The Mathematics of Wheels and Spirographs®

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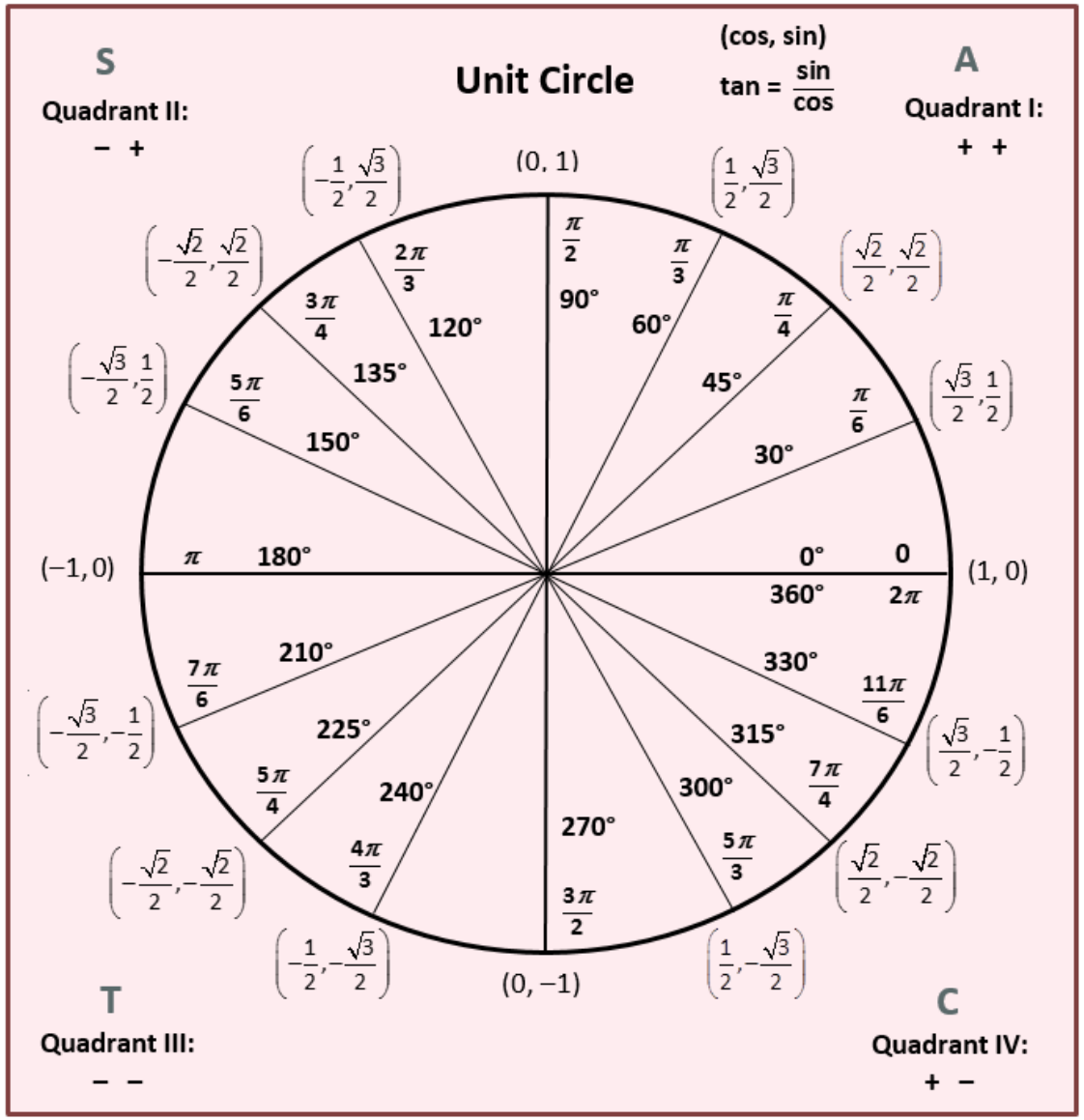
Warning: ☺ This presentation was supposed to have been interactive with lots of audience participation, so comments and slides have been added to suggest possible answers. If you have other comments and/or suggestions, please let me know. Thank you and enjoy!

In the beginning…

X values are cosines of angles.

Y values are sines of angles.

So let's graph that unit circle… Parametrically.



Mode

NORMAL FLOAT AUTO a+bi RADIAN MP
PRESS [<] OR [>] TO SELECT AN OPTION
MATHPRINT CLASSIC
NORMAL SCI ENG
FLOAT 0 1 2 3 4 5 6 7 8 9
RADIAN DEGREE
FUNCTION PARAMETRIC POLAR SEQ
THICK DOT-THICK THIN DOT-THIN
SEQUENTIAL SIMUL
REAL a+bi re^(θi)
FULL HORIZONTAL GRAPH-TABLE
FRACTION TYPE: n/d Un/d
ANSWERS: AUTO DEC
STAT DIAGNOSTICS: OFF ON
STAT WIZARDS: ON OFF
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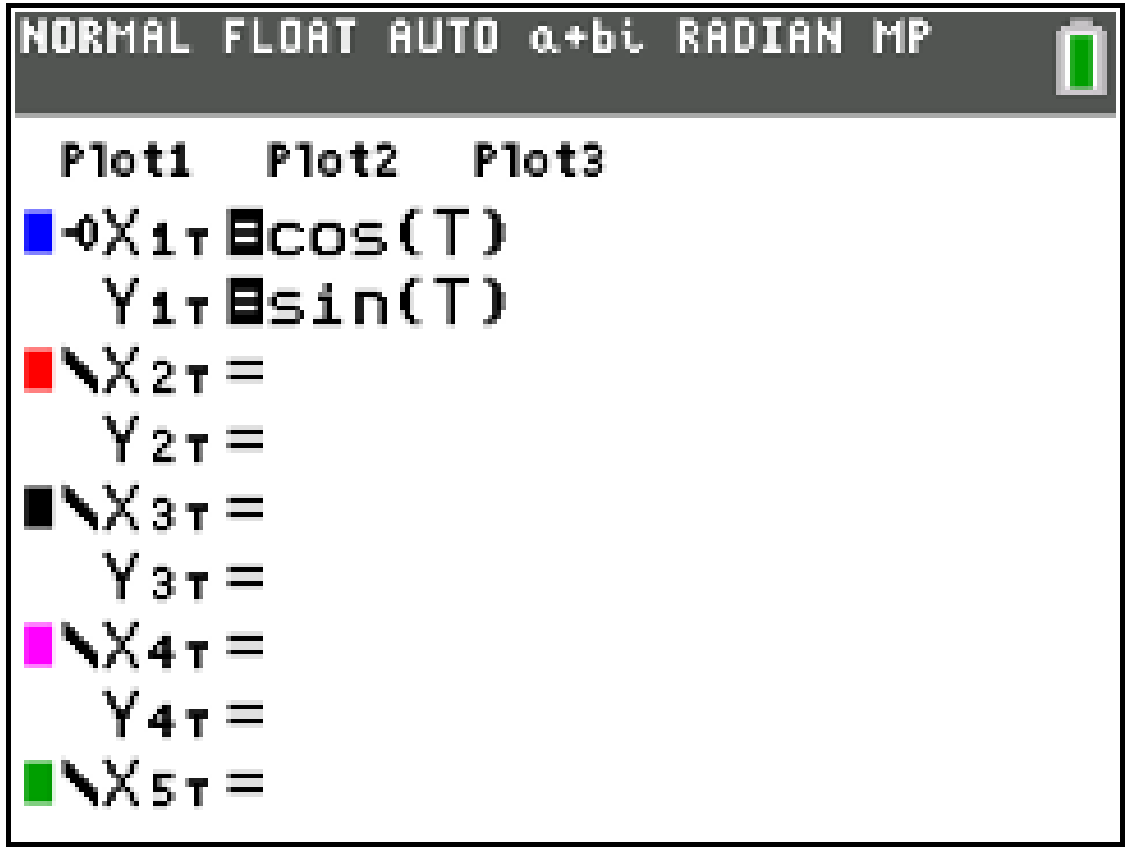
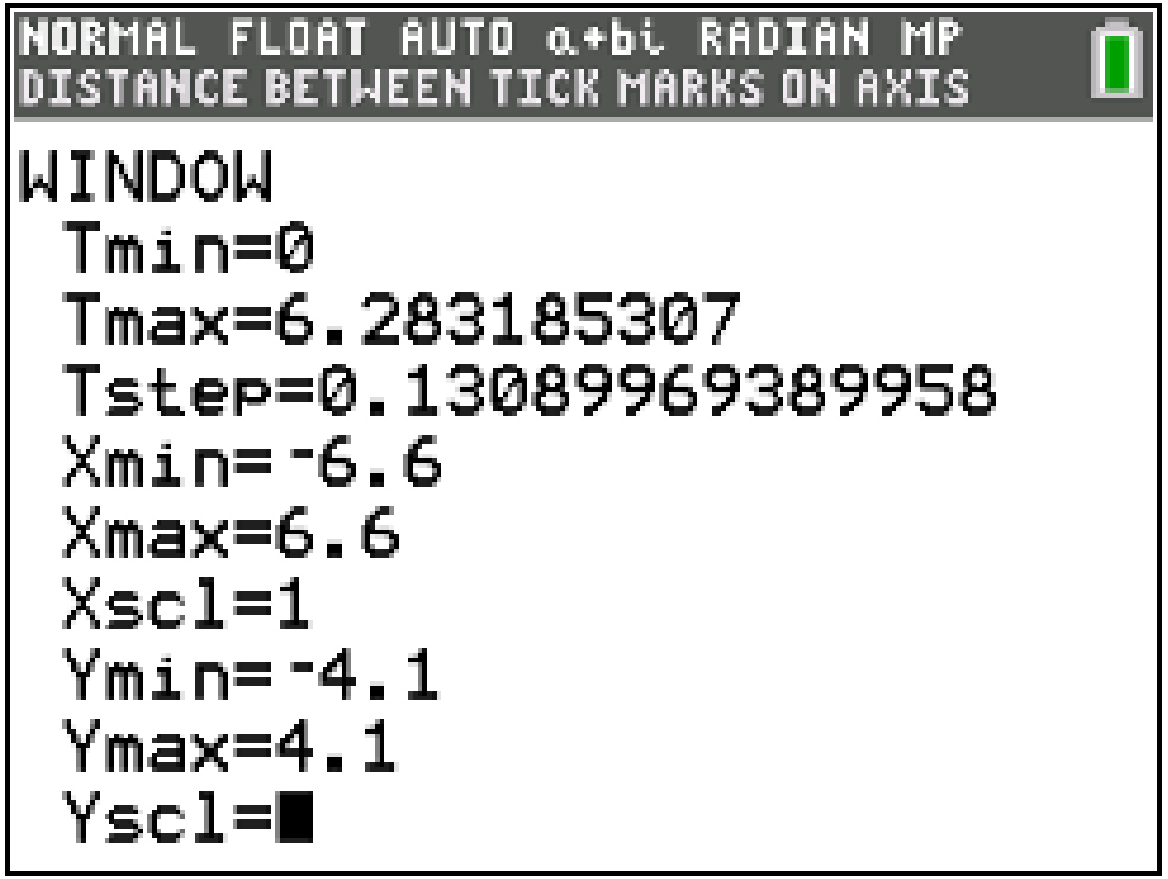
LANGUAGE: ENGLISH

