Explorations to Accompany
Limits: Understand, Define, "Do," Apply—
A logical way to convey the concept to calculus students, enabled by technology
To have been presented at the
(canceled) T³ Conference Dallas, 3/14/20, Session 229

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The attached Explorations were written for my students. Working in cooperative groups, the students are introduced to concepts and techniques before a more formal classroom presentation. They can also be used in "guided discovery" format, where students are given, say, two minutes to work on Problem 1. Then this problem is discussed before students are given another minute or two for the next problem, etc. They can even be used as a basis for your classroom presentation without actually giving students copies of the Exploration. They make excellent preview work for students in a "flipped-class" environment.

These Explorations form the basis for the Power Point presentation at the T³ Dallas Conference. The numbers refer to the chapter and section in the presenter's textbook *Calculus: Concepts and Applications.* Titles, in order of presentation, are:

- 1-1a Instantaneous Rate of Change of a Function [**limit** of average rates]
- 2-2a Definition of **Limit**
- 2-2b Epsilon, Delta **Limit** Proof
- 2-5a **Limit** as *x* "Approaches" Infinity [i.e., *x* gets far away from zero]
- 2-7c Epsilon and Delta Techniques Project [provides precalculus review]
- 6-5a Introduction to L'Hospital's Rule [for *calculating* **limits**]
- 1-3a Introduction to Definite Integrals [*x* times *y,* where *y varies* with *x*]
- 1-4a Definite Integrals by Trapezoidal Rule [**limit** of trapezoid areas]
- 5-9a Volume by Plane Disc Slices [**limit** of Riemann sums, dV = (area)(dx)]
- 5-9b Volume by Plane Washer Slices [**limit** of Riemann sums, dV = (area)(dx)]
- 8-4a Volume by Cylindrical Shells [**limit** of Riemann sums, dV = (area)(dx)]

Solutions for these Explorations are in an accompanying PDF file.

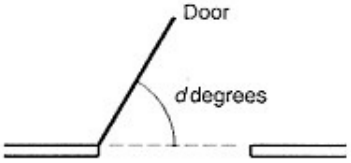
You have permission to use these Explorations with students in your own school. Please do not use them in any way that would infringe on the copyrights. The presenter would welcome your comments and suggestions for making the Explorations and text more useful to your students.

Calculus: Concepts and Applications, originally published by Key Curriculum Press, is now with Kendall Hunt. A free 30-day on-line preview of the text and ancillary material (including all 150+ Explorations) can be accessed at www.flourishkh.com.

Name: _____ Group Members: _____

Foerster Calculus Exploration 1-1a: Instantaneous Rate of Change of a Function Date: _____

Objective: Explore the instantaneous rate of change of a function.



The diagram shows a door with an automatic closer. At time $t = 0$ seconds someone pushes the door. It swings open, slows down, stops, starts closing, then slams shut at time $t = 7$ seconds. As the door is in motion the number of degrees, d, it is from its closed position depends on t.

1. Sketch a reasonable graph of d versus t.

2. Suppose that d is given by the equation

$$d = 200t \cdot 2^{-t}.$$

Plot this graph on your grapher. Sketch the results here.

3. Make a table of values of d for each second from $t = 0$ through $t = 10$. Round to the nearest 0.1°.

t	d
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

4. At time $t = 1$ second, does the door appear to be opening or closing? How do you tell?

5. What is the average rate at which the door is moving for the time interval [1, 1.1]? Based on your answer, does the door seem to be opening or closing at time $t = 1$? Explain.

6. By finding average rates for time intervals [1, 1.01], [1, 1.001], and so on, make a conjecture about the *instantaneous* rate at which the door is moving at time $t = 1$ second.

7. Read Section 1-1. What do you notice?!

8. In calculus you will learn by four methods:
   - algebraically,
   - numerically,
   - graphically,
   - verbally (talking and writing).

   Write a paragraph telling what you have learned as a result of doing this Exploration that you did not know before. Use the back of this sheet.

