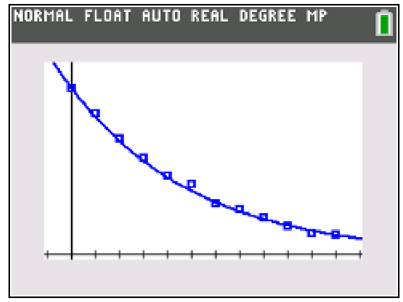
Exponential Dice



Math Objectives

- Students will analyze data determined by a simulation involving tossing dice.
- Students will find and analyze exponential growth and decay functions.
- Students will model with mathematics.
- Students will try to make a connection with how to understand these topics in IB Mathematics courses and on their final assessments.

Vocabulary

- exponential decay function
- exponential growth function
- half-life
- doubling time
- simulation

About the Lesson

- This lesson is aligning with the curriculum of IB Mathematics Applications and Interpretations (AI) SL/HL and IB Mathematics Approaches and Analysis (AA) SL/HL
- This falls under the IB Mathematics Content Topic 2 Functions:
  AI 2.5b Exponential growth and decay models
  AA 2.9a Exponential functions and their graphs
  AA 2.9b $f(x) = a^x, a > 0,\ f(x) = e^x$
- This lesson involves using a simulation to generate data that can be modeled by exponential growth and decay functions.
- As a result, students will:
  - Apply this information to real world situations by fitting a function to data using simulation, regression and theoretical analysis, and analyzing properties of growth and decay functions (i.e. doubling and half-life).

Teacher Preparation and Notes

- This activity is done with the use of the TI-84 family as an aid to the problems, specifically using Lists and the Probability Simulation App.

Activity Materials

- Compatible TI Technologies:

TI-84 Plus*, TI-84 Plus Silver Edition*, TI-84 Plus C Silver Edition, TI-84 Plus CE

** with the latest operating system (2.55MP) featuring MathPrint[TM] functionality.*

Tech Tips:

- This activity includes screen captures taken from the TI-84 Plus CE. It is also appropriate for use with the rest of the TI-84 Plus family. Slight variations to these directions may be required if using other calculator models.
- Watch for additional Tech Tips throughout the activity for the specific technology you are using.
- Access free tutorials at http://education.ti.com/calculators/pd/US/Online-Learning/Tutorials

Lesson Files:

Student Activity
Exponential_Dice_Student-84.pdf
Exponential_Dice_Student-84.doc

Teacher Note: It would be beneficial for students to clear all lists and functions. Press 2nd + and select **ClearAllLists**. Press y=, move to any equation that is defined and press clear.

Any quantity that grows or decays at a fixed rate at regular intervals grows or decays exponentially. Many real world phenomena can be modeled by exponential functions to show how things grow or decay over time. Examples of such phenomena include the studies of the population growth of people, bacteria, and viruses; the decay of radioactive substances; the change of temperatures; and the accumulation of interest or the payment of credit.

In Problem 1, a simulation involving dice will generate a data set that can be modeled by a function in the form $Y = a \cdot b^x$. You will use the Probability Simulation App on the TI-84 CE to help you find a value of a and then determine a possible value of b.
In the simulation, we toss a large number of dice, remove all the dice with certain face value(s) such as 6's, 3's and 4's, etc., and then repeat these two steps until only a few dice are left. To run the simulation, you will have to repeat the process detailed below on the Prob Sim App several times.

Imagine that a stomach bug is spreading through your school and you are trying to keep track of the number of students who have not yet become ill. You can suppose that:

- Each die represents a person.
- Each toss represents a week.
- If a ***6*** comes up, a student becomes ill, so remove that die from the population.

Teacher Tip: To help students understand what is happening during the simulation, demonstrate the process by conducting several trials of the simulation using a set of 30 – 50 dice. Even better, have each group of 2 – 4 students perform such a demonstration.

Running the Probability Simulation App:

1) Press the apps key on the handheld and scroll down to 0: Prob Sim and press enter or just press 0.
2) Scroll down to 2: Roll Dice or just press 2.
3) Since $a =$ initial number of dice, you will start with 200 dice to roll.
4) At the bottom of the Prob Sim App you see tabs, to start rolling the dice, press the ROLL tab, this will roll the first die.
5) Now the tabs at the bottom have changed to increase the number of rolls by 1, 10, or 50. Press the corresponding tabs until your initial roll count is 200.
6) Press the right arrow button and find the frequency of the sixes rolled. You will now subtract this amount from the starting number of dice (200) as this stands for the number of ill students removed.

7) Fill in the table below and repeat this process until very few students remain (at least 10 trials). You can start the simulation again by pressing the ESC tab (y =), ESC (y =), YES (y =), 2: ROLL DICE. Your dice count (number of dice remaining) should be the result after step 6.