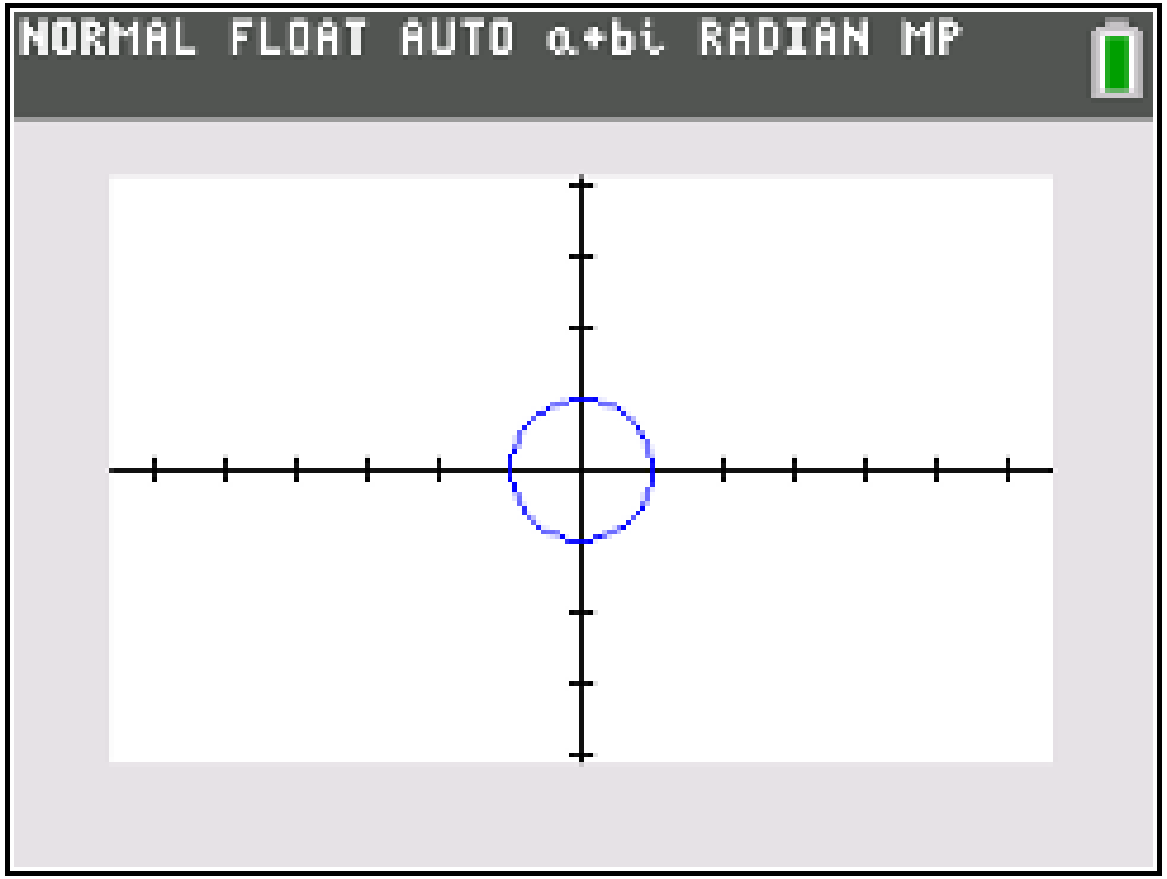
Y=


NORMAL FLOAT AUTO a+bi RADIAN MP
Plot1 Plot2 Plot3
X1T=cos(T)
Y1T=sin(T)
X2T=
Y2T=
X3T=
Y3T=
X4T=
Y4T=
X5T=

Window



Graph



What’s the difference between this Unit Circle and a Ferris Wheel?

Different radius

(audience suggestion – 20 m)

Different position

(audience suggestion – start at the bottom of the wheel)

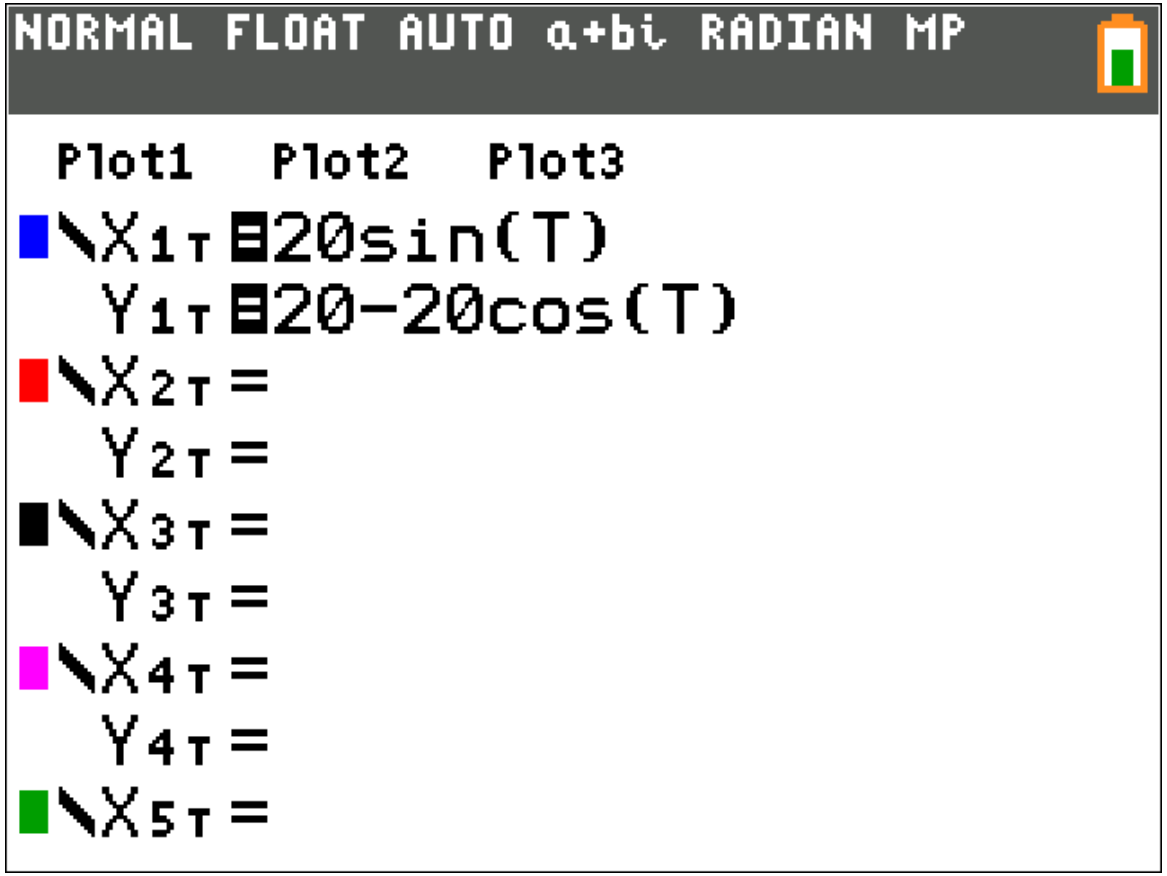
How do we fix that?

Audience Discussion: To fix the radius, change the coefficient of the trig functions to 20.
To raise the Ferris wheel so we don't go underground, add 20 to Y(T).
This results in $X_{1T} = 20\cos(T)$ and $Y_{1T} = 20 + 20\sin(T)$

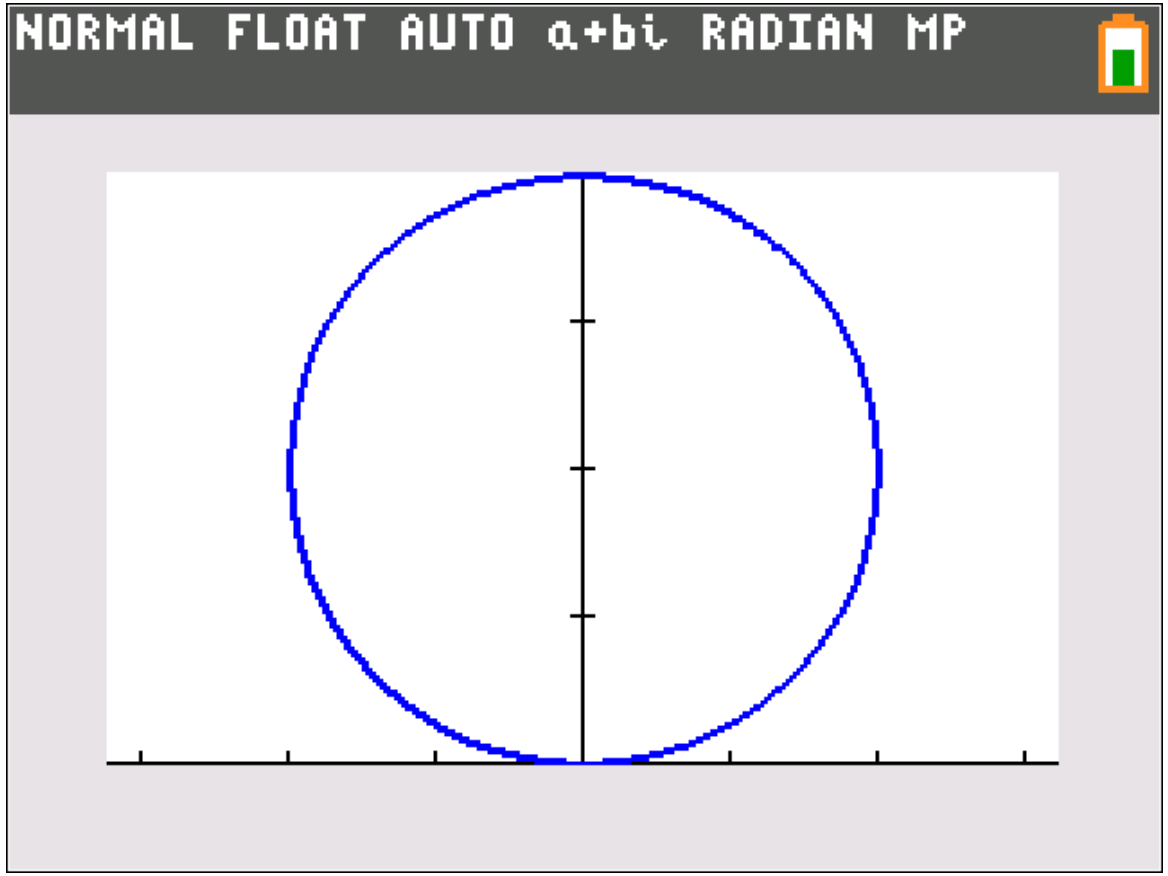
The window needs to be fixed too, so change the window so that Xmin = -20, Xmax = 20, Ymin = 0 and Ymax = 40. This Ferris wheel looks more like an ellipse, so press Zoom and ZSquare to make the Ferris wheel look more realistic.

However, this Ferris wheel doesn't start at ground level. In other words, $Y_{1T}(0) = 20$ and not 0. So let's look at what we would like it to do. We would like the height (y values) to start low and end high. This could be accomplished with a phase shift, but a negative cosine function has this same property, so I will change the y equation to $Y_{1T} = 20 – 20\cos(T)$. If the Ferris wheel is still moving counterclockwise, the x values start at 0, increase to x = 20, then decrease to = -20 and come back to 0. The sine function does the same thing, so change x equation to $X_{1T} = 20\sin(T)$.

Equations



Graph



What’s the difference between this Unit Circle and a Ferris Wheel?

Different speed?

(Skyview Atlanta claims 4 rotations in 15 minutes…)

Different direction?

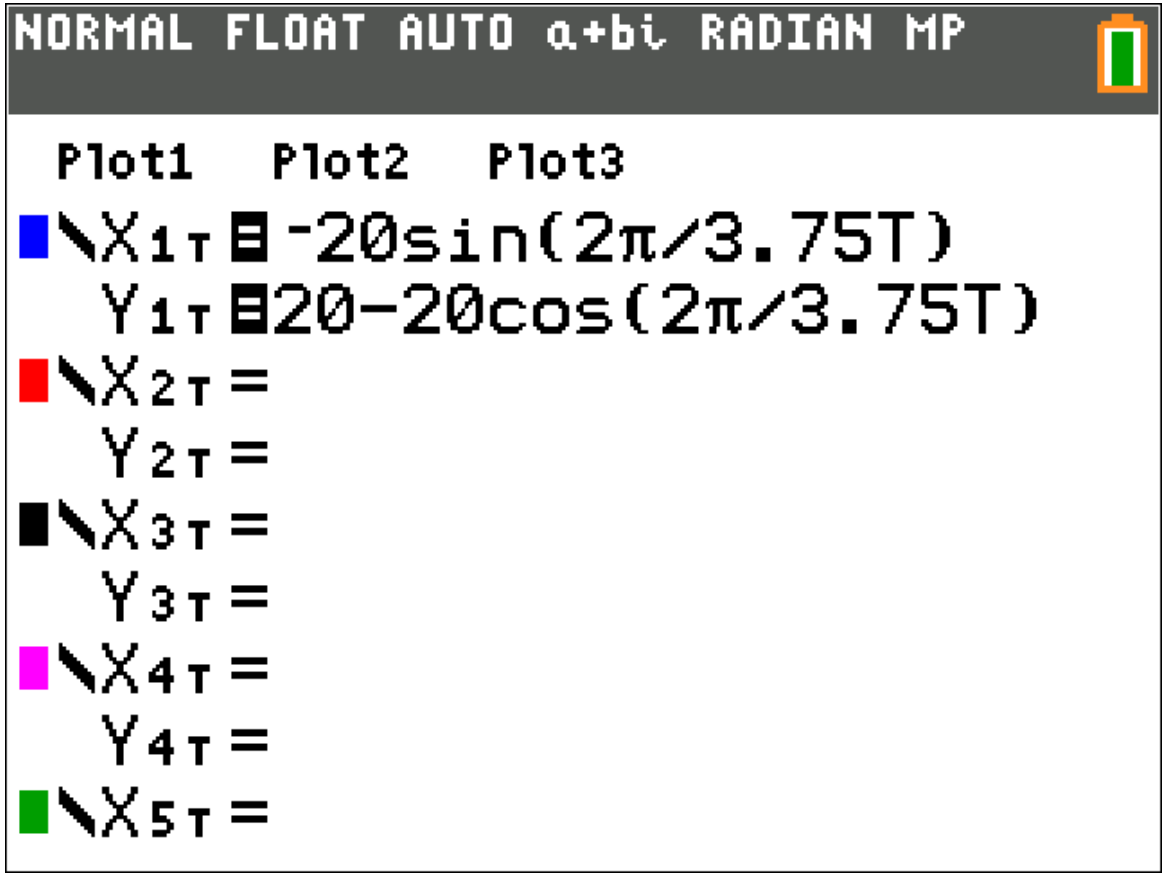
(Shall we go clockwise?)