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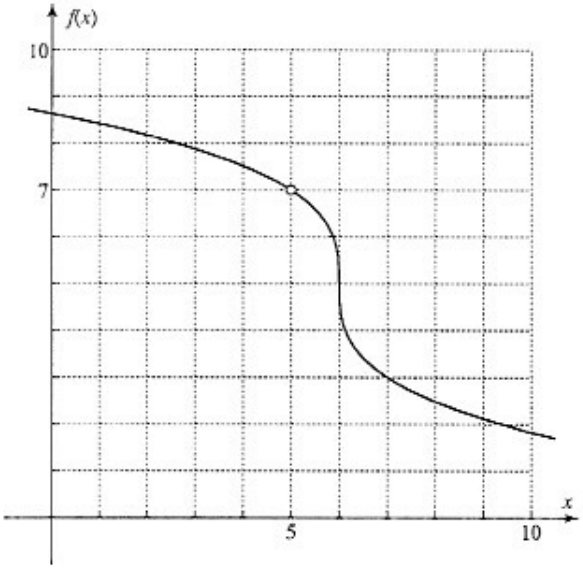
Name _____ Group _____ Date _____

Exploration 2-2a: The Definition of Limit

Objective: Interpret graphically and algebraically the definition of limit.

Let f be the function whose graph is shown here.



1. The limit of $f(x)$ as x approaches 5 is equal to 7. Write the definition as it applies to f at this point.

2. Let $\varepsilon = 1$. From the graph, estimate how close to 5 on the left side x must be kept in order for $f(x)$ to be within ε units of 7.

3. From the graph, estimate how close to 5 on the right side x must be kept in order for $f(x)$ to be within $\varepsilon = 1$ unit of 7.

4. For $\varepsilon = 1$, approximately what is the maximum δ could equal in the definition of limit in order for $f(x)$ to be within ε units of 7 whenever x is within δ units of 5 (but not equal to 5)?

5. The equation of the function graphed is
$f(x) = 5 - 2(x - 6)^{1/3}$, for $x \neq 5$.
Calculate precisely the value of δ from Problem 4.

6. If $\varepsilon = 0.01$, calculate precisely the maximum δ could equal in order for $f(x)$ to be within ε units of 7 whenever x is within δ units of 5 (but not equal to 5).

7. Substitute $(7 - \varepsilon)$ for $f(x)$ and $(5 + \delta)$ for x. Solve for δ in terms of ε. Use the result to show that there is a *positive* value of δ for any $\varepsilon > 0$, no matter how small ε is, and thus that 7 really *is* the limit of $f(x)$ as x approaches 5.

8. What did you learn as a result of doing this Exploration that you did not know before? (Over.)

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Name _____ Group Members: _____

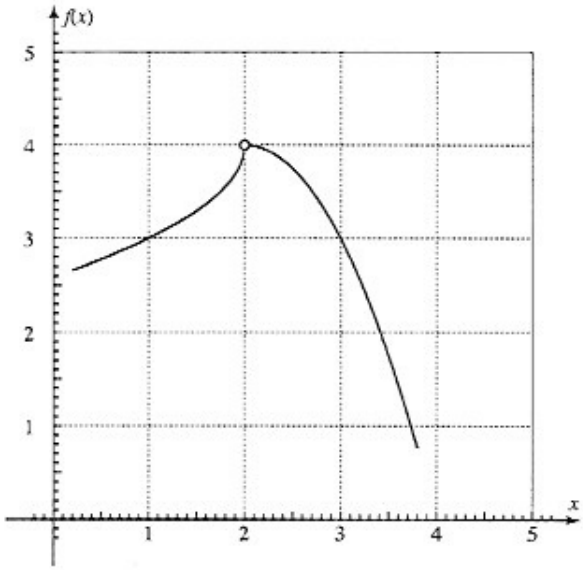
Exploration 2-2b: Epsilon, Delta Limit Proof

Date: _____

Objective: Show that you know, understand, and can use the definition of limit.

