Monday Night Calculus

Local Linearity and L'Hospital's Rule

10/12 Question

1. Find the limit, if it exists.

(a) $\displaystyle \lim_{x\to(\pi/2)} \frac{\cos x}{x-\frac{\pi}{2}}$

$$\lim_{x\to(\pi/2)} \cos x = 0 \quad \text{and} \quad \lim_{x\to(\pi/2)} \left(x-\frac{\pi}{2}\right) = 0$$

Therefore, $\displaystyle \lim_{x\to(\pi/2)} \frac{\cos x}{x-\frac{\pi}{2}}$ is in an indeterminate form of type $\frac{0}{0}$

$$\lim_{x\to(\pi/2)} \frac{\cos x}{x-\frac{\pi}{2}} = \lim_{x\to(\pi/2)} \frac{-\sin x}{1} \qquad \text{L'Hospital's Rule}$$

$$= \frac{-\sin\left(\frac{\pi}{2}\right)}{1} = -1 \qquad \text{Evaluate limits}$$

(b) $\displaystyle \lim_{x\to(\pi/2)} \frac{\sin x - 1}{x-\frac{\pi}{2}}$

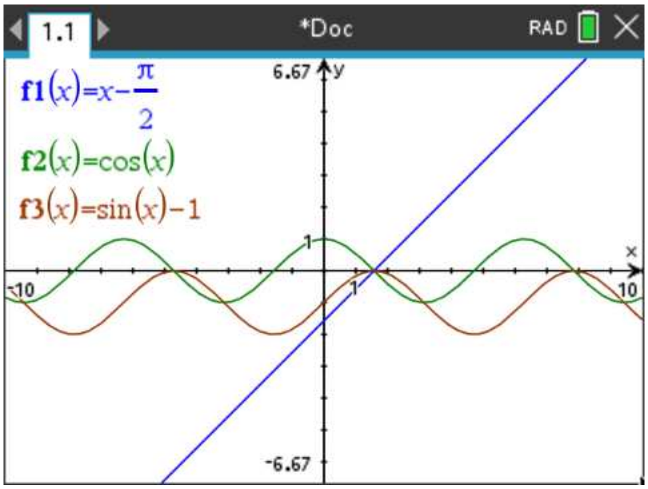
$$\lim_{x\to(\pi/2)} (\sin x - 1) = 1 - 1 = 0 \quad \text{and} \quad \lim_{x\to(\pi/2)} \left(x-\frac{\pi}{2}\right) = 0$$

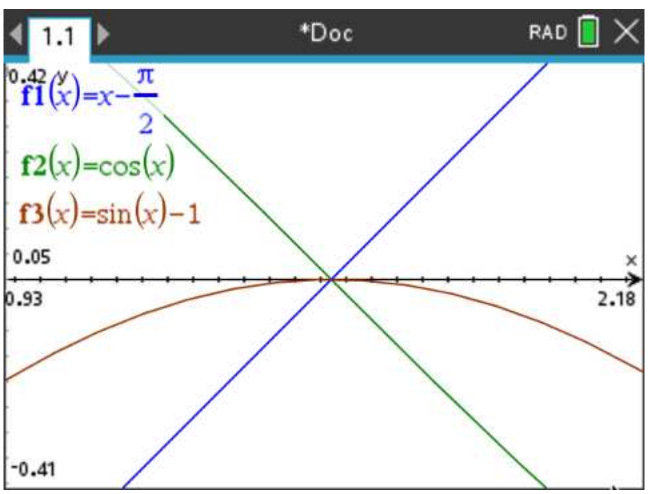
Therefore, $\displaystyle \lim_{x\to(\pi/2)} \frac{\sin x - 1}{x-\frac{\pi}{2}}$ is in an indeterminate form of type $\frac{0}{0}$

$$\lim_{x\to(\pi/2)} \frac{\sin x - 1}{x-\frac{\pi}{2}} = \lim_{x\to(\pi/2)} \frac{\cos x}{1} \qquad \text{L'Hospital's Rule}$$

$$= \frac{\cos\frac{\pi}{2}}{1} = \frac{0}{1} = 0 \qquad \text{Evaluate limits}$$

2. Sketch the graphs of $f(x) = x - \frac{\pi}{2}$, $g(x) = \cos x$, and $h(x) = \sin x - 1$ in the same viewing window. Zoom in on the graphs at the point $\left(\frac{\pi}{2}, 0\right)$ and use local linearity to explain how the graphs relate to the limits found in parts (a) and (b).





As $x \to \pi/2$:

- The slope of the graph of $y = \sin x - 1$ approaches $m = 0$.
- The slope of the graph of $y = \cos x$ approaches $m = -1$.
- The slope of the graph of $y = x - \frac{\pi}{2}$ approaches $m = 1$.

These observations support the results obtained using L'Hospital's Rule.