Audience Discussion: If the Ferris wheel makes 4 revolutions in 15 minutes, that's 8π radians in 15 minutes or 2π radians in 3.75 seconds. To change the speed, change the coefficient of T to $2\pi/(3.75)$.

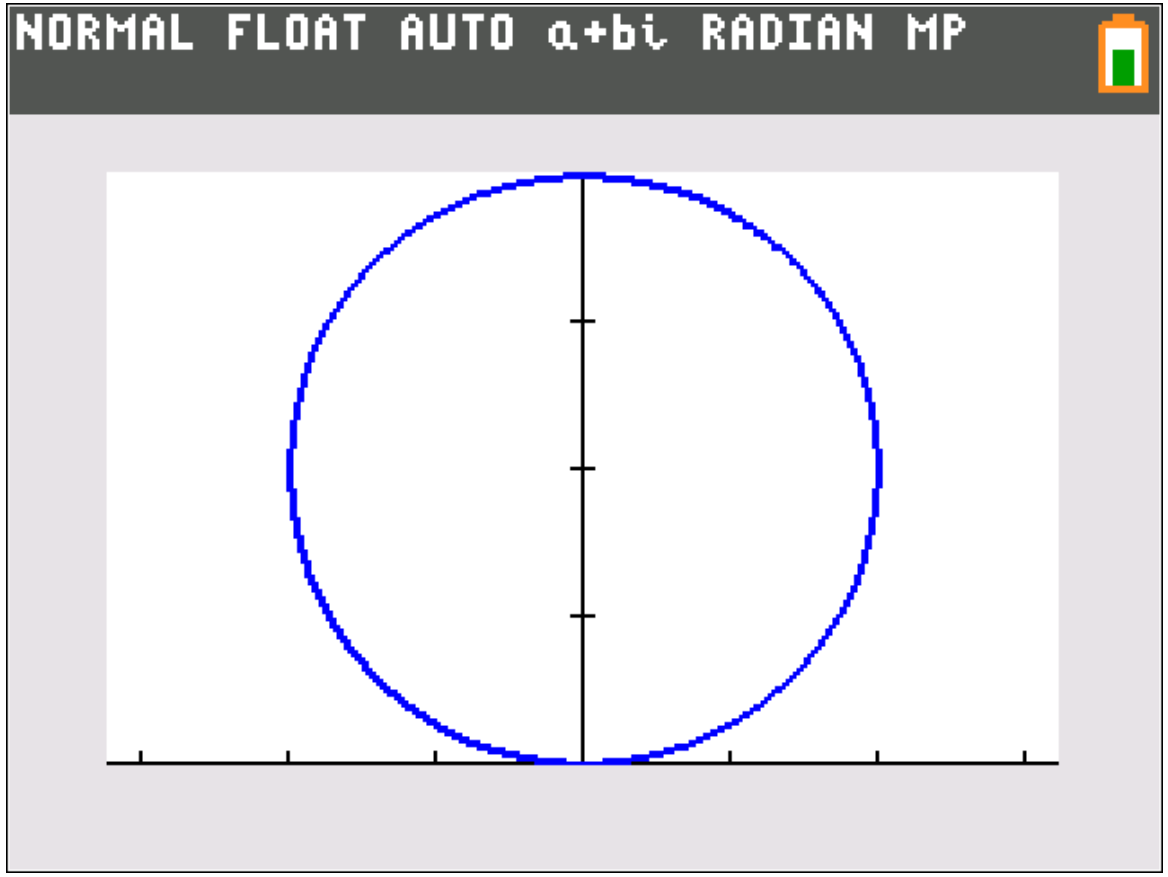
If we want to have the Ferris wheel go clockwise, we change the x values to start in the negative direction. This means the new equations are:

$X_{1T} = -20\sin(2\pi T/3.75)$ and $Y_{1T} = 20 - 20\cos(2\pi T/3.75)$

Equations



Graph



The entire wheel changes position.

How?

(Audience rolls paper plates on tables until someone suggests that we add a linear term to the x values.)

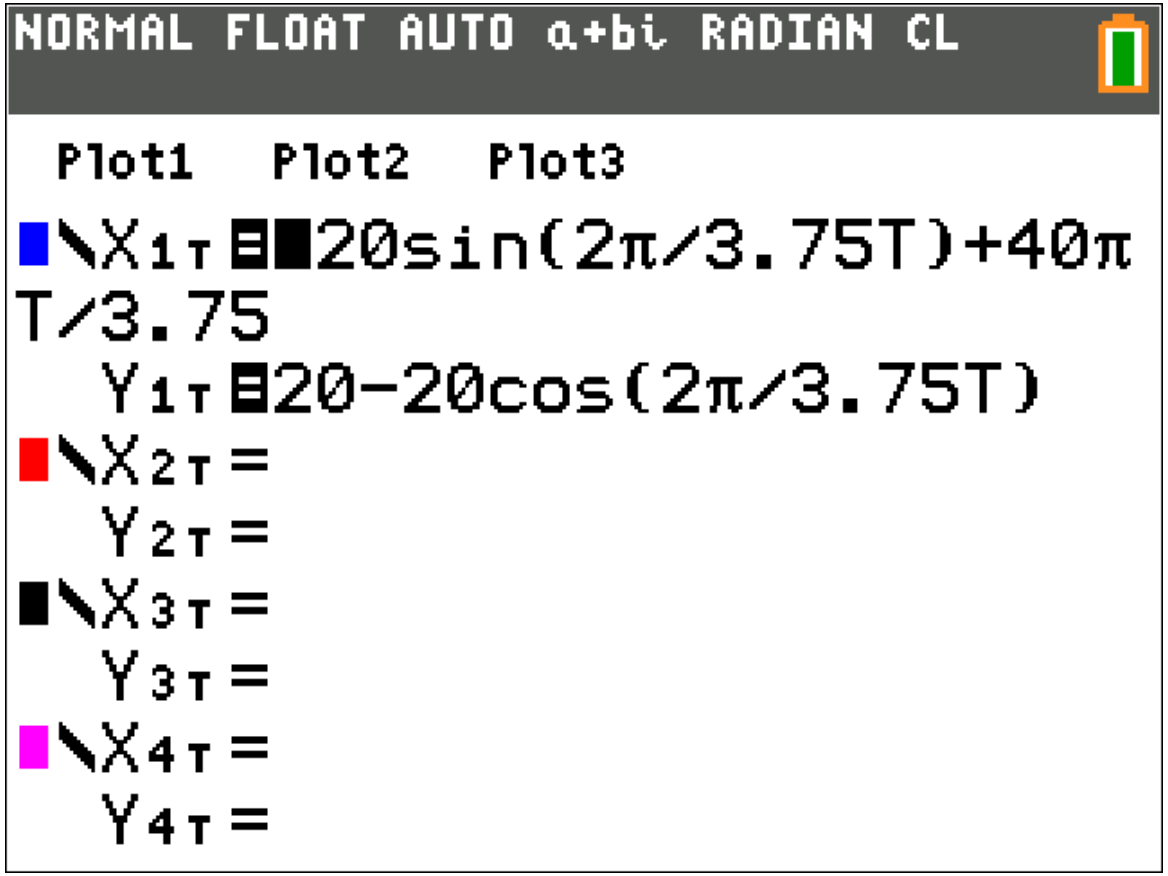
What's the difference between the Unit Circle and a car wheel?

Audience Discussion: If we keep the same wheel that we had earlier, the wheel travels 1 circumference in 3.75 seconds so since distance = rate·time, we need to add $(40\pi/3.75)T$ to the x term to get:

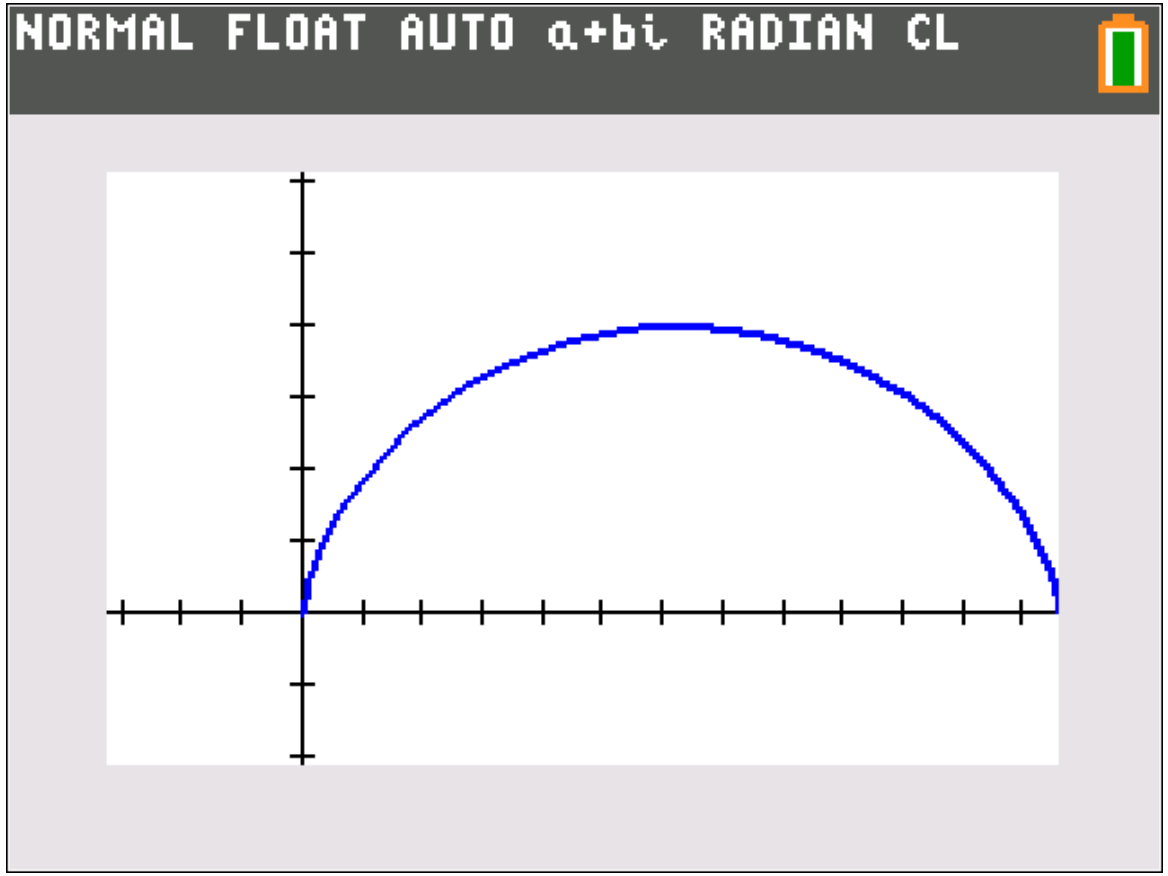
$X_{1T} = -20\sin(2\pi T/3.75) + (40\pi/3.75)T$ and $Y_{1T} = 20 - 20\cos(2\pi T/3.75)$

And since the wheel should travel 40π units, we need to change the window to let $X_{max} = 40\pi$. And, once again, we need to ZSquare to get a more reasonable view.