1. Write the definition of limit.

For Problems 2–6, the figure shows the graph of a **piecewise function**. Note that $f(2)$ is undefined.



2. What does $L = \lim_{x \to 2} f(x)$ seem to equal? _____

3. Let $\varepsilon = 0.6$ in the definition of limit. On the figure, plot lines at $L + \varepsilon$ and at $L - \varepsilon$. From the points where these lines intersect the graph, estimate to one decimal place the maximum distance to the left and to the right of 2 that x could be kept, and still make $f(x)$ stay within ε units of 4.

   Left: _____ Right: _____

4. From your answers to Problem 3, write the maximum value of δ in the definition of limit such that $f(x)$ stays within ε units of 4 whenever x is kept within δ units of 2 (but not *equal* to 2).

   Max. $\delta \approx$ _____

5. The equation of function f is

$$f(x) = \begin{cases} 4 - \sqrt{2 - x} & \text{if } x < 2 \\ 4 - (x - 2)^2 & \text{if } x > 2 \end{cases}$$

   Substitute $4 - 0.6$ for $f(x)$ in the equation for the more restrictive branch. For x, substitute either $(2 - \delta)$ or $(2 + \delta)$, whichever is appropriate. Then solve the resulting equation for δ. Does the answer agree with your answer to Problem 4?

6. Repeat Problem 5 in general, substituting $4 - \varepsilon$ for $f(x)$ and $2 - \delta$ for x, and showing that there is a positive value of δ for *any* positive value of ε, no matter how small, thus proving that 4 is the limit of $f(x)$ as x approaches 2.

7. What did you learn as a result of doing this Exploration that you did not know before? (Over.)

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Name _____ Group _____ Date _____

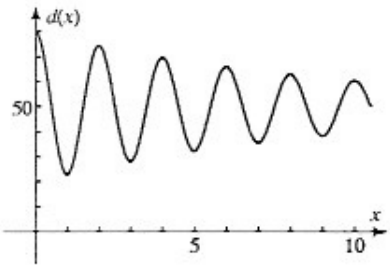
Exploration 2-5a: Limit as x Approaches Infinity

Objective: Discover what it means for a function to approach a limit as x approaches infinity.

A pendulum is pulled away from its rest position and let go. As it swings back and forth its distance, $d(x)$ cm, from the wall is given by the equation

$$d(x) = 50 + 30(0.9)^x \cdot \cos \pi x,$$

where x is time in seconds since it was let go. The graph of function d shows that friction decreases the amplitude of the swings as time goes on.



1. Plot $y_1 = d(x)$ on your grapher using the window shown. Does the graph agree with the figure?
2. The number 50 is the **limit of $d(x)$ as x approaches infinity**. Plot lines $y_2 = 52$ and $y_3 = 48$. If x is large enough, the graph stays within these two lines. By experimenting with the window, find the smallest possible number D for which $d(x)$ stays within 2 units of 50 if x is kept greater than D.
3. The quantity $30(0.9)^x$ in the equation for $d(x)$ is the **amplitude** of the cosine function. Plot $y_4 = 50 + 30(0.9)^x$. Is this graph really an upper bound for $d(x)$? _____
4. Find a new value of $x = D$ that makes $30(0.9)^D = 2$. Sketch the result on the graph in Problem 3. Does $d(x)$ really stay within 2 of 50 if x is kept $> D$?
5. Find quickly a value of $D > 0$ such that $d(x)$ stays within 0.1 unit of 50 for all $x > D$. Show how you get your answer.

Because infinity is not a number, you cannot make x close to infinity. So the limit as x approaches infinity must be modified to say that x is kept far enough away from zero. Here is a formal definition.

$$L = \lim_{x \to \infty} f(x)$$

if and only if

for any number $\varepsilon > 0$ (no matter how small)
there is a number $D > 0$ such that
if x is kept $> D$,
then $f(x)$ stays within ε units of L.

