**Monday Night Calculus**

**Implicit Differentiation**

10/26 Question

**1.** Let $y = \tan^{-1} x$

**(a)** Find a formula for the derivative of the arctangent function, that is, find an expression for $\dfrac{dy}{dx}$.

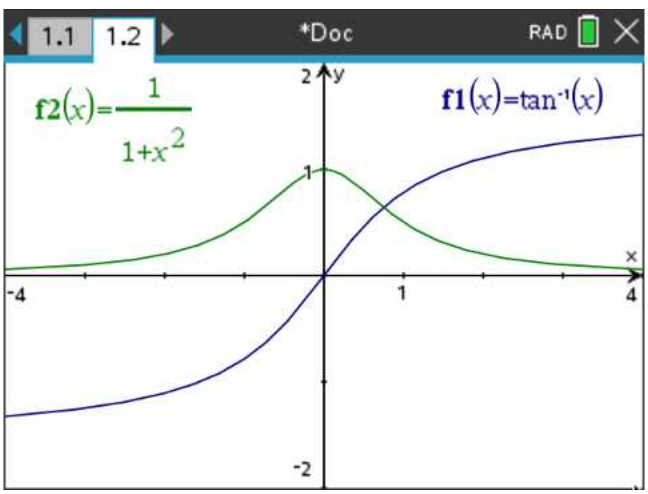
$y = \tan^{-1} x \quad \Rightarrow \quad \tan y = x$

$\sec^2 y \, \dfrac{dy}{dx} = 1$ — Differentiate implicitly with respect to $x$

$\dfrac{dy}{dx} = \dfrac{1}{\sec^2 y} = \dfrac{1}{1 + \tan^2 y} = \dfrac{1}{1 + x^2}$ — Solve for $\frac{dy}{dx}$; trig identity

$\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1 + x^2}$

**(b)** Graph the arctangent function and its derivative in the same viewing window. Describe the relationship between the two graphs.




This figure shows the graph of $f(x) = \tan^{-1} x$ and its derivative $f'(x) = \dfrac{1}{1 + x^2}$

Notice that $f$ is always increasing and, therefore, $f'(x)$ is always positive.

As $x \to \pm\infty$, $\tan^{-1} x \to \pm\dfrac{\pi}{2}$.

The graph of $f$ flattens out as $x$ increases and decreases without bound.
This behavior is indicated by: $f'(x) \to 0$ as $x \to \pm\infty$.

**2.** Find $\dfrac{dy}{dx}$ by implicit differentiation.

**(a)** $\sin(xy) = 1 + \cos y$

$$\cos(xy)(xy' + y) = -\sin y \cdot y'$$

$$xy' \cos(xy) + y' \sin y = -y\cos(xy)$$

$$y'(x\cos(xy) + \sin y) = -y\cos(xy)$$

$$y' = \frac{-y\cos(xy)}{x\cos(xy) + \sin y}$$

**(b)** $e^y \cos x = x + \cos y$

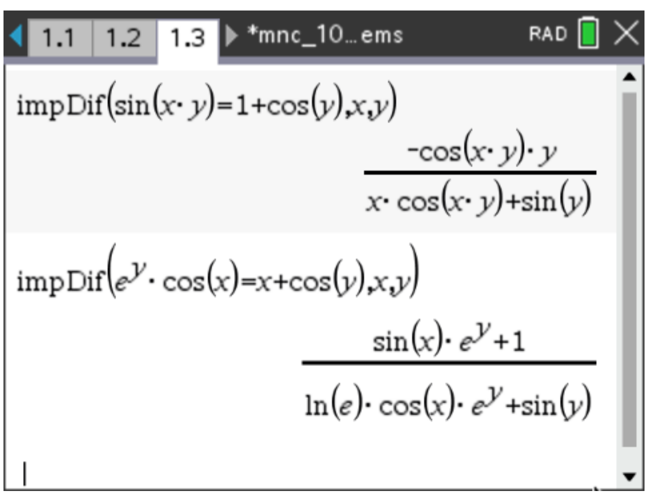
$$e^y(-\sin x) + e^y y' \cos x = 1 + (-\sin y)y'$$

$$y'e^y \cos x + y' \sin y = e^y \sin x + 1$$

$$y'(e^y \cos x + \sin y) = e^y \sin x + 1$$

$$y' = \frac{1 + e^y \sin x}{e^y \cos x + \sin y}$$

Here is some confirmation of these results using the TI-nspire.