Equations*



Graph



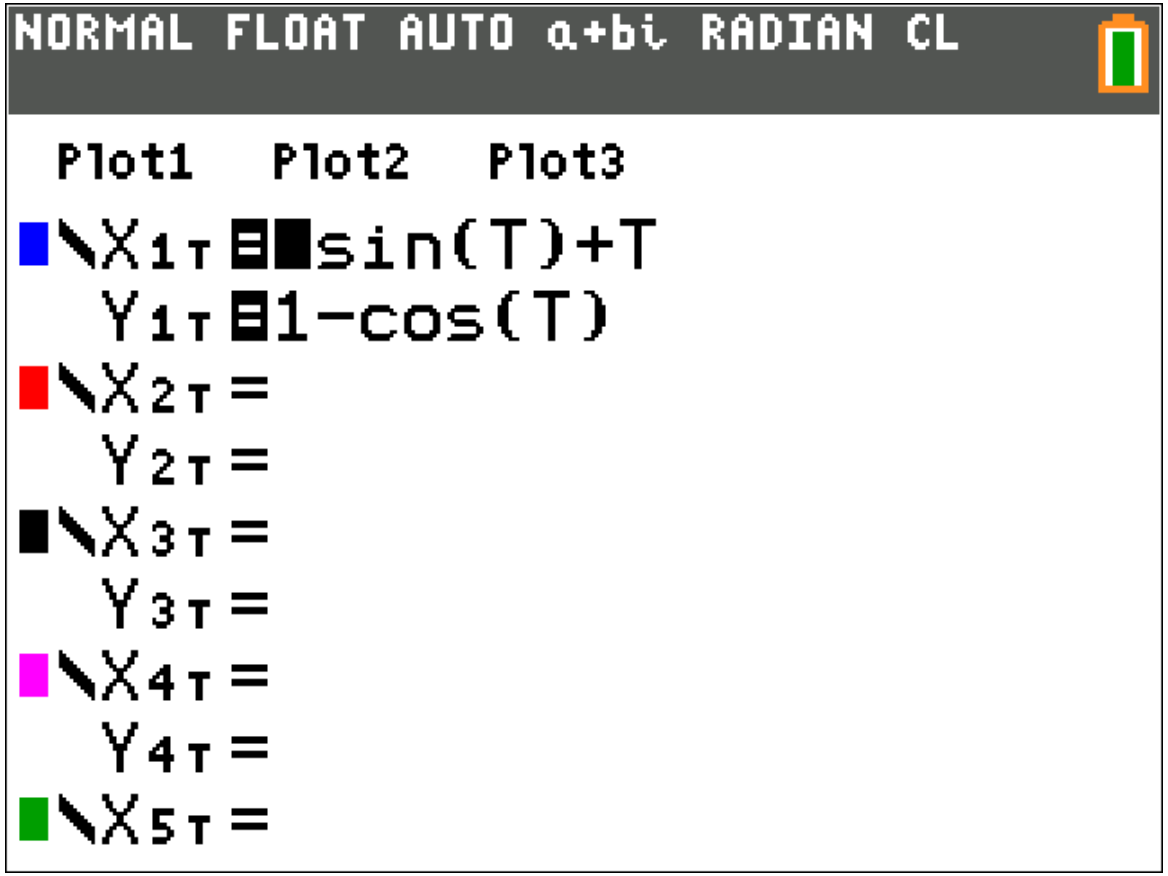
*Mode changed to Classic.

A different audience Discussion: If we use a unit circle, we would start with:
$X_{1T} = -\sin(T)$ and $Y_{1T} = 1 - \cos(T)$ to get the wheel rotating in the correct direction. Then we need to add the linear term.

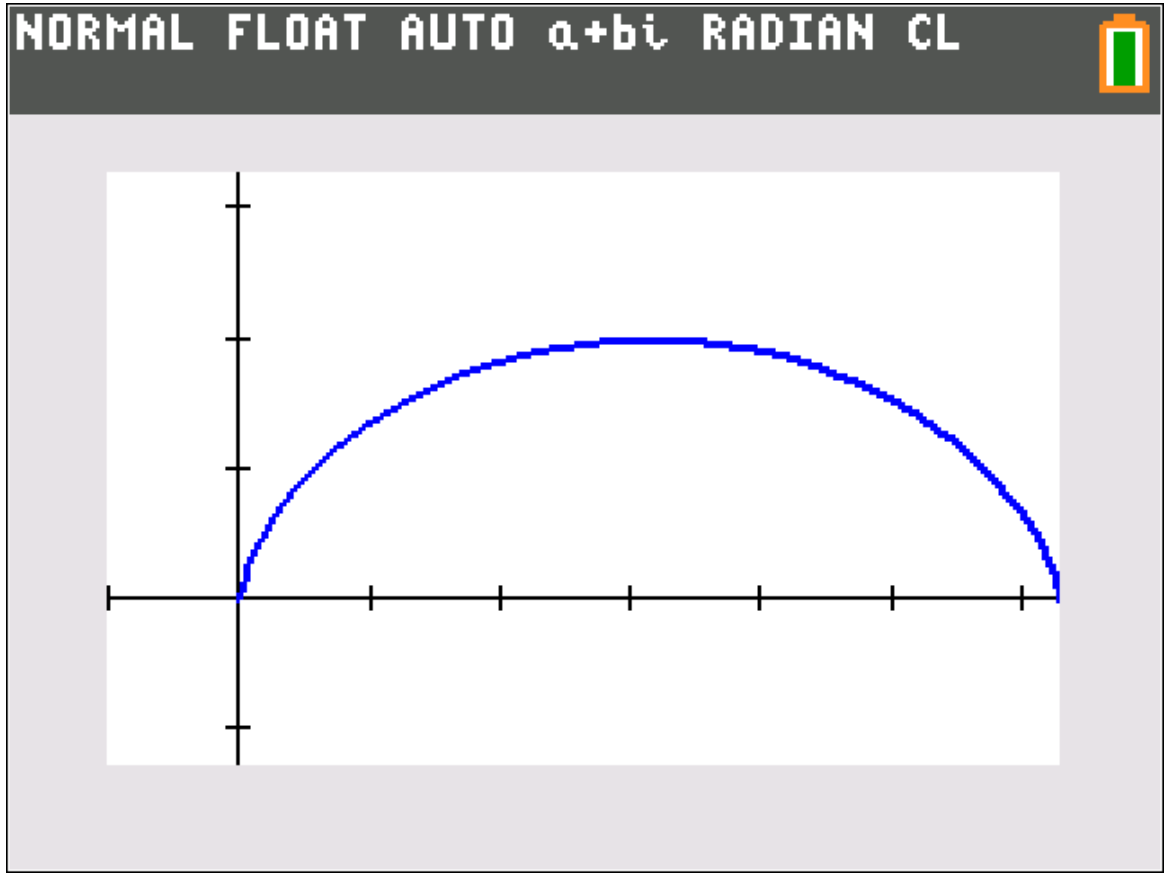
Since the wheel travels a distance of 2π (the circumference) in 1 rotation (or 2π radians) , its velocity is 1 unit/sec so we need to add 1T to the X(T) equation, changing it to $X_{1T} = -\sin(T) + T$.

And since the wheel should travel 2π units, we need to change the window to let $X_{max} = 2\pi$. And, once again, we need to ZSquare to get a more reasonable view.

Equations



Graph

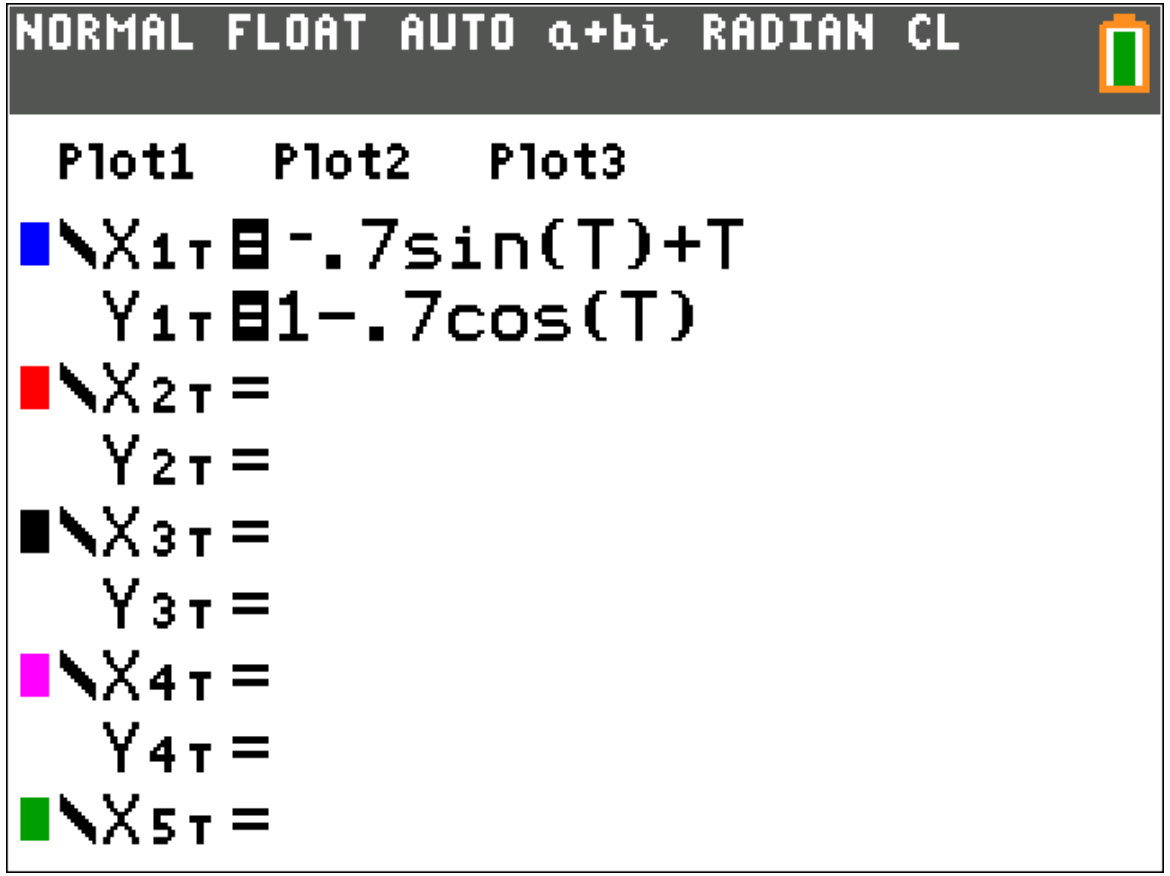


What if a point is not on the rim?