6. For the function d in this Exploration, explain why this statement is false: "The larger x gets, the closer $d(x)$ gets to 50." Explain how the words "stays within" in the definition of limit avoid the misconception that $d(x)$ is always getting closer to the limit.
7. Tell the real-world meaning of the limit 50 for the function d in this Exploration.
8. What did you learn as a result of doing this Exploration that you did not know before?

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Name _____ Group Members: _____

2-7c: Epsilon and Delta Techniques Project

Date: _____

Objective: Make sure you understand the epsilon-delta definition of limit.

1. Let $f(x) = x^3$. Find $\lim_{x \to 2} f(x)$.
   How close must x be kept to 2 in order to make $f(x)$ stay within 0.1 unit of $f(2)$? How close must x be kept to 2 in order to make $f(x)$ to stay within ε units of $f(2)$? How does your answer to this question show that $f(2)$ really *is* the limit of $f(x)$ as x approaches 2?

2. Let $g(x) = \sin \pi x$. Evaluate $g(0.5)$. Sketch the graph of g, showing at least one full cycle. How close must x be kept to 0.5 in order to make $g(x)$ stay within 0.01 unit of $g(0.5)$? Show that there is a positive number δ for any $\varepsilon > 0$ such that if x is within δ units of 0.5 (but not equal to 0.5), then $g(x)$ is within ε units of $g(0.5)$.

3. Let $h(x) = 2^{x-3} + 5$. Find $h(3)$. How close must x be kept to 3 in order to make $h(x)$ stay within ε units of $h(3)$?

4. Let $r(x) = \dfrac{\sin x - 0.5}{x - \pi/6}$
   What form does $r(\pi/6)$ take? What name is given to forms like this? See if you can figure out the limit of $r(x)$ as x approaches $\pi/6$. See if you can figure out what key(s) to press on your calculator to find this number.

(Continued on the back.)

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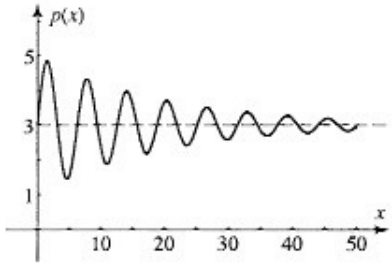
5. Let $p(x) = 3 + 2(0.95^x) \sin x$, graphed here.



Explain why $p(x)$ is within 0.2 unit of 3 whenever $x > D$, where D is the value of x that makes $2(0.95^x) = 0.2$. Find this value of D. Plot these graphs using a window with about $D - 10$ to $D + 10$ for x and $[2.7, 3.3]$ for y. Sketch.

$y_1 = p(x)$, $y_2 = 3.2$, $y_3 = 2.8$, $y_4 = 3 + 2(0.95^x)$

6. Show that for any $\varepsilon > 0$, there is a $D > 0$ such that if $x > D$, then $p(x)$ is within ε units of 3. What does this fact allow you to conclude about the number 3? Why does the line $y = 3$ satisfy the definition of a horizontal asymptote, even though the graph of p crosses the line an infinite number of times?

7. Let $T(x) = \tan \frac{\pi}{2}x$. Find

$$\lim_{x \to 1^-} T(x) \quad \text{and} \quad \lim_{x \to 1^+} r(x)$$

Sketch the graph of function T in the neighborhood of $x = 1$. Why can't you say that $\lim_{x \to 1} T(x) = \infty$? Why can you say that $\lim_{x \to 1} T(x)$ is "infinite?"

8. For function T in Problem 7, how close to 1 on the negative side must you keep x in order for $T(x)$ to be greater than 1000? Show that for any positive number E, no matter how large, there is a number $\delta > 0$ such that if x is within δ units of 1 on the left, but not equal to 1, then $T(x) > E$.

9. What did you learn as a result of doing this Exploration that you did not know before?

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Name _____ Group _____ Date _____

Exploration 6-5a Introduction to l'Hospital's Rule

Objective: Find the number that dy/dx approaches if both dy/dt and dx/dt approach zero at a particular value of t.

