.

The 95% approximate confidence interval is (0.660, 0.740).

The 99% approximate confidence interval is (0.647, 0.753).

Being more confident means a wider interval is required.

There are variations in confidence intervals between samples and that most, but not all, confidence intervals contain p .

NORMAL FLOAT AUTO REAL RADIAN MP

1-PropZInt

x:350
n:500
C-LEVEL:0.9

CALCULATE

NORMAL FLOAT AUTO REAL RADIAN MP

1-PropZInt

(0.66629, 0.73371)

$\hat{p}$ = 0.7
n = 500

1-PropZInt(350,500,0.95)

Done

NORMAL FLOAT AUTO REAL RADIAN MP

1-PropZInt

(0.65983, 0.74017)

$\hat{p}$ = 0.7
n = 500

1-PropZInt(350,500,0.99)

Done

NORMAL FLOAT AUTO REAL RADIAN MP

1-PropZInt

(0.64721, 0.75279)

$\hat{p}$ = 0.7
n = 500

Understanding sample sizes and margins of error for sample proportions

Question

Determine the sample size required to achieve a margin of error of 2% in an approximate 95% confidence interval for the proportion, p , of people in Queensland who have brown eyes, if the sample proportion, $\hat{p}$, is 0.7.

Solution

The margin of error, M , is given by $M = z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ and is the radius of the confidence interval.

For an approximate 95% confidence interval, substituting

$$M = 0.02 \text{ and } \hat{p} = 0.7 \text{ gives } 0.02 = 1.9600\sqrt{\frac{0.7(1-0.7)}{n}}.$$

Solve for n in the equation $0.02 = 1.9600\sqrt{\frac{0.7(1-0.7)}{n}}$:

- Press $\boxed{\text{math}}$ > **MATH** > **Numeric Solver ...**
- For E1, enter 0.02.
- For E2, enter $1.96\sqrt{\frac{0.7(1-0.7)}{N}}$.
- Press **NEXT** [f5] and set a starting value $N = 1$ (or any small positive value).
- Press **SOLVE** [f5] to find a solution to the equation $E1 = E2$ for n .

Answer: achieve a margin of error of 2%, a sample size of 2017 is required.

NORMAL FLOAT AUTO REAL RADIAN MP
ENTER EQUATION E1=E2
EQUATION SOLVER
E1: 0.02
E2: $1.96 \cdot \sqrt{\frac{0.7(1-0.7)}{N}}$
NEXT

NORMAL FLOAT AUTO REAL RADIAN MP
SOLUTION IS MARKED ■
 $0.02 = 1.96 \cdot \sqrt{\frac{0.7(1-0.7)}{N}}$
■ $N = 2016.8400000009$
bounds = $\{-1E99, 1E99\}$
■ $E1 - E2 = 0$

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