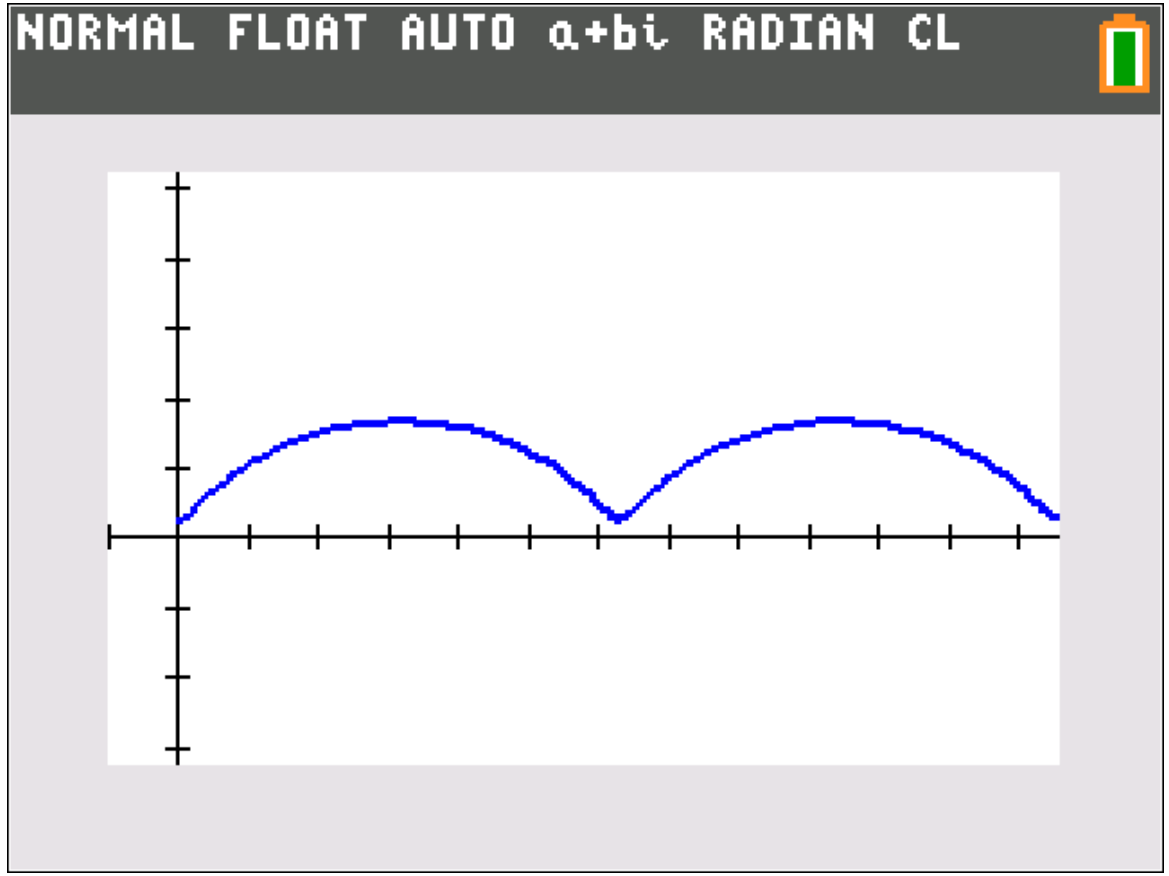
(The rate the wheel rotates won't change but the radius will. So just make the radius of the wheel smaller – or larger if you want to model a train.)

What's the difference between the Unit Circle and a wheel?

Equations*

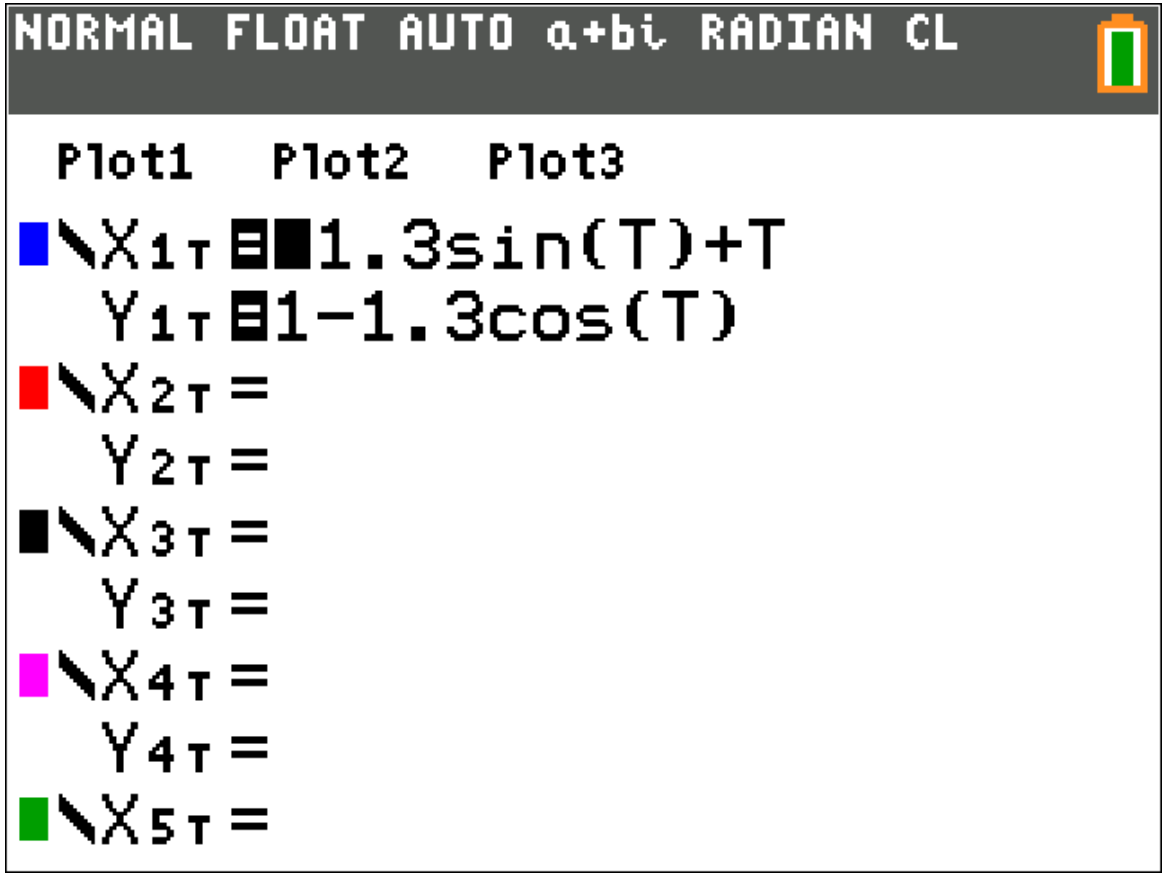


Graph

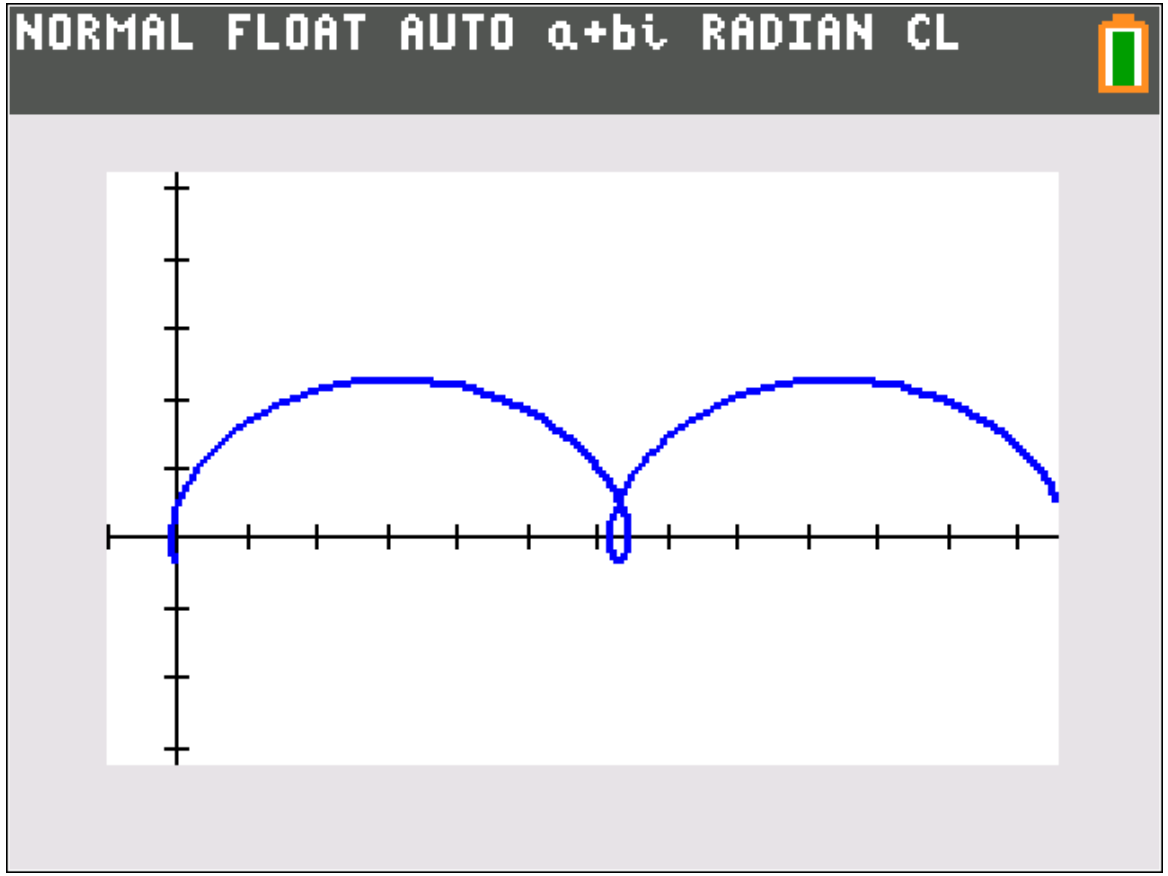


*Window changed to fit 2 rotations.

Equations



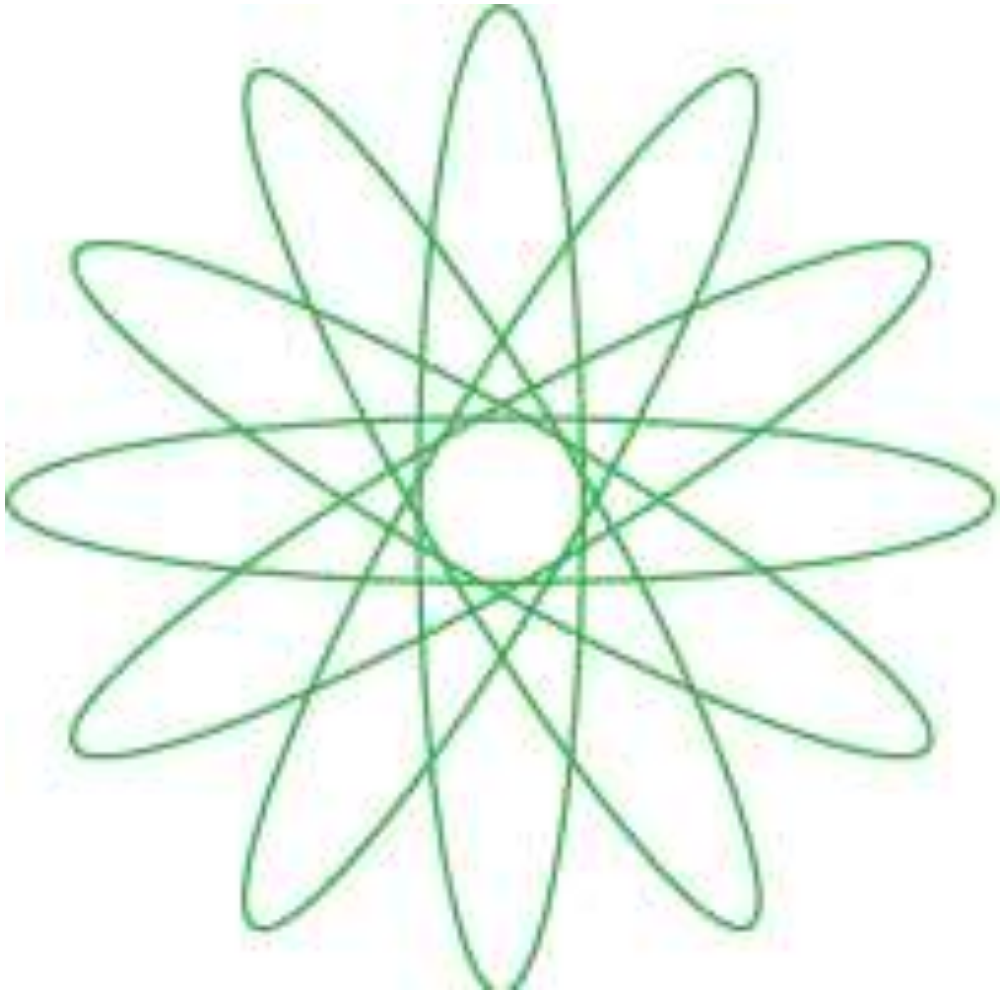
Graph



*Window changed to fit 2 rotations.

Let’s make one to see what happens…

So what about **Spirographs®**



Let’s make one to see what happens…

Here are some links to **Spirograph®** videos to remind you how they work.