1. Plot $f(x) = \dfrac{x^3 - 8x^2 + 21x - 14}{\ln x}$
   Use a friendly window that includes $x = 1$ as a grid point. Sketch the graph.

2. Explain why $f(1)$ is undefined. What does the limit of $f(x)$ seem to be as x approaches 1?

3. Find $\displaystyle\lim_{x \to 1} \frac{\frac{d}{dx}(x^3 - 8x^2 + 21x - 14)}{\frac{d}{dx}(\ln x)}$
   What do you notice?

4. Look up **l'Hospital's rule** in your text. Write the statement of l'Hospital's rule.

5. What form does $\displaystyle\lim_{x \to \infty} x^2 e^{-x}$ take? Tell why this form is indeterminate.

6. L'Hospital's rule also works for the form ∞/∞. Transform the limit in Problem 5 to this form.

7. Use l'Hospital's rule to evaluate the limit in Problem 6.

8. Show that l'Hospital's rule applies to
   $$\lim_{x \to 2} \frac{1 - \cos 3(x - 2)}{(x - 2)^2}$$
   Use the rule to find the limit.

9. Confirm your answer to Problem 8 graphically.

10. Confirm your answer to Problem 8 by table.

11. What did you learn as a result of doing this Exploration that you did not know before? (Over.)

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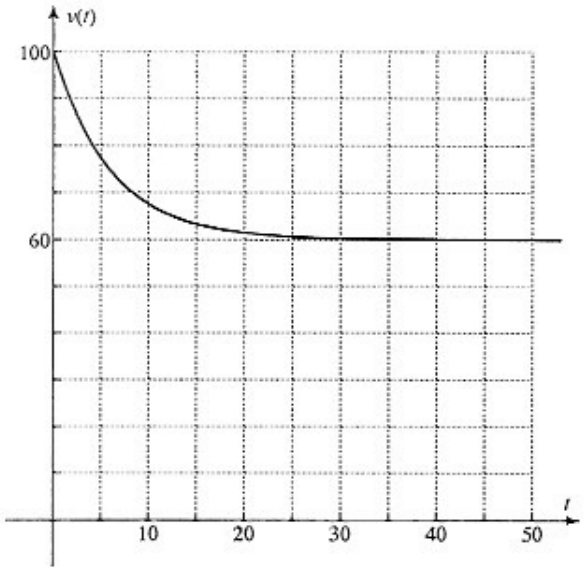
Name: _____ Group Members: _____

Foerster Calculus Exploration 1-3a: Introduction to Definite Integrals Date: _____

Objective: Find out what a definite integral is by working a real-world problem concerning speed of a car.

As you drive on the highway you accelerate to 100 feet per second to pass a truck. After you have passed, you slow down to a more moderate 60 ft/sec. The diagram shows the graph of your velocity, $v(t)$, as a function of the number of seconds, t, since you started slowing.



1. What does your velocity seem to be between $t = 30$ and $t = 50$ seconds? How far do you travel in the time interval [30, 50]?

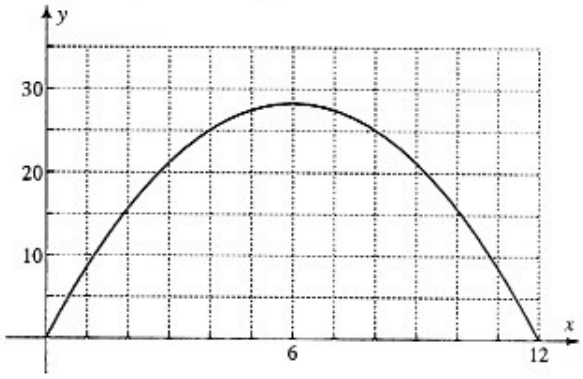
2. Explain why the answer to Problem 1 can be represented as the area of a *rectangular* region of the graph. Shade this region.

