Monday Night Calculus, November 1, 2021

1. If a function f is continuous for all x and if f has a relative maximum at $(-1, 4)$ and a relative minimum at $(3, -2)$, which of the following statements must be true? (Rachel Elaine)

(A) The graph of f has a point of inflection somewhere between $x = -1$ and $x = 3$.

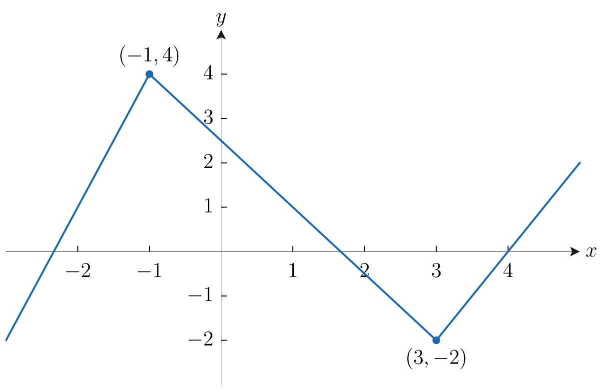
(B) $f'(-1) = 0$

(C) The graph of f has a horizontal asymptote.

(D) The graph of f has a horizontal tangent line at $x = 3$.

(E) The graph of f intersects both axes.

Solution



2. Let f be a function satisfying $f'(x) = 4x^4 + [f(x)]^2 + 2^x$ (for all x). Which of the following statements must be true? (Ruth LN Wunderlich)

(A) $f'(0) = 1$

(B) $f(x) > 0$ for all x

(C) The tangent line to the graph of f at every point has a positive slope.

(D) f is not differentiable at $x = 0$

(E) None of the above

Solution

(A) $f'(0) = 4 \cdot 0^4 + [f(0)]^2 + 2^0 = [f(0)]^2 + 1$

(B) $f(x) > 0$?

(C) $f'(x) = 4x^4 + [f(x)]^2 + 2^x$

(D) $f'(0) = 4 \cdot 0^4 + [f(0)]^2 + 2^0$

(E) None of the above

3. Consider the curve given by the equation $(2y+1)^3 - 24x = -3$. (Jenny Foreman)

(a) Show that $\dfrac{dy}{dx} = \dfrac{4}{(2y+1)^2}$

(b) Write an equation of the tangent line to the curve at the point $(-1,-2)$.

(c) Evaluate $\dfrac{d^2y}{dx^2}$ at the point $(-1,-2)$.

(d) The point $\left(\frac{1}{6}, 0\right)$ is on the curve. Find the value of $(y^{-1})'(0)$.

Solution

(a) $3(2y+1)^2 \cdot (2y') - 24 = 0$

$$2y' = \frac{24}{3(2y+1)^2} = \frac{8}{(2y+1)^2}$$

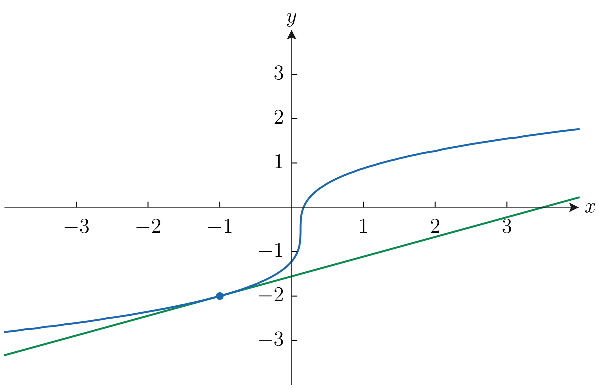
$$y' = \frac{4}{(2y+1)^2}$$

(b) Verify $(-1,-2)$ is on the curve.

$$(2(-2)+1)^3 - 24(-1) = (-3)^3 + 24 = -3$$

$$\left.\frac{dy}{dx}\right|_{(x,y)=(-1,-2)} = \frac{4}{(2(-2)+1)^2} = \frac{4}{(-3)^2} = \frac{4}{9}$$

An equation of the tangent line: $y + 2 = \dfrac{4}{9}(x+1)$



(c) $y' = \dfrac{4}{(2y+1)^2} = 4(2y+1)^{-2}$

$$\frac{d^2y}{dx^2} = 4(-2)(2y+1)^{-3} \cdot 2y' = \frac{-16}{(2y+1)^3} \cdot \frac{4}{(2y+1)^2} = \frac{-64}{(2y+1)^5}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(x,y)=(-1,-2)} = \frac{-64}{(2(-2)+1)^5} = \frac{-64}{(-3)^5} = \frac{64}{243}$$

(d)

Derivative of an Inverse Function

If f is a one-to-one differentiable function, its inverse f^{-1} is also differentiable and

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

provided the denominator is not 0.

