Monday Night Calculus

Function Analysis using Graphical Stems

11/9 Question

The graph of $f'$, the derivative of a differentiable function $f$, is shown for $-12 \leq x \leq 12$. The graph consists of four line segments and a semicircle.

1. (a) Find all values of $x$ in the interval $-12 < x < 12$, if any, at which $f$ has a critical point. Classify each critical point as the location of a relative minimum, relative maximum, or neither. Justify your answers.

   $f'(x) = 0$: $x = -9, -5, 5$

   $f'(x)$ DNE: none

   Critical points of $f$: $x = -9, -5, 5$

   $f$ has a relative maximum at $x = -9$ because $f'$ changes from positive to negative there.

   $f$ has neither a relative minimum nor a relative maximum at $x = -5$ because $f'$ does not change sign there.

   $f$ has a relative minimum at $x = 5$ because $f'$ changes from negative to positive there.

(b) Find the values of $x$ in the interval $-12 < x < 12$ at which $f$ has an inflection point. Explain your reasoning.

   $f$ has an inflection point at the points where $x = -7, -5, 0, 7$ because $f'$ changes from increasing to decreasing or vice versa at these values.

(c) For $-12 < x < 12$, find the open intervals on which $f$ is decreasing and concave up. Explain your reasoning.

   $f$ is decreasing where $f'$ is negative: $(-9, -5), (-5, 5)$.

   $f$ is concave up where $f'$ is increasing: $(-7, -5), (0, 7)$.

   $f$ is decreasing and concave up: $(-7, -5), (0, 5)$
(d) For $-12 < x < 12$, find the open intervals on which $f$ is increasing and concave down. Explain your reasoning.

$f$ is increasing where $f'$ is positive: $(-12, -9), (5, 12)$
$f$ is concave down where $f'$ is decreasing: $(-12, -7), (-5, 0), (7, 12)$
$f$ is increasing and concave down: $(-12, -9), (7, 12)$

2. (a) It is known that $f(4) = -6$. Find an equation of the line tangent to the graph of $f$ at $x = 4$.

Point: $(4, -6)$
Equation of the half-circle: $y = -\sqrt{25 - x^2}$
$f'(4) = -\sqrt{25 - 4^2} = -3$
An equation of the tangent line:
$y + 6 = -3(x - 4) \Rightarrow y = -3x + 6$

(b) Find $f''(4)$.

On the interval $(-5, 5)$: $f''(x) = -(25 - x^2)^{1/2}$

$f''(x) = -\frac{1}{2}(25 - x^2)^{-1/2}(-2x) = \frac{x}{\sqrt{25 - x^2}}$

$f''(4) = \frac{4}{\sqrt{25 - 4^2}} = \frac{4}{3}$

Note: An alternate solution involves a radius from the center of the circle to the point $(4, -3)$.

3. Let $g$ be the function defined by $g(x) = f''(x)$. Sketch a graph of $g$ over the open interval $-12 < x < 12$. 

![Graph of f, f', and f'']
4. (a) Find a positive value \( a \) such that \( f'(a) = f''(a) \). For this value of \( a \), find \( f'''(a) \).

Consider the graphs of \( f' \) and \( f'' \).

\( f'(6) = 3 \) and \( f''(6) = 3 \)

\( f'''(6) = 0 \)

(b) Is there a negative value \( x \) such that \( f'(x) = f''(x) \)? Explain why or why not.

Consider the graphs of \( f' \) and \( f'' \).

There are two values of \( x < 0 \) such that \( f'(x) = f''(x) \).

\( f'(-8) = f''(-8) = -2 \) and there is a value \( c, -5 < c < -4 \), such that \( f'(c) = f''(c) \).

\[-\sqrt{25 - x^2} = \frac{x}{\sqrt{25 - x^2}} \Rightarrow x^2 - x - 25 = 0\]

\[x = \frac{-(-1) \pm \sqrt{1 - 4(1)(-25)}}{2(1)} = \frac{1 \pm \sqrt{101}}{2} = -4.525, \ 5.525\]

\[x = \frac{1 - \sqrt{101}}{2}\]