Monday Night Calculus

Derivatives of Trigonometric Functions

9/28 Question

1. Let $f(x) = 2x \sin x$.

(a) Find an equation of the tangent line to the graph of f at the point $\left(\frac{\pi}{2}, \pi\right)$.

(b) Illustrate your answer to part (a) by graphing f and the tangent line in the same viewing window.

Solution

(a) $f'(x) = (2x) \cdot \frac{d}{dx}(\sin x) + \frac{d}{dx}(2x) \cdot (\sin x)$ Product Rule

$= (2x) \cdot (\cos x) + 2 \cdot \sin x$ Basic derivatives

$= 2x \cos x + 2 \sin x$ Simplify

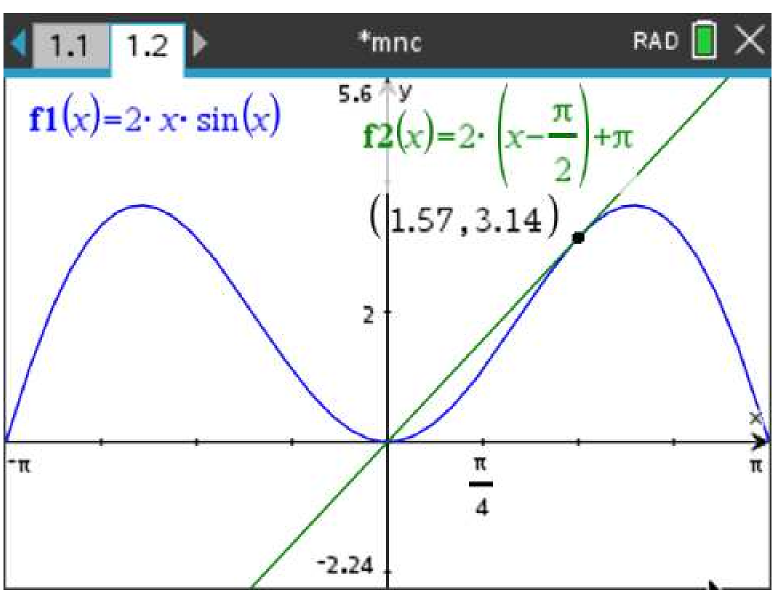
$$f'\left(\frac{\pi}{2}\right) = 2 \cdot \frac{\pi}{2} \cdot \cos \frac{\pi}{2} + 2 \sin \frac{\pi}{2}$$

$$= \pi \cdot 0 + 2 \cdot 1 = 2$$

An equation of the line tangent to the graph of f at the point $\left(\frac{\pi}{2}\right)$:

$$y - \pi = 2\left(x - \frac{\pi}{2}\right) \quad \Rightarrow \quad y = 2\left(x - \frac{\pi}{2}\right) + \pi$$

(b) A graph of f and the tangent line:



(b) Suppose $f\left(\frac{\pi}{3}\right) = -4$ and $f'\left(\frac{\pi}{3}\right) = 3$.

Let $g(x) = f(x)\sin x$, $h(x) = \dfrac{\cos x}{f(x)}$, and $j(x) = (g \circ h)(x)$.

(a) Find $g'\left(\frac{\pi}{3}\right)$.

(b) Find $h'\left(\frac{\pi}{3}\right)$.

(c) Find $j'\left(\frac{\pi}{3}\right)$.

Solution

(a)
$$g'(x) = f(x) \cdot \frac{d}{dx}(\sin x) + \frac{d}{dx}(f(x)) \cdot \sin x$$
$$= f(x) \cdot \cos x + f'(x) \cdot \sin x$$

$$g'\left(\frac{\pi}{3}\right) = f\left(\frac{\pi}{3}\right) \cdot \cos\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right) \cdot \sin\left(\frac{\pi}{3}\right)$$
$$= (-4) \cdot \left(\frac{1}{2}\right) + (3) \cdot \left(\frac{\sqrt{3}}{2}\right) = -2 + \frac{3\sqrt{3}}{2}$$

(b)
$$h'(x) = \frac{f(x)\dfrac{d}{dx}(\cos x) - \cos x \dfrac{d}{dx}(f(x))}{[f(x)]^2}$$
$$= \frac{-f(x)\sin x - f'(x)\cos x}{[f(x)]^2}$$

$$h'\left(\frac{\pi}{3}\right) = \frac{-f\left(\frac{\pi}{3}\right) \cdot \sin\left(\frac{\pi}{3}\right) - f'\left(\frac{\pi}{3}\right) \cdot \cos\left(\frac{\pi}{3}\right)}{\left[f\left(\frac{\pi}{3}\right)\right]^2}$$

$$= \frac{-(-4)\left(\frac{\sqrt{3}}{2}\right) - (3)\left(\frac{1}{2}\right)}{(-4)^2} = \frac{2\sqrt{3} - 3/2}{16} = \frac{1}{32}(4\sqrt{3} - 3)$$

(c) $j(x) = (g \circ h)(x) = g(h(x))$

$$j'(x) = g'(h(x)) \cdot h'(x)$$

$$h\left(\frac{\pi}{3}\right) = \frac{\cos\left(\frac{\pi}{3}\right)}{f\left(\frac{\pi}{3}\right)} = \frac{1/2}{-4} = -\frac{1}{8}$$

$$\begin{aligned}
j'\left(\frac{\pi}{3}\right) &= g'\left(h\left(\frac{\pi}{3}\right)\right) \cdot h'\left(\frac{\pi}{3}\right) \\
&= g'\left(-\frac{1}{8}\right) \cdot \left(-\frac{1}{32}(3 + 4\sqrt{3})\right) \\
&= \left(f\left(-\frac{1}{8}\right) \cdot \cos\left(-\frac{1}{8}\right) + f'\left(-\frac{1}{8}\right) \cdot \sin\left(-\frac{1}{8}\right)\right) \cdot \left(\frac{1}{32}(4\sqrt{3} - 3)\right)
\end{aligned}$$

(c) A particle moves along a horizontal line so that its position at time t, $t \geq 0$, is given by

$$s(t) = 4\cos t \sin t - 4\sin t$$

Find the first value of $t > 0$ for which the particle is at rest.

Solution

$$v(t) = s'(t) = 4[\cos t \cos t + (-\sin t)\sin t] - 4\cos t$$

$$= 4(\cos^2 t - \sin^2 t) - 4\cos t$$

$$= 4(2\cos^2 t - 1 - \cos t) = 4(2\cos t + 1)(\cos t - 1)$$

$v(t) = 0$

$$2\cos t + 1 = 0 \quad \Rightarrow \quad \cos t = -\frac{1}{2} \quad \Rightarrow \quad t = \frac{2\pi}{3}$$

$$\cos t - 1 = 0 \quad \Rightarrow \quad \cos t = 1 \quad \Rightarrow \quad t = 2\pi$$

The first value of $t > 0$ for which the particle is at rest is $t = \frac{2\pi}{3}$