$\left(\frac{1}{6}, 0\right)$ is on the graph of f, then $\left(0, \frac{1}{6}\right)$ is on the graph of f^{-1}.

$$(y^{-1})'(0) = \frac{1}{y'(y^{-1}(0))} = \frac{1}{y'\left(\frac{1}{6}\right)}$$

$$y'\left(\frac{1}{6}\right) = \frac{4}{(2 \cdot 0 + 1)^2} = 4$$

$$(y^{-1})'(0) = \frac{1}{4}$$

4. Sand is falling from a rectangular box, whose base measures 40 inches by 20 inches, at a constant rate of 300 cubic inches per minute. (Rick Eileen Aaron, edited)

(a) Find the rate of change of depth of sand in the box.

(b) The sand is forming a conical pile. At the instant when the pile is 23 inches high and the diameter is 16 inches, the diameter of the base is increasing at a rate of 1.5 inches per minute.

(i) Find the rate of change of the area of the circular base at this instant.

(ii) Find the rate of change of the height of the conical pile at this instant.

Solution

(a) $V = 40 \cdot 20 \cdot D$ — Volume of the box

$$\frac{dV}{dt} = 800 \cdot \frac{dD}{dt}$$ Derivative with respect to t

$$\frac{dD}{dt} = \frac{-300}{800} = -\frac{3}{8} \text{ in/m}$$ Solve for $\frac{dD}{dt}$

(b) **(i)** $A = \pi r^2 = \pi \left(\frac{D}{2}\right)^2 = \frac{\pi}{4} D^2$ — Area of the circular base

$$\frac{dA}{dt} = \frac{\pi}{4} \cdot 2D \cdot \frac{dD}{dt}$$ Derivative with respect to t

$$= \frac{\pi}{4} \cdot 2 \cdot 16 \cdot \frac{3}{2}$$ Given values

$$= 12\pi \text{ in}^2/\text{m}$$ Simplify

(ii) $V = \frac{1}{3}\pi r^2 h = \frac{\pi}{3}\left(\frac{D}{2}\right)^2 h = \frac{\pi}{12} D^2 h$ — Expression for volume

$$\frac{dV}{dt} = \frac{\pi}{12}\left(2D\frac{dD}{dt}h + D^2\frac{dh}{dt}\right)$$ Derivative with respect to t

$$300 = \frac{\pi}{12}\left(2 \cdot 16 \cdot \frac{3}{2} \cdot 23 + 16^2 \cdot \frac{dh}{dt}\right)$$ Given values

$$\frac{dh}{dt} = \frac{225 - 69\pi}{16\pi} = 0.164 \text{ in/m}$$ Solve for $\frac{dh}{dt}$

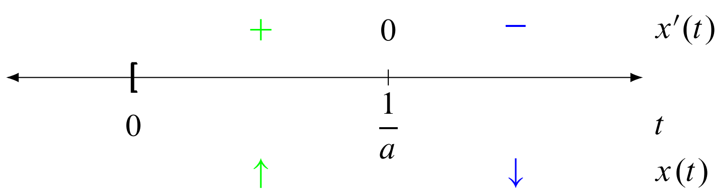
5. A particle moves along the x-axis so that at any time $t \geq 0$ its position is given by $x(t) = te^{-at}$, where a is a positive constant. At what time is the particle's position farthest to the right? (Melissa Helbert)

Solution

$$x'(t) = 1 \cdot e^{-at} + t \cdot (-ae^{-at})$$
$$= e^{-at}(1 - at)$$

$x'(t) = 0$: $1 - at = 0 \quad \Rightarrow \quad t = \dfrac{1}{a}$

$x'(t)$ DNE: none



The particle is farthest to the right when $t = \dfrac{1}{a}$.