So what about **Spirographs®**

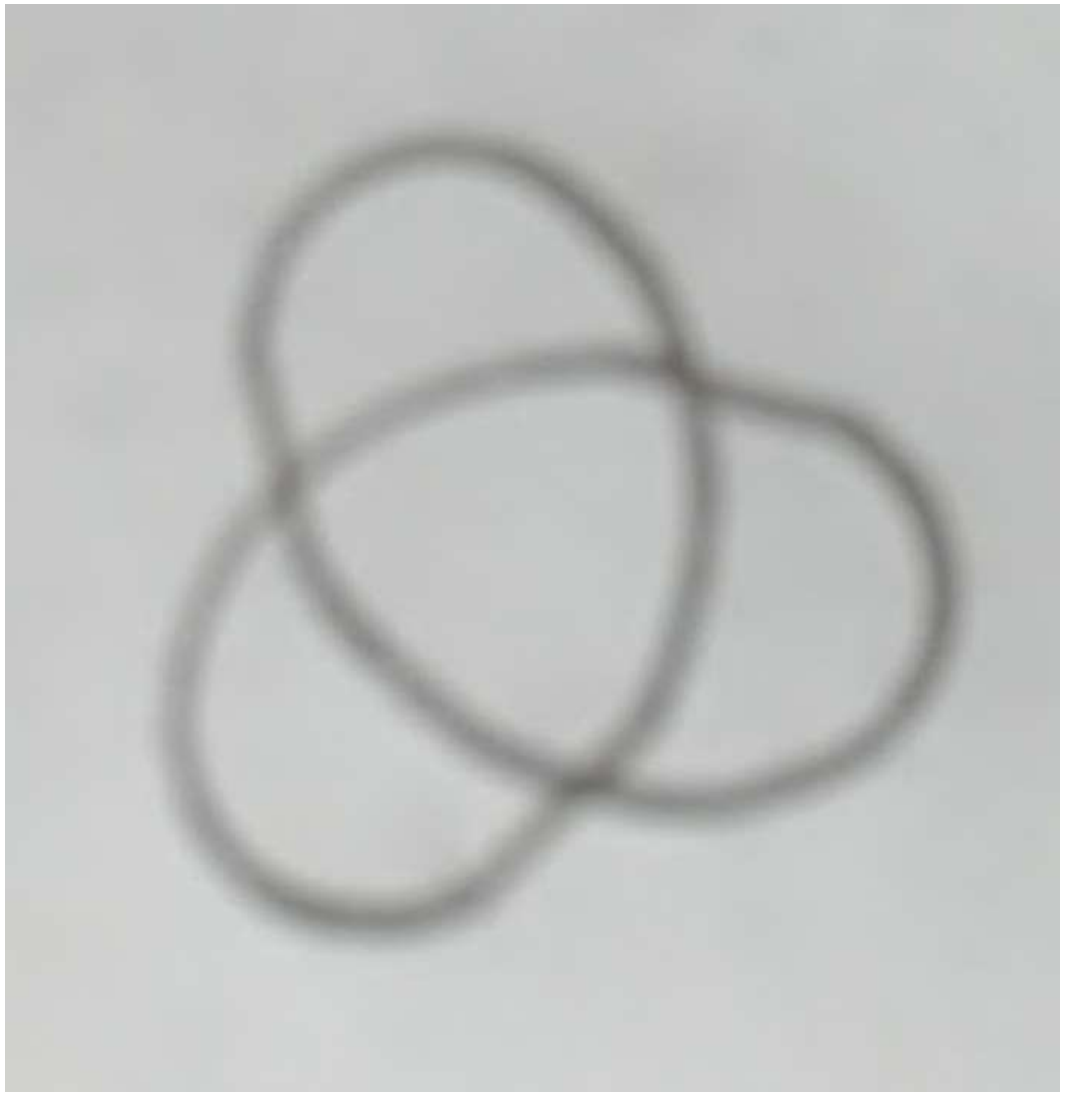
https://www.youtube.com/watch?v=ZkNU3fvELFg

https://www.youtube.com/watch?v=yKCS7KwehQI&t=222s

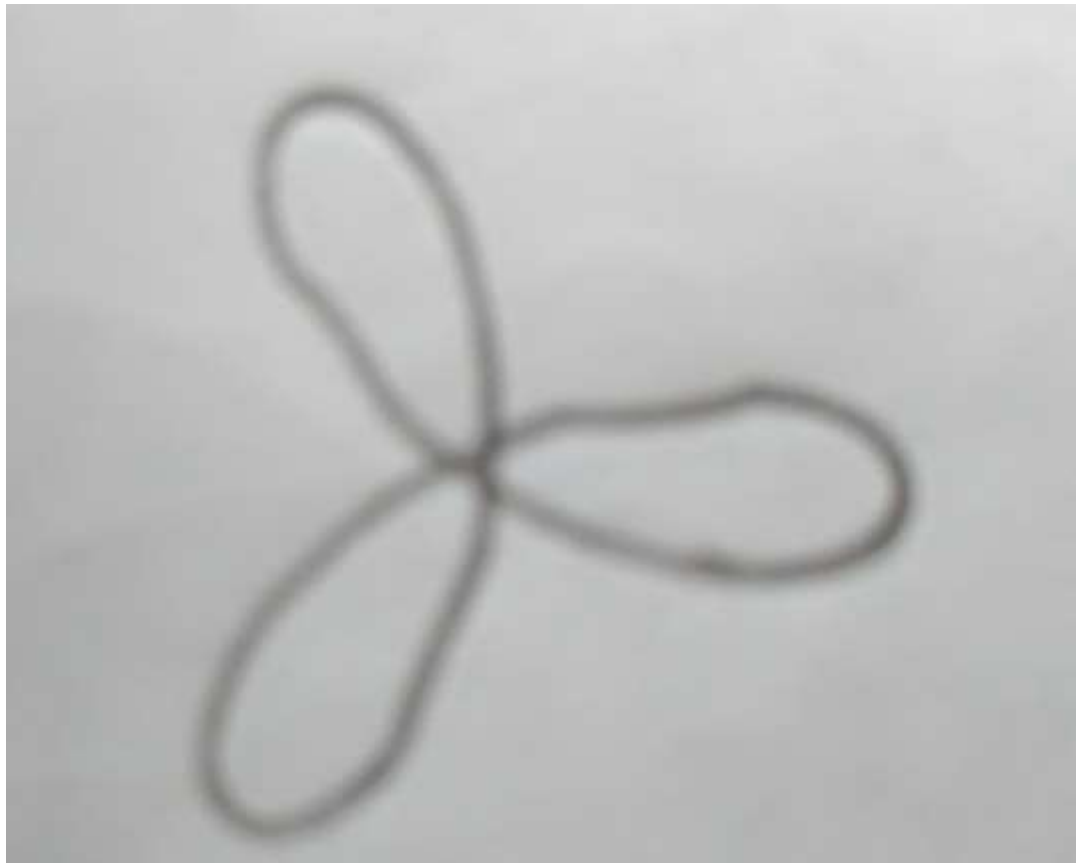
Let’s make one to see what happens…

(Audience is given a small toy that draws a Spirograph® type figure. This particular toy was purchased at Dollar Tree but has been discontinued.)

So what about **Spirographs®**



Here are 2 such designs.



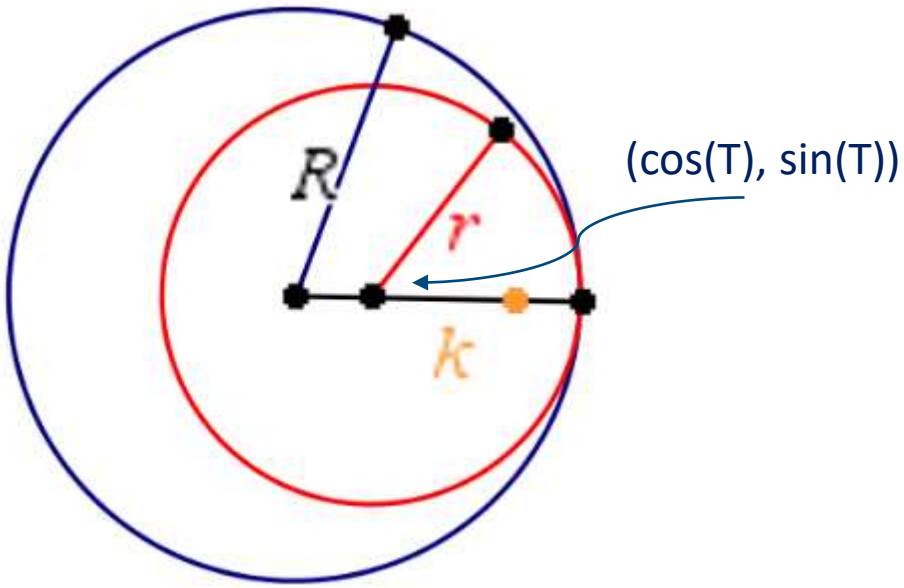
Audience Discussion: It appears that we are adding a circle to a circle. If we placed the pencil at the center of the inner disk, we would get a circle with radius of R – r. Let's call that distance 1 and we would have:

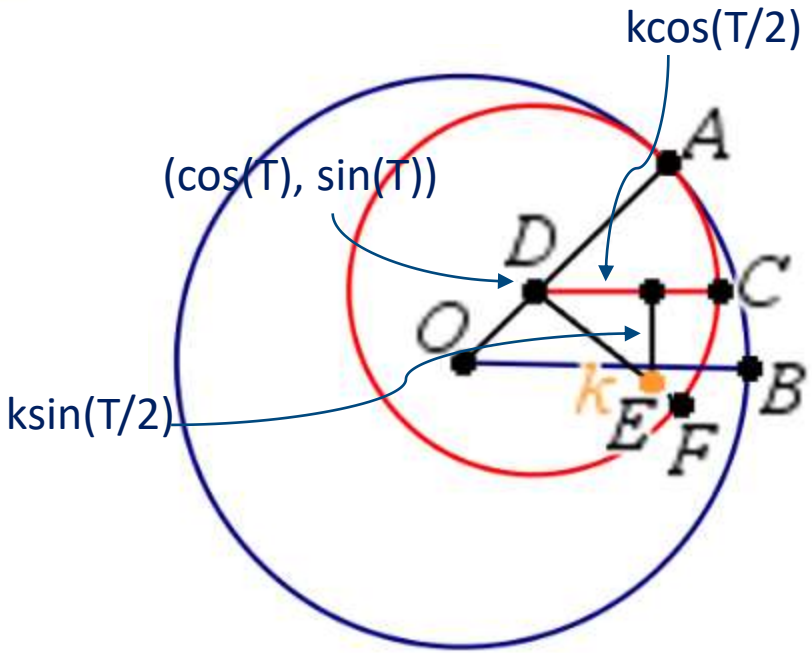
$X_{1T} = \cos(T)$ and $Y_{1T} = \sin(T)$

However, we place the pencil some distance k from the center of the disk.

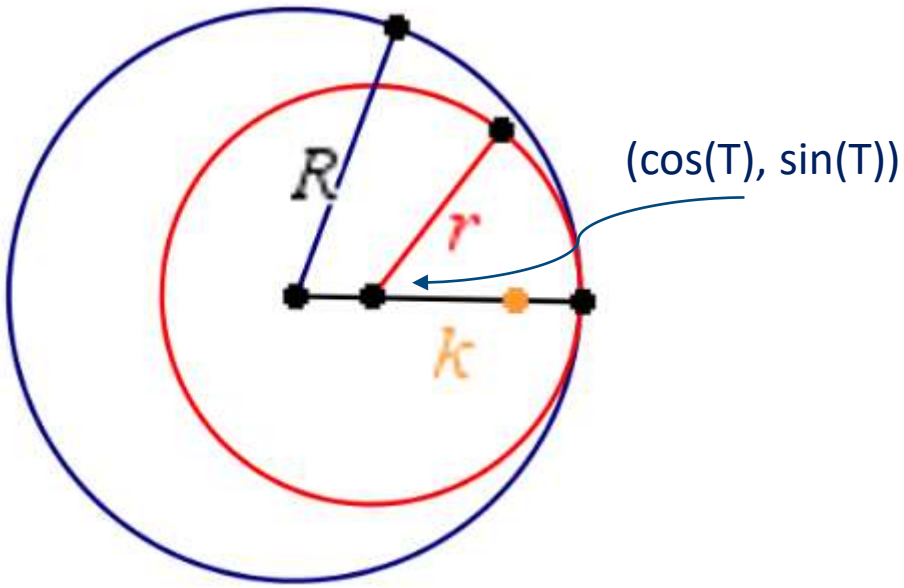
When the disk travels through an angle of θ (∠AOB in the figure) the point k units away (called E in the figure) lies on an angle of –θ/2 (with our toy). This means the new coordinate of E is

$X_{1T} = \cos(T) + k\cos(T/2)$ and $Y_{1T} = \sin(T) - k\sin(T/2)$

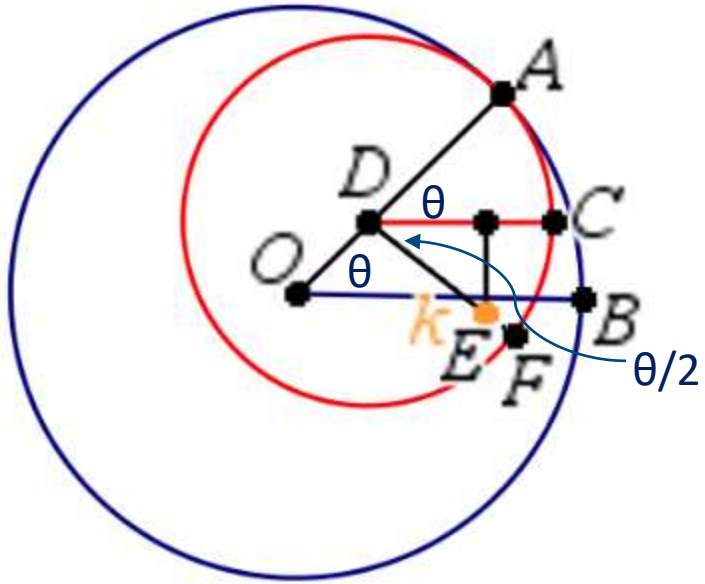




Audience Discussion: Our particular toy has 30 teeth in the outer gear and 20 teeth on the disk. Therefore after 60 teeth have intersected, the figures will be in the same position again. This means the disk will have made 2 trips around the outer gear and that results in 3 “loops”. This also means the ratio of the radii is a 3/2, so without loss of generality, I’ll call the large gear 3 and the disk radius 2.