3. The distance you travel between $t = 0$ and $t = 20$ can also be represented as the area of a region bounded by the (curved) graph. Count the number of squares in this region. Estimate the area of parts of squares to the nearest 0.1 square space. For instance, how would you count this partial square?

4. How many feet does each small square on the graph represent? How far, therefore, did you go in the time interval [0, 20]?

5. Problems 3 and 4 involve finding the product of the x-value and the y-value for a function where y may *vary* with x. Such a product is called the **definite integral** of y with respect to x. Based on the units of t and $v(t)$, explain why the definite integral of $v(t)$ with respect to t in Problem 4 has feet for its units.

6. The graph shows the cross-sectional area, y square inches, of a football as a function of the distance, x inches, from one of its ends. Estimate the definite integral of y with respect to x.



7. What are the units of the definite integral in Problem 6? What, therefore, do you suppose the definite integral represents?

8. What have you learned as a result of doing this Exploration that you did not know before? (Over.)

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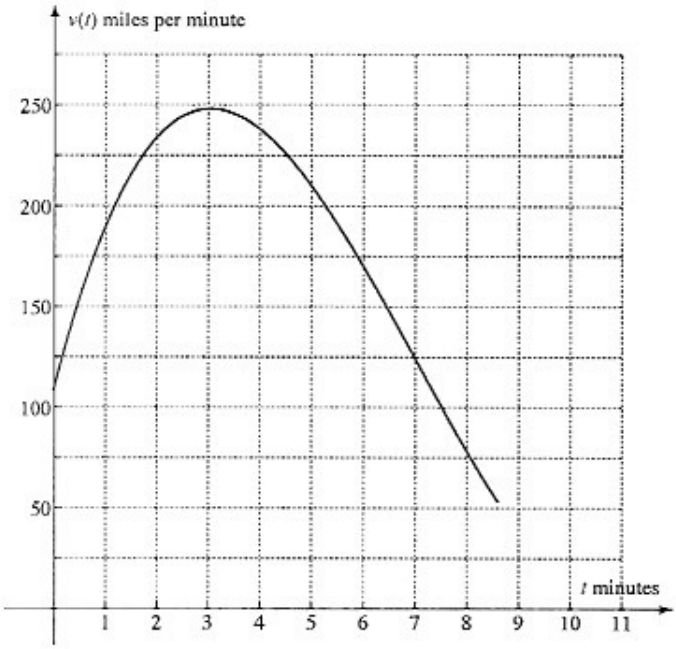
Name: _____ Group Members: _____

Foerster Calculus Exploration 1-4a: Definite Integrals by Trapezoidal Rule Date: _____

Objective: Estimate the definite integral of a function numerically instead of graphically by counting squares.

Rocket Problem: Ella Vader (Darth's daughter) is driving in her rocket ship. At time $t = 0$ minutes she fires her rocket engine. The ship speeds up for awhile, then slows down as Alderaan's gravity takes its effect. The graph of her velocity, $v(t)$ miles per minute, is shown below.



1. What mathematical concept would be used to estimate the distance Ella goes between $t = 0$ and $t = 8$?

2. Estimate the distance in Problem 1 geometrically.

3. Ella figures that her velocity is given by

$$v(t) = t^3 - 21t^2 + 100t + 110.$$

Plot this graph on your grapher. Does the graph confirm or refute what Ella figures? Tell how you arrive at your conclusion.

4. Divide the region under the graph from $t = 0$ to $t = 8$, which represents the distance, into four strips of equal width. Draw four trapezoids whose areas approximate the areas of these strips, and whose parallel sides go from the x-axis to the graph. By finding the areas of these trapezoids, estimate the distance Ella goes. Does the answer agree with Problem 2?

5. The technique in Problem 4 is the **trapezoidal rule**. Put a program into your grapher to use this rule. The function equation may be stored as y_1. The input should be the starting time, the ending time, and the number of trapezoids. The output should be the value of the definite integral. Test your program by using it to answer Problem 4.