**3.** Find an equation of the tangent line to the curve at the given point.

$x^2 + y^2 = (2x^2 + 2y^2 - x)^2 \qquad (0, -\tfrac{1}{2})$

$$2x + 2yy' = 2(2x^2 + 2y^2 - x)(4x + 4yy' - 1)$$

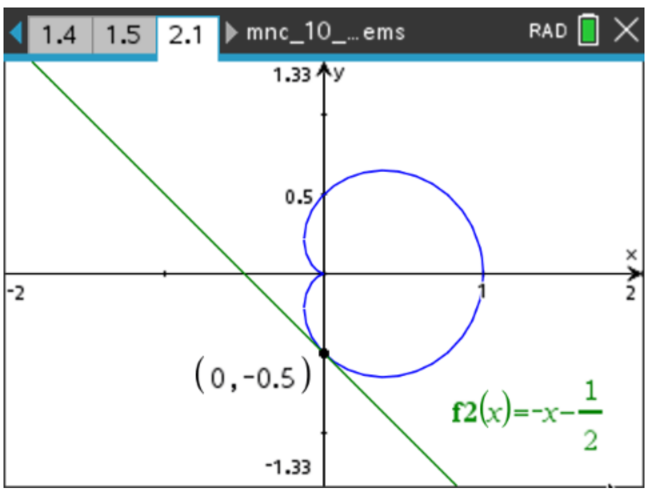
$$2(0) + 2\left(-\frac{1}{2}\right)y' = 2\left(2(0)^2 + 2\left(-\frac{1}{2}\right)^2 - 0\right)\left(4(0) + 4\left(-\frac{1}{2}\right)y' - 1\right)$$

$$-y' = (1)(-2y' - 1) \quad \Rightarrow \quad y' = -1$$

An equation of the tangent line:

$$y + \frac{1}{2} = (-1)(x - 0) \quad \Rightarrow \quad y = -x - \frac{1}{2}$$

This figure shows the graph of the expression and the tangent line.



**4.** The graph of the equation $y^2 = x^3 + 3x^2$ is called the Tschirnhausen cubic.

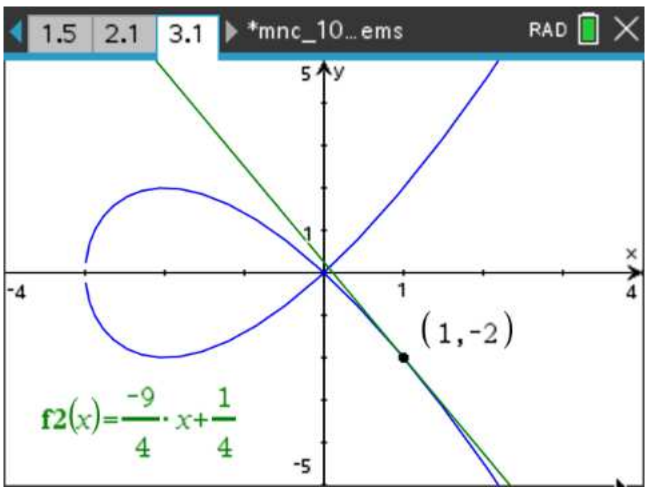
**(a)** Find an equation of the tangent line to this graph at the point $(1, -2)$.

$$2yy' = 3x^2 + 6x \quad \Rightarrow \quad y' = \frac{3x^2 + 6x}{2y}$$

At the point $(1, -2)$: $y' = \dfrac{3(1)^2 + 6(1)}{2(-2)} = -\dfrac{9}{4}$

An equation of the tangent line:

$$y + 2 = -\frac{9}{4}(x - 1) \quad \Rightarrow \quad y = -\frac{9}{4}x + \frac{1}{4}$$



**(b)** Find the points on this graph where the tangent line is horizontal.

$$y' = \frac{3x^2 + 6x}{2y} = 0 \quad \Rightarrow \quad 3x^2 + 6x = 3x(x+2) = 0 \quad \Rightarrow \quad x = 0, \; -2$$

$x = 0 \quad \Rightarrow \quad y = 0 \quad \Rightarrow \quad y'$ is in an indeterminate form.

Claim: The graph has two tangent lines at the origin. Slopes?

Therefore, the graph has horizontal tangent lines at the points: $(-2, -2), \; (-2, 2)$

**(c)** Graph the Tschirnhausen cubic and the horizontal tangent lines in the same viewing window.