The reason our second angle is $-\theta/2$ is that : Once the disk has traveled through angle of θ, ($m\angle AOB = \theta$) the arc AB has length 3θ and since arc AB meshed with arc AF, AF has length 3θ also. This means that angle ADF has measure $3\theta/2$. Since OB is parallel to DC, $m\angle ADC = \theta$ also, leaving $m\angle CDF = \theta/2$ and since this angle is clockwise instead of counterclockwise, we can call it $-\theta/2$.



Try This…

$$X_{1T} = \cos(T) + 1\cos(T/2)$$

$$Y_{1T} = \sin(T) - 1\sin(T/2)$$

(Change the 1 to match your figure.)

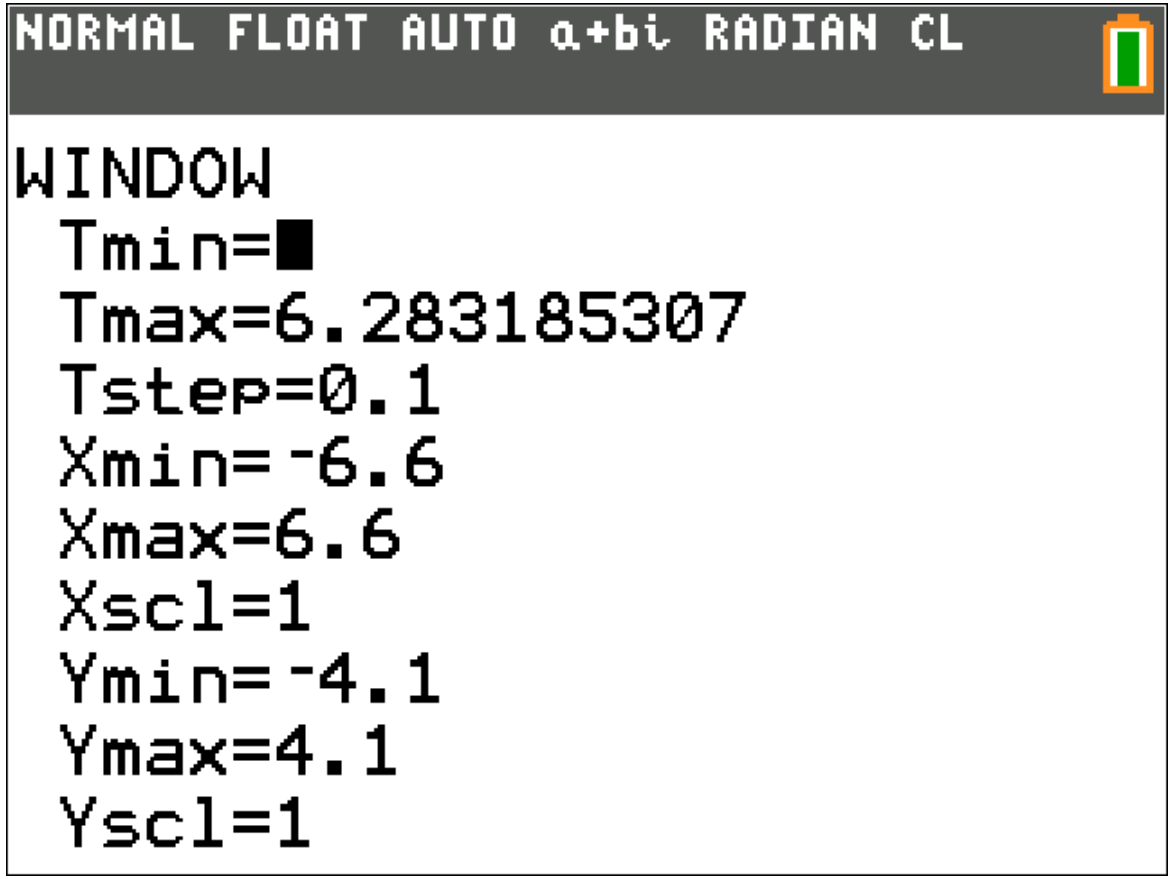
Try This…

$$X_{1T} = \cos(T) + \{1, 1.5, 1.8\}\cos(T/2)$$

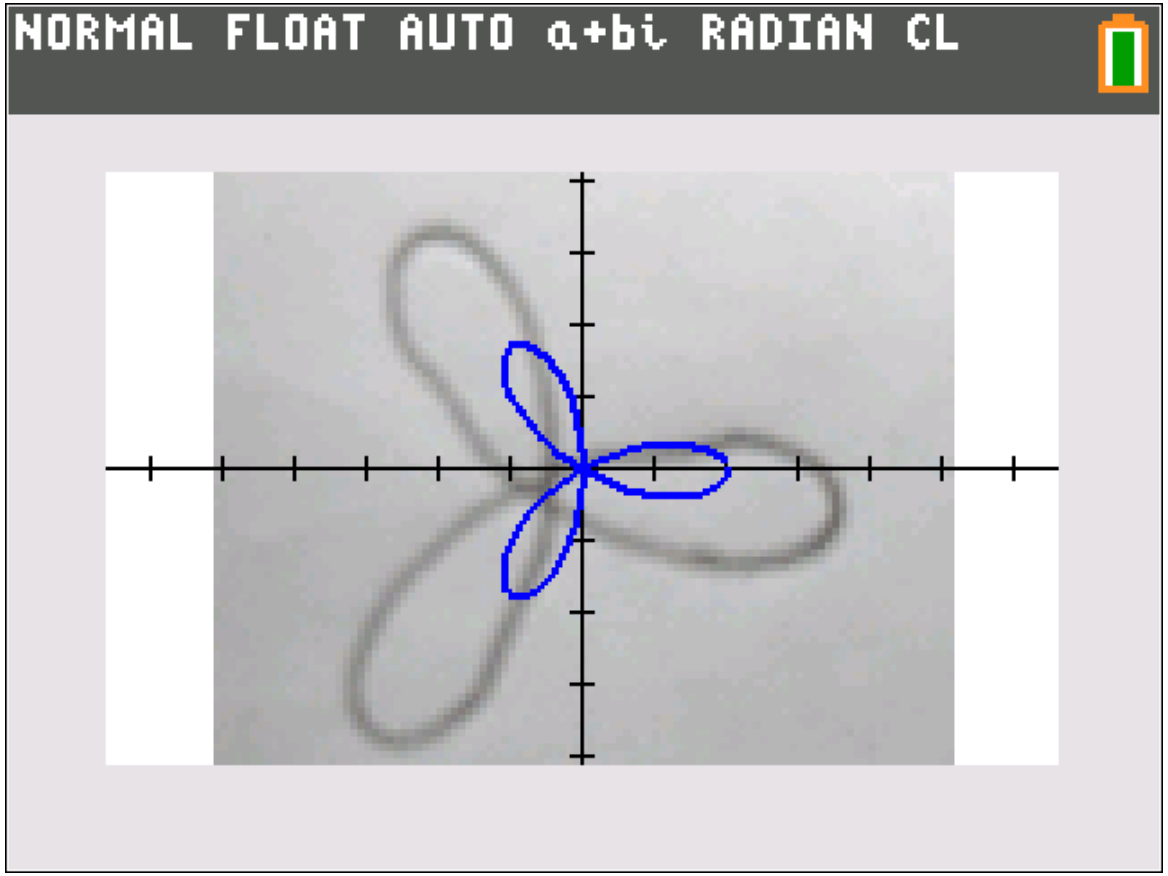
$$Y_{1T} = \sin(T) - \{1, 1.5, 1.8\}\sin(T/2)$$

Note that the set brackets allow one to draw multiple graphs with one command, instead of having to go to Y= and change the coefficient in the equation each time.

