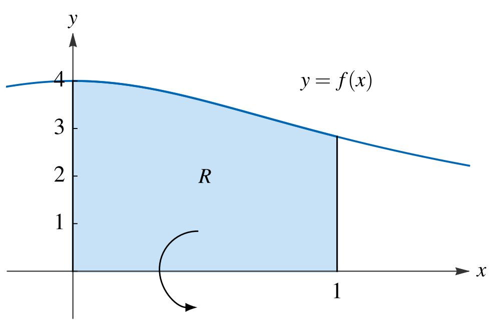
Monday Night Calculus

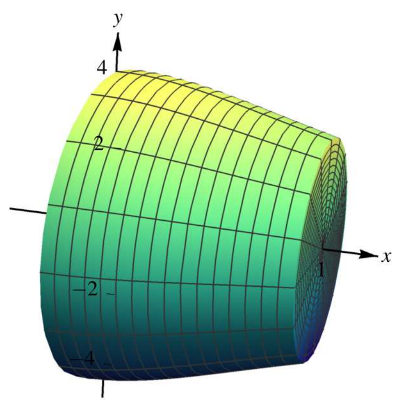
Volume

Exercises

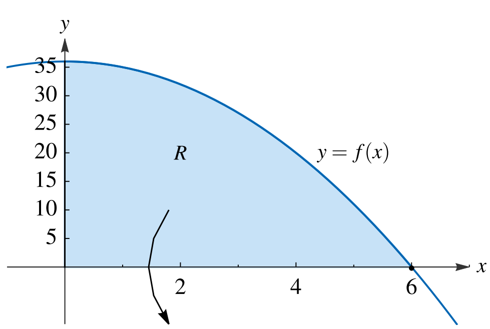
1. Let R be the region in the first quadrant bounded by the graph of $y = \dfrac{4}{\sqrt{1+x^2}}$, the coordinate axes, and the vertical line $x = 1$. Find the volume of the solid obtained when R is rotated about the x-axis.

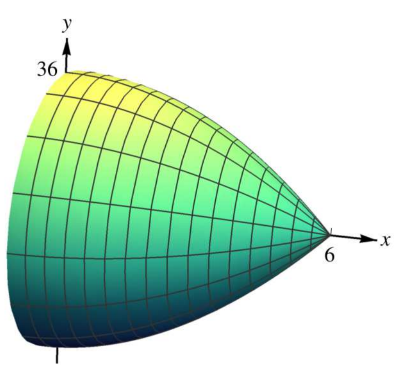
$$V = \pi \int_0^1 \left(\frac{4}{\sqrt{1+x^2}}\right)^2 dx = 16\pi \int_0^1 \frac{1}{1+x^2}\, dx$$
$$= 16\pi \Big[\tan^{-1} x\Big]_0^1 = 16\pi[\tan^{-1} 1 - \tan^{-1} 0]$$
$$= 16\pi\left[\frac{\pi}{4} - 0\right] = 4\pi^2$$





2. (a) Let R be the region in the first quadrant bounded by the graph of $y = 36 - x^2$ and the coordinate axes. A container has the shape of the solid formed by rotating the region R about the x-axis. If the units on the axes are centimeters, how many liters of water does the container hold?





$$V = \pi \int_0^6 (36 - x^2)^2 \, dx = \pi \int_0^6 (1296 - 72x^2 + x^4) \, dx$$

$$= \pi \left[1296x - 24x^3 + \frac{x^5}{5} \right]_0^6$$

$$= \pi \left[\left(1296 \cdot 6 - 24 \cdot 6^3 + \frac{6^5}{5} \right) - (0) \right] = \frac{20736\pi}{5}$$

$= 13028.813$ cubic cm or 13.029 liters

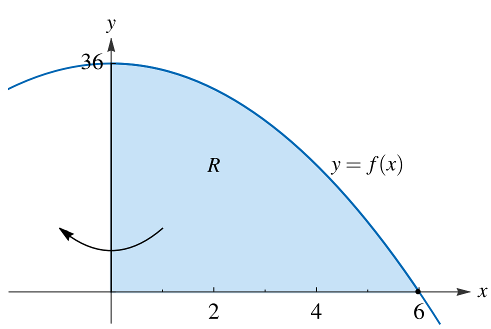
(b) Suppose a second container has the shape of the solid formed by rotating the region R (described in part (a)) about the y-axis. Find the resulting volume of the container.