6. Use the program from Problem 5 to estimate the definite integral using 20 trapezoids.

7. The *exact* value of the definite integral is the *limit* of the estimates by trapezoids as the width of each trapezoid approaches zero. By using the program from Problem 5 make a conjecture about the exact value of the definite integral.

8. What is the fastest Ella went? At what time was that?

9. Approximately what was Ella's rate of change of velocity when $t = 5$? Was she speeding up or slowing down at that time?

10. At what time does Ella stop? Based on the graph, does she stop abruptly or gradually?

11. What have you learned as a result of doing this Exploration that you did not know before? (Over.)

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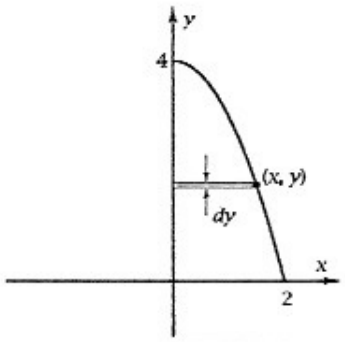
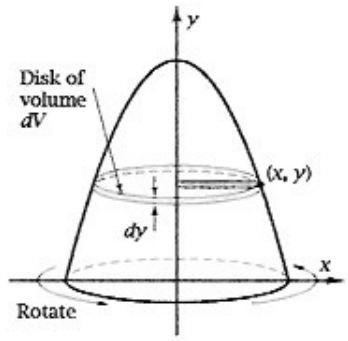
Name: _____ Group Members: _____

Foerster Calculus Exploration 5-9a: Volume by Plane Disk Slices Date: _____

Objective: Calculate exactly the volume of a solid of variable cross-sectional area.



Draw the region.
Slice a strip perpendicular
to the axis of rotation.



Rotate the region and the strip to form a solid.
The rotating strip forms a flat disk.

The region in Quadrant I bounded by the parabola

$$y = 4 - x^2$$

is rotated about the y-axis to form a solid paraboloid, as shown above.

1. A representative slice of the region is shown in the left-hand figure. As the slice turns, it generates a **plane disk**, as shown in the right-hand figure. Let (x, y) be a sample point on the graph within the slice. Find the volume, dV, of the slice in terms of the coordinates of the sample point.

2. By appropriate algebra, transform dV so that it is terms of y alone.

3. Find, exactly, the volume V of the solid by adding the volumes of the slabs and taking the limit as the thickness of the slabs goes to zero (i.e., integrate).

4. As a rough check on your answer, compare it with the volume of a circumscribed cylinder and with the volume of an inscribed cone.

5. Based on your answer to Problem 4, make a conjecture about the volume of a paraboloid in relationship to the volume of the circumscribed cylinder.

6. What did you learn as a result of doing this Exploration that you did not know before?

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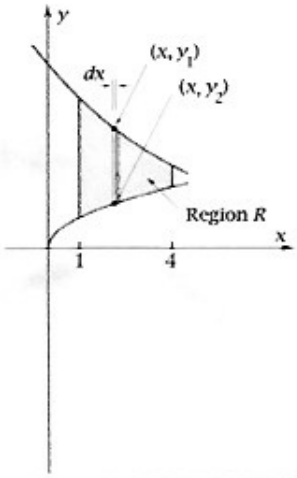
Name: _____ Group Members: _____

Exploration 5-9b: Volume by Plane Washer Slices

Date: _____

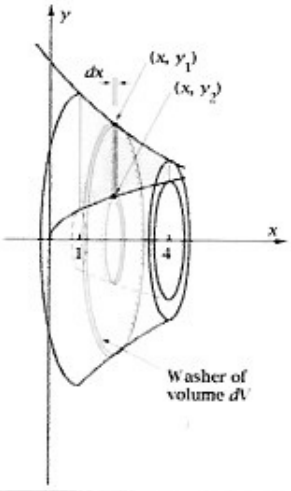
Objective: Find the volume of a solid of revolution that has a hole in it (Section 8-5).



