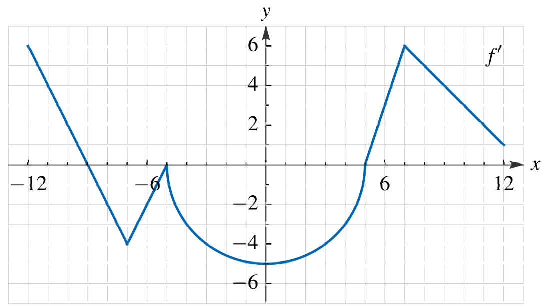
Monday Night Calculus

Function Analysis using Graphical Stems

11/9 Question

The graph of f', the derivative of a differentiable function f, is shown for $-12 \le x \le 12$. The graph consists of four line segments and a semicircle.



1. (a) Find all values of x in the interval $-12 < x < 12$, if any, at which f has a critical point. Classify each critical point as the location of a relative minimum, relative maximum, or neither, Justify your answers.

$f'(x) = 0$: $x = -9, \ -5, \ 5$

$f'(x)$ DNE: none

Critical points of f: $x = -9, \ -5, \ 5$

f has a relative maximum at $x = -9$ because f' changes from positive to negative there.

f has neither a relative minimum nor a relative maximum at $x = -5$ because f' does not change sign there.

f has a relative minimum at $x = 5$ because f' changes from negative to positive there.

(b) Find the values of x in the interval $-12 < x < 12$ at which f has an inflection point. Explain your reasoning.

f has an inflection point at the points where $x = -7, \ -5, \ 0, \ 7$ because f' changes from increasing to decreasing or vice versa at these values.

(c) For $-12 < x < 12$, find the open intervals on which f is decreasing and concave up. Explain your reasoning.

f is decreasing where f' is negative: $(-9, -5), \ (-5, 5)$.

f is concave up where f' is increasing: $(-7, -5), \ (0, 7)$.

f is decreasing and concave up: $(-7, -5), \ (0, 5)$

(d) For $-12 < x < 12$, find the open intervals on which f is increasing and concave down. Explain your reasoning.

f is increasing where f' is positive: $(-12, -9),\ (5, 12)$

f is concave down where f' is decreasing: $(-12, -7),\ (-5, 0),\ (7, 12)$

f is increasing and concave down: $(-12, -9),\ (7, 12)$

2. **(a)** It is known that $f(4) = -6$. Find an equation of the line tangent to the graph of f at $x = 4$.

Point: $(4, -6)$

Equation of the half-circle: $y = -\sqrt{25 - x^2}$

$f'(4) = -\sqrt{25 - 4^2} = -3$

An equation of the tangent line:

$y + 6 = -3(x - 4) \quad \Rightarrow \quad y = -3x + 6$

(b) Find $f''(4)$.

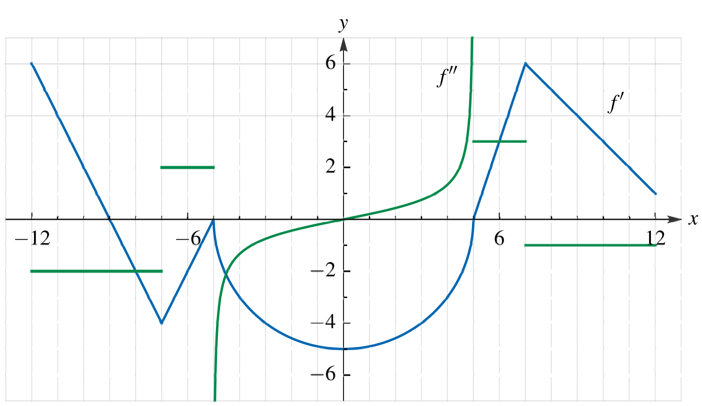
On the interval $(-5, 5)$: $\ f'(x) = -(25 - x^2)^{1/2}$

$$f''(x) = -\frac{1}{2}(25 - x^2)^{-1/2}(-2x) = \frac{x}{\sqrt{25 - x^2}}$$

$$f''(4) = \frac{4}{\sqrt{25 - 4^2}} = \frac{4}{3}$$

Note: An alternate solution involves a radius from the center of the circle to the point $(4, -3)$.

3. Let g be the function defined by $g(x) = f''(x)$. Sketch a graph of g over the open interval $-12 < x < 12$.



4. **(a)** Find a positive value a such that $f'(a) = f''(a)$. For this value of a, find $f'''(a)$.

Consider the graphs of f' and f''.

$f'(6) = 3$ and $f''(6) = 3$

$f'''(6) = 0$

(b) Is there a negative value x such that $f'(x) = f''(x)$? Explain why or why not.

Consider the graphs of f and f''.

There are two values of $x < 0$ such that $f'(x) = f''(x)$.

$f'(-8) = f''(-8) = -2$ and there is a value c, $-5 < c < -4$, such that $f'(c) = f''(c)$.

$$-\sqrt{25 - x^2} = \frac{x}{\sqrt{25 - x^2}} \quad \Rightarrow \quad x^2 - x - 25 = 0$$

$$x = \frac{-(-1) \pm \sqrt{1 - 4(1)(-25)}}{2(1)} = \frac{1 \pm \sqrt{101}}{2} = -4.525, \quad 5.525$$

$$x = \frac{1 - \sqrt{101}}{2}$$