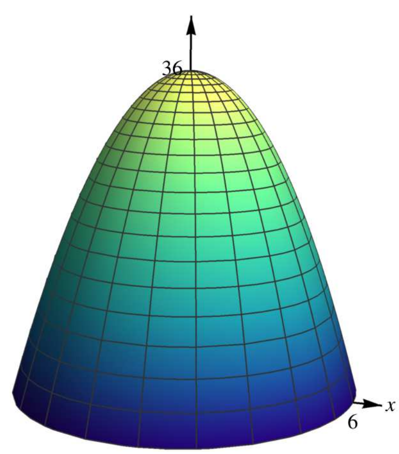
$$y = 36 - x^2 \quad \Rightarrow \quad x = \sqrt{36 - y}$$

$$V = \pi \int_0^{36} (\sqrt{36 - y})^2 \, dy = \pi \int_0^{36} (36 - y) \, dy$$

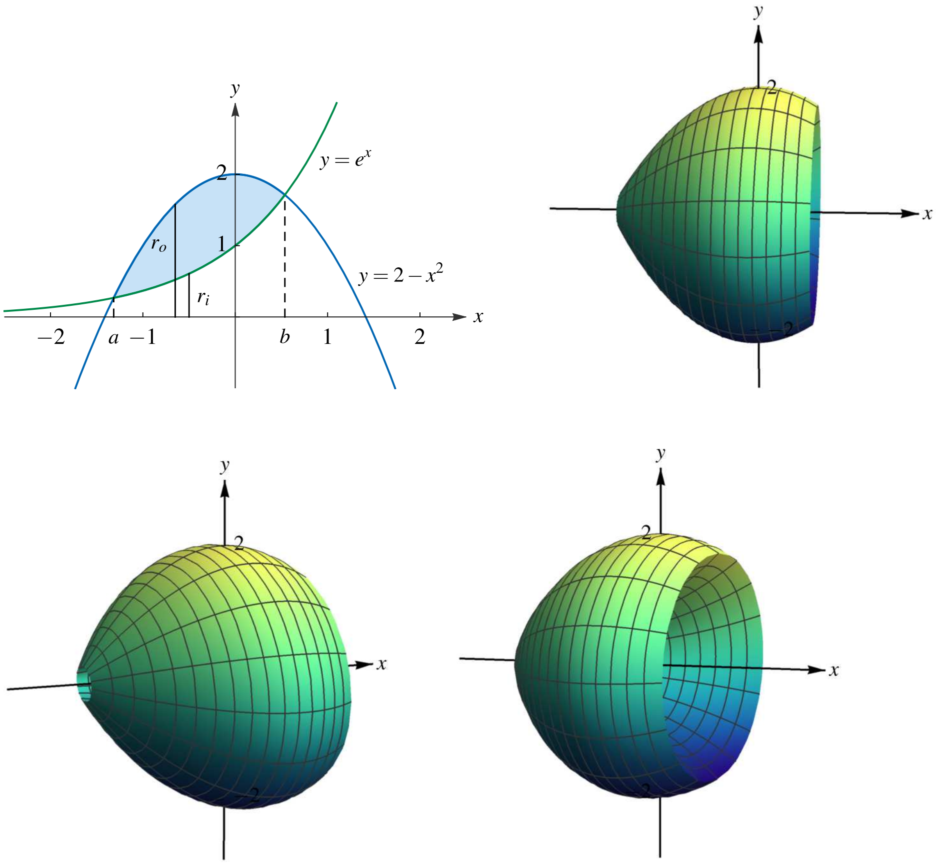
$$= \pi \left[36y - \frac{y^2}{2} \right]_0^{36} = \pi \left[36^2 - \frac{36^2}{2} \right] = 648\pi$$