Answer: Table values will vary, but should follow an exponential decay pattern.

Trial	Remaining Dice	Ratio
0	200	
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		

1. The first two columns contain the trial number and the number of dice remaining (number of students who have not yet become ill) after that trial.
   a. For the first function $Y_1 = a \cdot b^x$, explain why a is initial the number of dice.

Sample Answers: The initial value is $f(0) = a \cdot b^0 = a$.

In the third column, **ratio**, the ratios between consecutive entries in Column B, $\frac{b2}{b1}, \frac{b3}{b2}, \frac{b4}{b3}, \ldots$ etc., have been calculated.

b. Explain what each ratio represents.

Sample Answers: Each ratio is less than 1 and represents the fraction

$\frac{number\ of\ dice\ in\ trial(k+1)}{number\ of\ dice\ in\ trial(k)}$ for $k = 1, 2, 3, \ldots$

The value of b for the first function is the average value of these ratios. Find this average b.

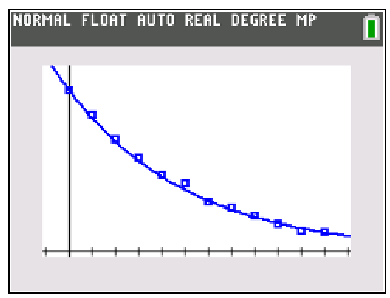
c. Explain why this value of b is a reasonable choice for the base of an exponential decay function to model this data.

Sample Answers: The average of the fractions of dice remaining from one trial to the next is a good estimate for the fixed rate of change that is the base for a decay model.

d. Record your first function here: $Y_1 =$ ______________________.

Sample Answers: $Y_1 = a \cdot b^x$ where *a* = 200 [entered] and *b* = mean(ratio) [calculated.] Students can enter $Y_1 = a \cdot b^x$ or the numerical values of a and b since 200 is stored in *a* and mean(ratio) is stored in *b*.

We now want to graph the data in the table you have created. Quit the simulation and from your home screen, press **stat**, **1: edit**. Enter the trial column data into L_1 and the remaining dice data into L_2. Now press **zoom**, **9: ZoomStat**.
Go to your y = screen and enter your function by typing $Y_1 = a \cdot b^x$, then press **graph**.



e. Explain how well the graph of your function fits the data.

Sample Answers: The graph of Y_1 is always above (below) the data points, so it does not fit the data very well. OR The graph of Y_1 has some data points slightly above the graph and other data points slightly below the graph, so it fits the data very well.

f. According to this model, approximate the percent of the dice that are being removed during each trial.

Sample Answers: $(1 - b) \cdot 100$; if $b = 0.85$ for example, then approximately 15% of the number of dice at one trial are removed for the next trial.

Now, find the regression equation to fit the data by going back to you home screen, press **stat**, **CALC**, **0: ExpReg**. Make sure the Xlist is L_1, the Ylist is L_2, and the Stor RegEq is Y_2, then press **enter** to calculate.

2. a. Record your regression function here: $Y_2 =$ ______________________.

Sample Answers: The regression function has the form $Y_2 = c \cdot d^x$ where *c* is around 200 and *d* is around 0.83.

Press graph to view the graph of the regression function on the scatterplot.

b. Discuss how the graph of the exponential regression function compares to that of your first function.

Sample Answers: The graph of Y_2 is always higher (lower) than that of Y_1; The graphs intersect and Y_2 is higher for larger values of x while Y_1 is higher for smaller values of x; etc.

Theoretically, you would expect that $\frac{1}{6}$ of the current number of dice would be removed at every trial.

3. a. For this situation, state the theoretical value for b.

Sample Answers: $b = \left(1 - \frac{1}{6}\right) = \frac{5}{6}$

b. Record your third function here: $Y_3 =$ ______________________ using this theoretical value of b and the initial value a you selected for the first function.

Sample Answers: $f3(x) = 200 \cdot \left(\frac{5}{6}\right)^x$

Graph this third function into Y_3 on the Y = screen.

4. Compute and interpret the following quantities;

a. $Y_1(6) - Y_3(6)$ and $Y_2(6) - Y_3(6)$

b. $Y_1(9) - Y_3(9)$ and $Y_2(9) - Y_3(9)$

c. $Y_1(12) - Y_3(12)$ and $Y_2(12) - Y_3(12)$

Note: It is possible to get an error message if fewer than 12 trials were needed in the simulation. If 18 or more trials were needed, you might want to compute these quantities when x = 15 or some larger value.

Sample Answers: Answers will vary. Type for example, $Y_1(6) - Y_3(6)$, and press enter to calculate the difference. The quantities $Y_1(k) - Y_3(k)$ and $Y_2(k) - Y_3(k)$ are the deviations between the values of the exponential functions based on the data and the theoretical function for any value of *k.* They will generally get smaller as the value of *k* increases.