Window



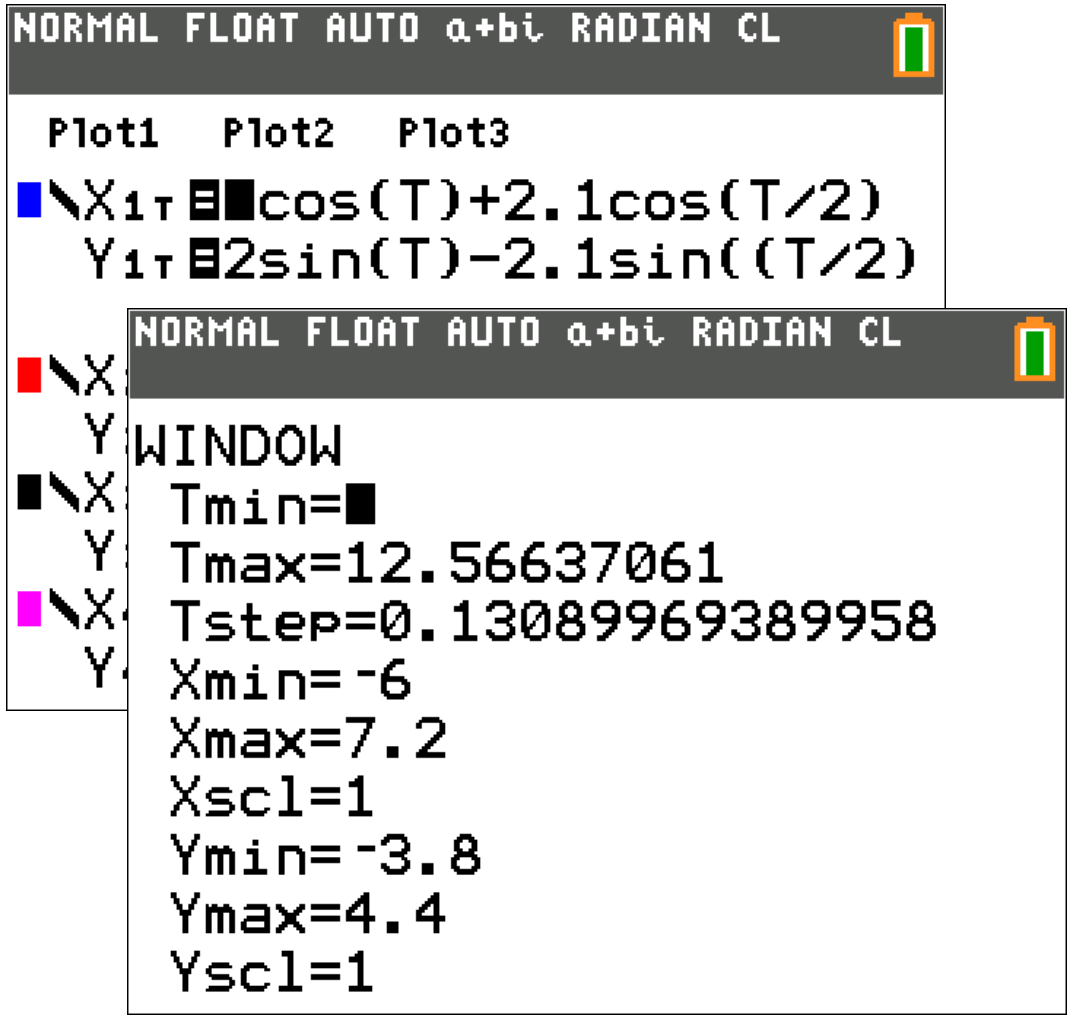
Graph



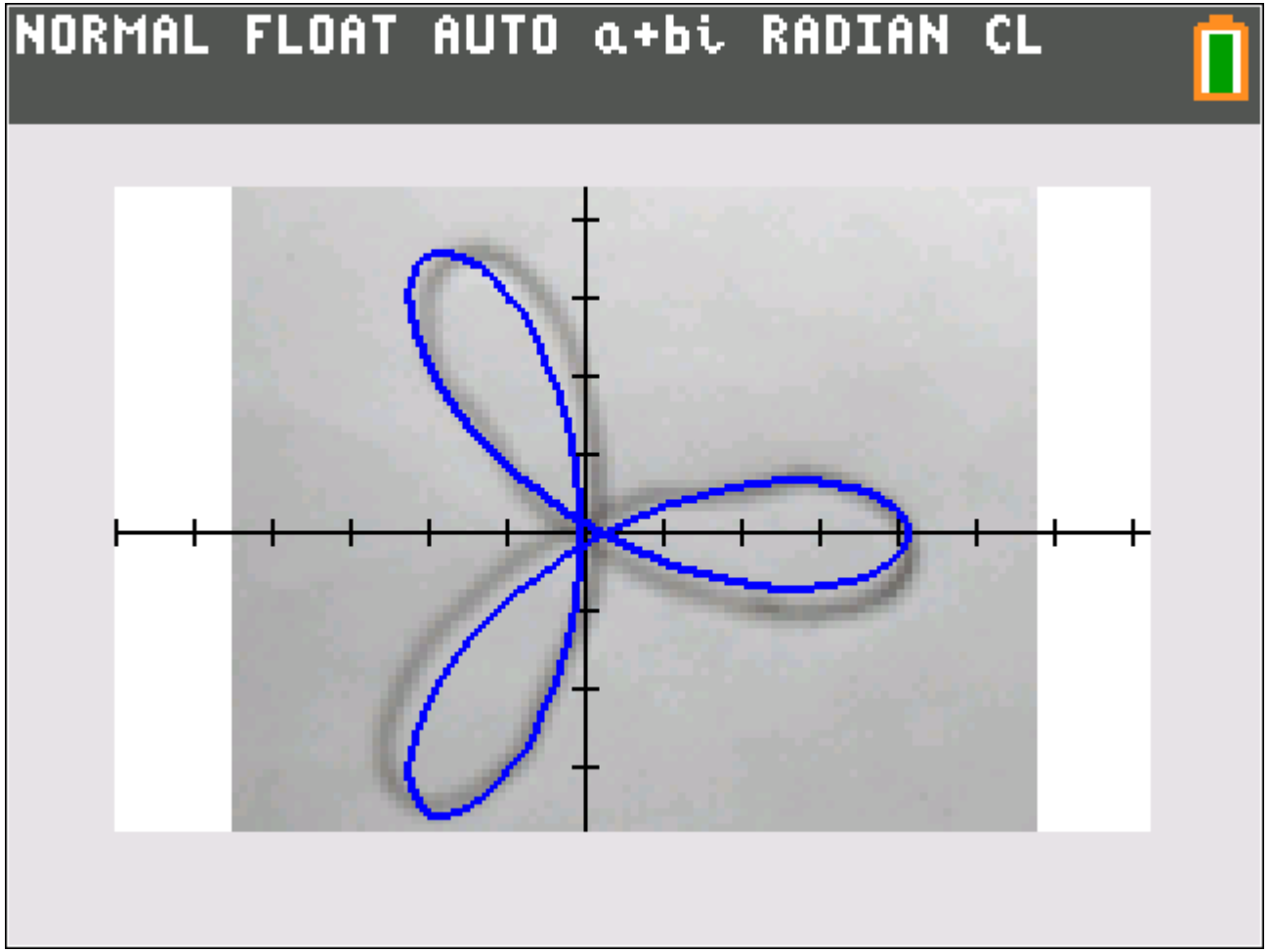
How can we alter our equation to match the picture?

(Move the axis left and down roughly .5 and double the size? And actually, it helped to increase the inner radius by .1 as well. And the graph is still slightly off due to the inaccuracy of the toy and the inaccuracy of the photographer… ☹)

Equation/Window



Graph

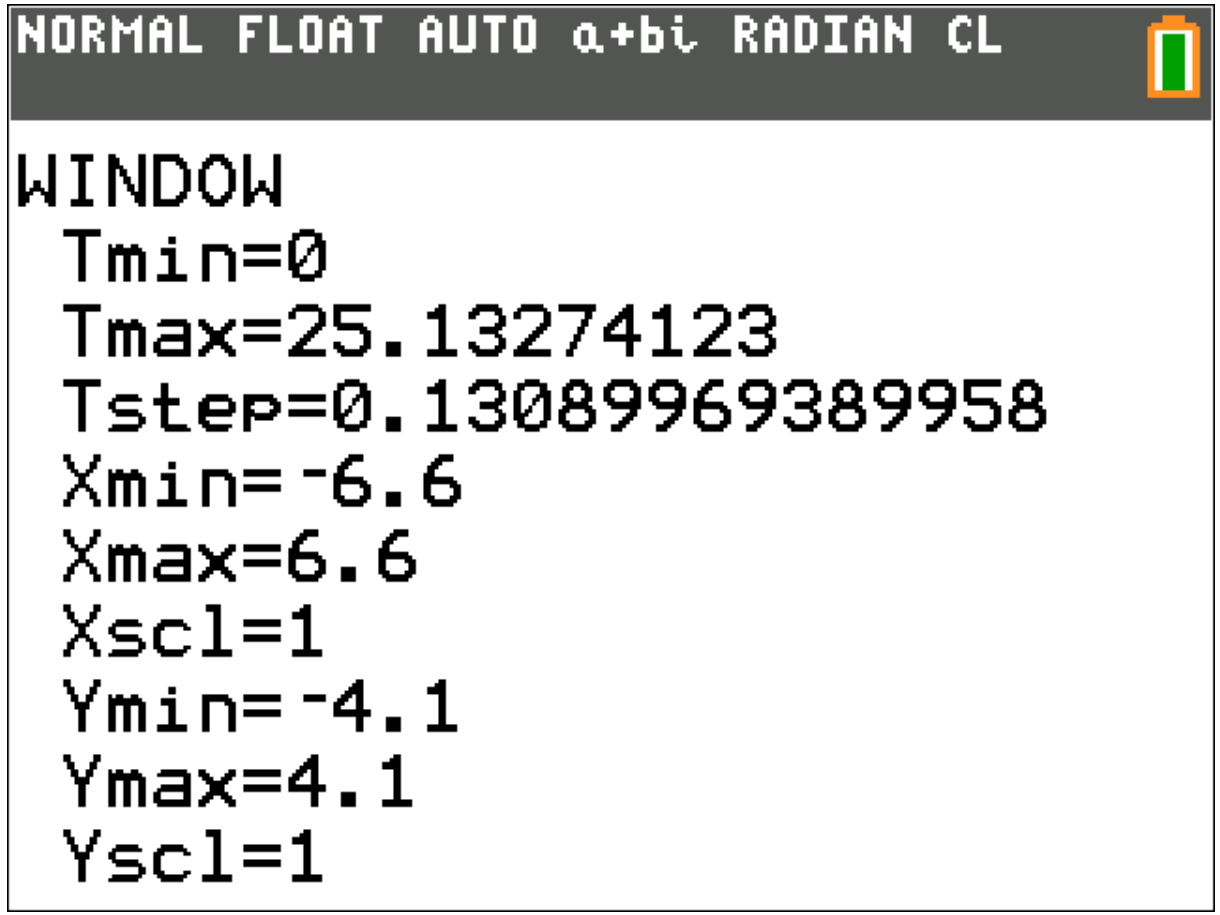


Try This…

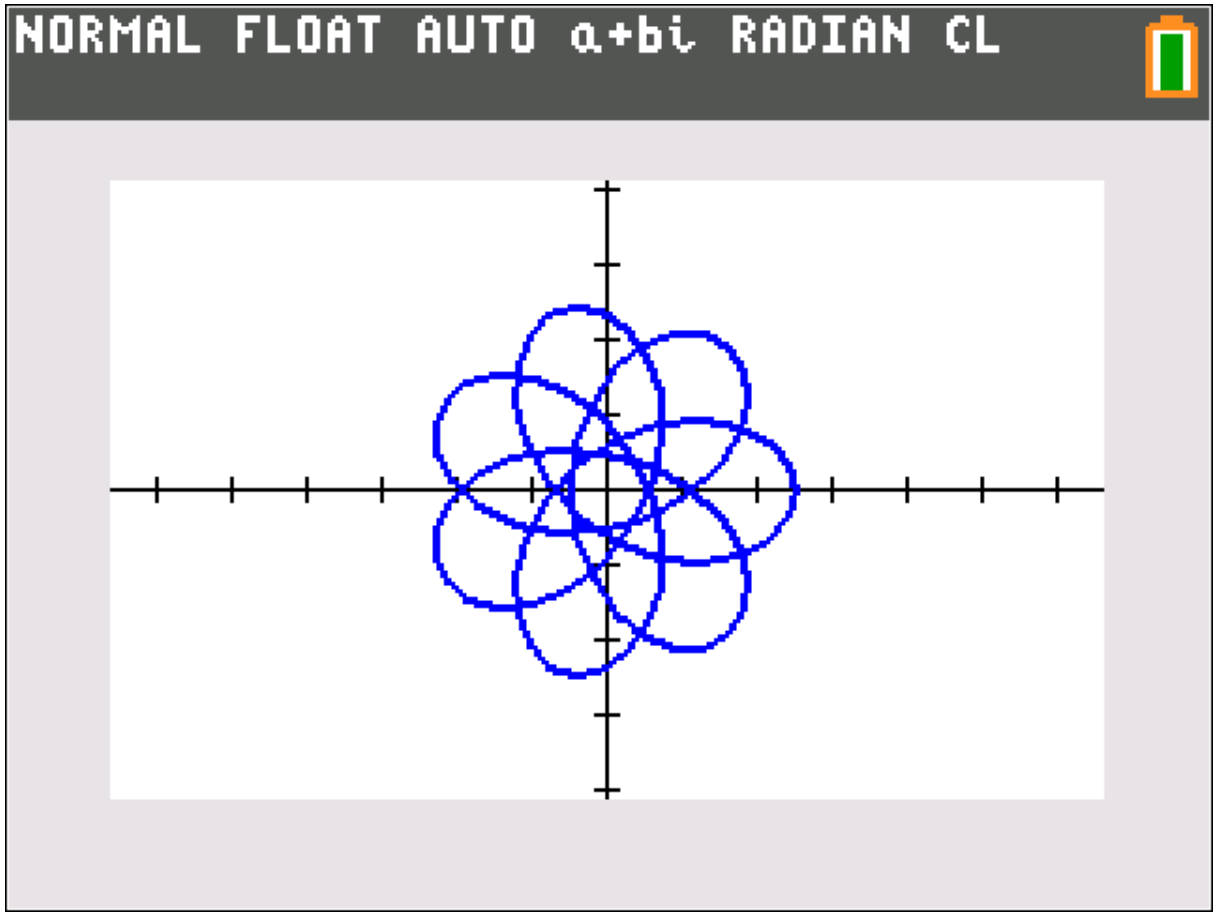
$$X_{1T} = \cos(T) + 1.5\cos(5/2T)$$

$$Y_{1T} = \sin(T) + 1.5\sin(5/2T)$$

Window



Graph



Try This…

$$X_{2T} = \cos(T) + 1.5\cos(4/3T)$$
$$Y_{2T} = \sin(T) + 1.5\sin(4/3T)$$

What do you notice about the sum of the numerator and the denominator of the second coefficient of T?

Audience Discussion: The sum of the numerator and denominator appears to give the number of “loops”.

Think about this and see if you can figure out why…

Thank you!

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