$= 2035.752$ cubic cm or 2.036 liters





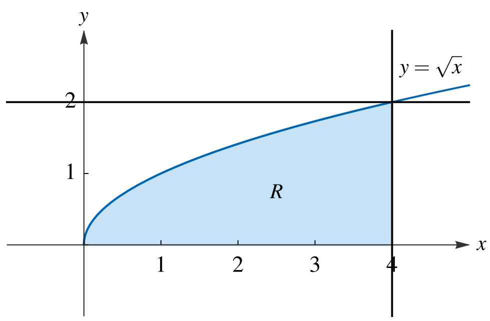
3. Let R be the region bounded by the graphs of $y = 2 - x^2$ and $y = e^x$. Find the volume of the solid generated when R is rotated about the x-axis.



$$2 - x^2 = e^x \quad \Rightarrow \quad a = -1.316, \quad b = 0.537$$

$$V = \pi \int_a^b [(2 - x^2)^2 - (e^x)^2]\, dx = 11.113$$

4. Let R be the region bounded by the graph of $y = \sqrt{x}$, the x-axis, and the vertical line $x = 4$. Let S_1 be the solid obtained by rotating the region R about the x-axis. Let S_2 be the solid obtained by rotating the region R about the line $y = 2$.



(a) Which solid, S_1 or S_2, has the greater volume? Show the calculations that support your conclusion.

$$V(S_1) = \pi \int_0^4 (\sqrt{x})^2\, dx = \pi \int_0^4 x\, dx$$

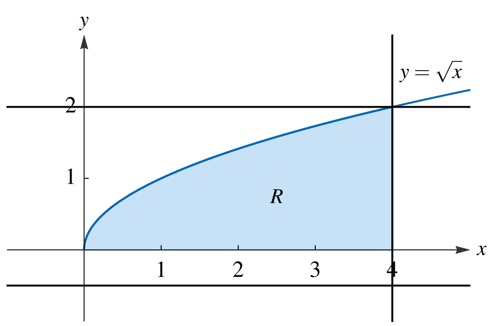
$$= \pi \left[\frac{x^2}{2}\right]_0^4 = \pi \cdot \frac{16}{2} = 8\pi$$

$$V(S_2) = \pi \int_0^4 \left[2^2 - (2 - \sqrt{x})^2\right] dx = \pi \int_0^4 (4 - (4 - 4\sqrt{x} + x))\, dx$$

$$= \pi \int_0^4 (4\sqrt{x} - x)\, dx = \pi \left[4 \cdot \frac{x^{3/2}}{3/2} - \frac{x^2}{2}\right]_0^4$$

$$= \pi \left(4 \cdot \frac{2}{3} \cdot 8 - 8\right) = \pi \left(\frac{64}{3} - 8\right) = \frac{40\pi}{3}$$

(b) There is a constant $c \neq 2$ such that the volume of the solid of revolution obtained by rotating the region R about the horizontal line $y = c$ is the same as the volume of S_2. Set up an equation involving integrals that could be used to solve for c, and use it to find c.



Solve the following equation for c:

$$\pi \int_0^4 \left[(\sqrt{x} - c)^2 - c^2\right] dx = \frac{40\pi}{3}$$

$$
\begin{aligned}
I &= \pi \int_0^4 \left[x - 2c\sqrt{x} + c^2 - c^2\right] dx = \pi \int_0^4 \left[x - 2c\sqrt{x}\right] dx \\
&= \pi \left[\frac{x^2}{2} - 2c\frac{x^{3/2}}{3/2}\right]_0^4 = \pi \left(8 - 2 \cdot c \cdot \frac{2}{3} \cdot 8\right) \\
&= \pi \left(8 - \frac{32c}{3}\right)
\end{aligned}
$$

Set this expression equal to the volume of the solid S_2.

$$
\begin{aligned}
\pi \left(8 - \frac{32c}{3}\right) &= \frac{40\pi}{3} \\
\pi \left(\frac{24 - 32c}{3}\right) &= \frac{40\pi}{3} \\
24 - 32c &= 40 \\
-32c &= 16 \quad \Rightarrow \quad c = -\frac{1}{2}
\end{aligned}
$$