Teacher Tip: You could ask the students to look carefully at the three graphs and decide which one, if any, best represents the data.

5. The **half-life** of a quantity whose value decreases with time is the length of time it takes for the quantity to decay to half of its initial value. Knowing the value of the half-life of various radioactive elements is sometimes used to determine the age of fossils and other natural objects.

   a. Find the half-life of this decay model using the exponential regression function, Y_2.

   Hint: You can use the “Numeric Solver” command from the home page pressing math, C: Numeric Solver.

Sample Answers: Answers will vary but can be found using "$nsolve(Y_2(x) = 100, x)$". Typical values are between 3 and 6.

   b. Find the half-life of this decay model using the theoretical exponential decay function, Y_3.

Sample Answers: Using "$nsolve(f3(x) = 100, x)$", $x \approx 3.80$.

Teacher Tip: You could mention that the theoretical value of the half-life is the solution to $\frac{1}{2} = b^x$ or $x = \frac{-\log 2}{\log b}$.

6. Suppose you ran another simulation where you removed all the 3’s and 4’s at each trial starting with 220 dice.

   a. Find the theoretical decay function, $g(x)$ for this situation.

   Record your answer here: $g(x)$ = _________

Sample Answers: $g(x) = 220 \cdot \left(\frac{2}{3}\right)^x$

b. Find the half-life of a decreasing quantity modeled by the function $g(x)$.

Sample Answers: $x \approx 1.71$

> **Teacher Tip:** Students could perform another simulation and use exponential regression to verify this theoretical model. It would be best to delete the graphs of Y1, Y2, and Y3 before moving to Problem 2, clicking on each equation, and clearing them.

Many things such as populations of people and animals grow at an exponential rate, In Problem 2, a simulation involving dice will generate a data set that can be modeled by a function in the form $Y = a \cdot b^x$. You will use the Probability Simulation App on the TI-84 CE to help you find a value of a and then determine a possible value of b.

In the simulation, we toss a small number of dice, add a die for each die with certain face value(s) such as 6's, 3's and 4's, etc. and then repeat these two steps until there are around 200 dice. To run the simulation, you will have to repeat the process detailed below on the Prob Sim App several times.

Imagine that you are keeping track of the deer population in a nearby animal park. You can suppose that

- Each die represents a deer
- Each toss represents a year.
- If a ***3 or 4*** comes up, a deer is born, so add a die to the population.

Running the Probability Simulation App:

1) Press the apps key on the handheld and scroll down to 0: Prob Sim and press enter or just press 0.
2) Scroll down to 2: Roll Dice or just press 2.
3) Since $a =$ initial number of dice, you will start with 3 – 5 dice to roll.
4) At the bottom of the Prob Sim App you see tabs, to start rolling the dice, press the ROLL tab, this will roll the first die.
5) Now the tabs at the bottom have changed to increase the number of rolls by 1, 10, or 50. Press the corresponding tabs until your initial roll count is 3 – 5.
6) Press the right arrow button and find the frequency of the threes and fours rolled. You will now add their sum total to the amount of starting dice (3 – 5) as this stands for the number of deer born.

7) Fill in the table below and repeat this process until around 200 dice are present (at least 7 – 10 trials). You can start the simulation again by pressing the ESC tab (y =), ESC (y =), YES (y =), 2: ROLL DICE. Your dice count (number of total dice) should be the result after step 6.

Answer: Table values will vary, but should follow an exponential growth pattern.

Trial	Total Dice	Ratio
0	3 to 5	
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		

7. The first two columns contain the trial number and the number of total dice (number of deer in the park population) after that trial.

   a. For the first function $Y_1 = a \cdot b^x$, explain why a is initial the number of dice.

Sample Answers: Each ratio is greater than 1 and represents the fraction

$$\frac{number\ of\ dice\ in\ trial(k+1)}{number\ of\ dice\ in\ trial(k)} \text{ for } k = 1, 2, 3, \ldots$$

In the third column, **ratio**, the ratios between consecutive entries in Column B, $\frac{b2}{b1}, \frac{b3}{b2}, \frac{b4}{b3}, \ldots$ etc., have been calculated.

   b. Explain what each ratio represents.

Sample Answers: The average of the fractions of dice added from one trial to the next is a good estimate for the fixed rate of change that is the base for a growth model.

The value of b for the first function is the average value of these ratios. Find this average b .

c. Explain why this value of b is a reasonable choice for the base of an exponential growth function to model this data.

Sample Answers: $Y_1 = a \cdot b^x$ where a = 4 [entered] and b = mean(ratio) [calculated]. Students can enter $Y_1 = a \cdot b^x$ or the numerical values of a and b since 4 is stored in a and mean(ratio) is stored in b.

d. Record your first function here: $Y_1 =$ ________________________.