The figure above shows the region R that is bounded above and below by the graphs of

$$y_1 = 6e^{-0.2x} \quad \text{and} \quad y_2 = \sqrt{x},$$

and on the left and right by the lines $x = 1$ and $x = 4$. The figure below shows the solid generated by rotating R about the x-axis. A strip of width dx generates a **plane washer** of thickness dx whose cross-section is a **circular ring** of outer radius y_1 and inner radius y_2.



1. Write the area of the circular ring as a function of y_1 and y_2. Use the result to write the volume, dV, of the washer.

2. Transform the equation for dV so that it is in terms of x alone.

3. Find V by adding up the dV's and taking the limit (i.e., integrate). Use the fundamental theorem to evaluate the integral *exactly*.

4. Get a decimal approximation for the volume you found in Problem 3.

5. Do the integration in Problem 3 numerically. Show that the answer is (virtually) the same as in Problem 4.

6. Check your answer with Example 3 in Section 8-5.

7. What have you learned as a result of doing this Exploration that you did not know before?

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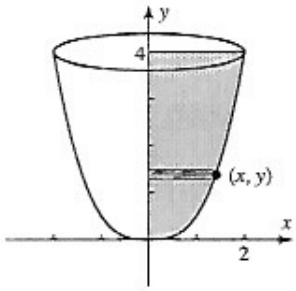
Name _____ Group Members: _____

Exploration 8-4a: Volumes by Cylindrical Shells

Date: _____

Objective: Find the volume of a solid of revolution by appropriate calculus.

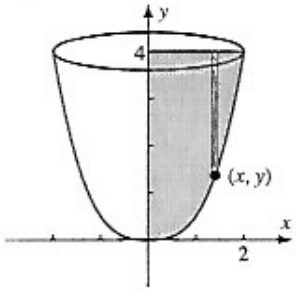
For Problems 1 and 2, the figure shows the region R in Quadrant I bounded by the graph of $y = 0.5x^3$ and the line $y = 4$. The region is rotated about the y-axis to form a solid.

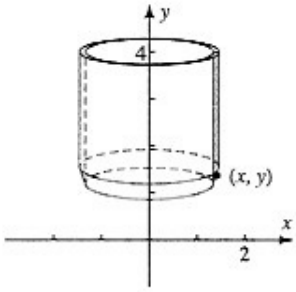


1. The strip shown is perpendicular to the axis of rotation. The sample point (x, y) is on the graph, within the strip. Draw the slice of the figure formed by the strip as it rotates. Then write dV, the volume of the slice, in terms of the sample point.

2. Do the algebra to get dV in terms of one variable. Then find the volume of the entire solid by appropriate integration. You may leave the answer as a multiple of π.

For Problems 3–4, the figure on the left shows the same solid, but with the strip of the region drawn *parallel* to the axis of rotation. As the region rotates, the strip generates a **cylindrical shell**, as shown on the right.

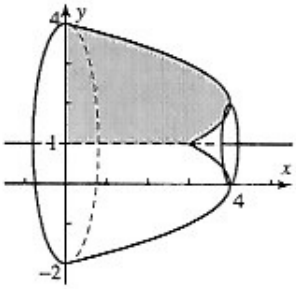




3. Write dV, the volume of the cylindrical shell, in terms of the sample point. Note that the volume equals the circumference of the shell at the sample point times the altitude of the shell at the sample point times the thickness of the shell.

4. Do the algebra to get dV in terms of one variable. Then write an integral equal to the volume of the entire solid. (What are the limits of integration?). Find the volume by doing the integration. If you don't get the same answer as in Problem 2, go back and check your work.

5. The figure shows the solid formed by rotating about the line $y = 1$ the region bounded by the graphs of $x = 4y - y^2$, the y-axis, and the line $y = 1$. Find the volume of the solid by slicing the region parallel to the axis of rotation and using the resulting cylindrical shells.



6. What did you learn as a result of doing this exploration that you did not know before? (Over.)

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