Note: These two examples of indeterminate forms $\frac{0}{0}$ are actually limits of difference quotients:

$f'\left(\frac{\pi}{2}\right)$ where $f(x) = \sin x$ and $g'\left(\frac{\pi}{2}\right)$ where $g(x) = \cos x$

3. Consider the limit $\lim_{x \to (\pi/2)} (\sec^2 x - \tan^2 x)$

(a) Find the value of the limit by writing $\sec x$ and $\tan x$ in terms of $\sin x$ and $\cos x$, and then using L'Hospital's Rule.

$$\lim_{x \to (\pi/2)} (\sec^2 x - \tan^2 x)$$

$$= \lim_{x \to (\pi/2)} \left(\frac{1}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x} \right) \qquad \text{Write in terms of } \sin x \text{ and } \cos x$$

$$= \lim_{x \to (\pi/2)} \frac{1 - \sin^2 x}{\cos^2 x} \qquad \text{Write as one fraction: } \tfrac{0}{0}$$

$$= \lim_{x \to (\pi/2)} \frac{-2 \sin x \cos x}{2 \cos x (-\sin x)} \qquad \text{L'Hospital's Rule}$$

$$= \lim_{x \to (\pi/2)} 1 = 1 \qquad \text{Evaluate limit}$$

(b) Find the value of the limit by using a trigonometric identity.

$$\lim_{x \to (\pi/2)} (\sec^2 x - \tan^2 x) = \lim_{x \to (\pi/2)} 1 = 1$$

4. Suppose f has a continuous derivative, and the line tangent to the graph of $y = f(x)$ at the point where $x = 5$ has the equation $y = 3x - 8$. Consider the limit

$$\lim_{x\to 5} \frac{f(x)^2 - 49}{x^2 - 25}$$

(a) Find the limit using L'Hospital's Rule.

$$\lim_{x\to 5}[f(x)^2 - 49] = f(5)^2 - 49 = 7^2 - 49 = 0$$

and

$$\lim_{x\to 5}(x^2 - 25) = 5^2 - 25 = 0$$

Therefore, $\lim_{x\to 5} \dfrac{f(x)^2 - 49}{x^2 - 25}$ is in an indeterminate form of type $\dfrac{0}{0}$

$$\lim_{x\to 5} \frac{f(x)^2 - 49}{x^2 - 25} = \lim_{x\to 5} \frac{2f(x)f'(x)}{2x} \qquad \text{L'Hospital's Rule}$$

$$= \frac{2 \cdot f(5) \cdot f'(5)}{2 \cdot 5} \qquad \text{Continuity}$$

$$= \frac{2 \cdot 7 \cdot 3}{10} = \frac{42}{10} = \frac{21}{5} \qquad \text{Evaluate limits}$$

(b) Find the limit by factoring the difference of squares in the numerator and denominator, and without using L'Hospital's Rule.

$$\lim_{x\to 5} \frac{f(x)^2 - 49}{x^2 - 25} = \lim_{x\to 5} \frac{(f(x)+7)(f(x)-7)}{(x+5)(x-5)} \qquad \text{Factor: difference of squares}$$

$$= \lim_{x\to 5} \frac{f(x)+7}{x+5} \cdot \lim_{x\to 5} \frac{f(x)-7}{x-5} \qquad \text{Limit law}$$

$$= \frac{14}{10} \cdot f'(5) = \frac{14}{10} \cdot 3 = \frac{21}{5} \qquad \text{Evaluate limits}$$

5. (Bonus Problems) Find the limit.

(a) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = 1^\infty$$

$$L = \left(1+\frac{1}{x}\right)^x \;\Rightarrow\; \ln L = \ln\left(1+\frac{1}{x}\right)^x = x\ln\left(1+\frac{1}{x}\right)$$

$$\lim_{x\to\infty}\ln L = \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) \qquad \infty\cdot 0 \text{ indeterminate form}$$

$$= \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \qquad \text{Rewrite: } \frac{0}{0}$$

$$= \lim_{x\to\infty}\frac{-\frac{1}{x^2}}{-\frac{1}{x^2}} \qquad \text{L'Hospital's Rule}$$

$$= \lim_{x\to\infty} 1 = 1 \qquad \text{Evaluate limit}$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e^1 = e$$

(b) $\lim_{x\to 0^+} (1-3x)^{2/x}$

$$\lim_{x\to 0^+} (1-3x)^{2/x} = 1^\infty$$

$$L = (1-3x)^{2/x} \quad \Rightarrow \quad \ln L = \frac{2}{x}\ln(1-3x)$$

$$\lim_{x\to 0^+} \ln L = \lim_{x\to 0^+} \frac{2\ln(1-3x)}{x}$$ Indeterminate form $\frac{0}{0}$

$$= \lim_{x\to 0^+} \frac{2\cdot\frac{-3}{1-3x}}{1}$$ L'Hospital's Rule

$$= \lim_{x\to 0^+} \frac{-6}{1-3x} = -6$$ Simplify; evaluate limit

$$\lim_{x\to 0^+} (1-3x)^{2/x} = e^{-6}$$