Sample Answers: The graph of Y_1 is always above (below) the data points, so it does not fit the date very well. OR The graph of Y_1 has some data points slightly above the graph and other data points slightly below the graph, so it fits the data very well.

We now want to graph the data in the table you have created. Quit the simulation and from your home screen, press **stat**, **1: edit**. Clear your data from Problem 1 by moving to the top of the first column and press **clear**, **enter**. Repeat the process for the second column. Now, Enter the trial column data into L_1 and the total dice data into L_2. Now press **zoom**, **9: ZoomStat**.

Go to your y = screen and enter your function by typing $Y_1 = a \cdot b^x$, then press **graph**.

e. Explain how well the graph of your function fits the data.

Sample Answers: The graph of Y_1 is always above (below) the data points, so it does not fit the data very well. OR The graph of Y_1 has some data points slightly above the graph and other data points slightly below the graph, so it fits the data very well.

f. According to this model, approximate the percent of the dice that are being added during each trial.

Sample Answers: $(b-1) \cdot 100$; $b = 1.35$ for example, then approximately 35% of the number of dice at one trial are added for the next trial.

Now, find the regression equation to fit the data by going back to you home screen, press **stat**, **CALC**, **0: ExpReg**. Make sure the Xlist is L_1, the Ylist is L_2, and the Stor RegEq is Y_2, then press **enter** to calculate.

8. a. Record your regression function here: $Y_2 =$ _______________________.

   Sample Answers: The regression function has the form $Y_2 = c \cdot d^x$ where *c* is around 4 and *d* is around 1.33.

Press graph to view the graph of the regression function on the scatterplot.

   b. Discuss how the graph of the exponential regression function compares to that of your first function.

   Sample Answers: The graph of Y_2 is always higher (lower) than that of Y_1; The graphs intersect and Y_2 is higher for larger values of x while Y_1 is higher for smaller values of x; etc..

Theoretically, you would expect that $\frac{1}{3}$ of the current number of dice would be added at every trial.

9. a. For this situation, state the theoretical value for b.

   Sample Answers: $b = \left(1 + \frac{1}{3}\right) = \frac{4}{3}$

   b. Record your third function here: $Y_3 =$ _______________________ using this theoretical value of b and the initial value a you selected for the first function.

   Sample Answers: $Y_3 = 4 \cdot \left(\frac{4}{3}\right)^x$

Graph this third function into Y_3 on the Y = screen.

10. Compute and interpret the following quantities;

    a. $Y_1(6) - Y_3(6)$ and $Y_2(6) - Y_3(6)$

    b. $Y_1(9) - Y_3(9)$ and $Y_2(9) - Y_3(9)$

c. $Y_1(12) - Y_3(12)$ and $Y_2(12) - Y_3(12)$

Sample Answers: Answers will vary. Type for example, $Y_1(6) - Y_3(6)$, and press enter to calculate the difference. The quantities $Y_1(k) - Y_3(k)$ and $Y_2(k) - Y_3(k)$ are the deviations between the values of the exponential functions based on the data and the theoretical function for any value of *k.* They will generally get smaller as the value of *k* increases.

Note: It is possible to get an error message if fewer than 12 trials were needed in the simulation. If 18 or more trials were needed, you might want to compute these quantities when x = 15 or some larger value.

Teacher Tip: You could ask the students to look carefully at the three graphs and decide which one, if any, best represents the data.

11. The **doubling time** of a quantity whose value increases over time is the length of time it takes for the quantity to double in size. It is applied to population growth, inflation, compound interest, the volume of tumors, and many other things that tend to grow over time.
    a. Find the doubling time of this growth model using the exponential regression function, Y_2.

Hint: You can use the "Numeric Solver" command from the home page pressing math, C: Numeric Solver.

Sample Answers: Answers will vary, but can be found using "$nsolve(Y_2(x) = 8, x)$". Typical values are between 2 and 4.

b. Find the doubling time of this growth model using the theoretical exponential growth function, Y_3.

Sample Answers: Using "$nsolve(Y_3(x) = 8, x)$"; $x \approx 3.80$.

Teacher Tip: You could mention that the theoretical value of the half-life is the solution to $2 = b^x \; or \; x = \frac{\log 2}{\log b}$.

12. Suppose you added a die for each of the 3's, 5's, and 6's at each trial starting with 3 dice.

    a. Find the theoretical growth function, $g(x)$ for this situation.

    Record your answer here: $g(x)$ = ______________________

Sample Answers: $g(x) = 3 \cdot \left(\frac{3}{2}\right)^x$.

    b. Find the doubling time of an increasing quantity modeled by the function $g(x)$.

Sample Answers: $x \approx 1.71$.

> **Teacher Tip:** Students could perform another simulation and exponential regression to verify this theoretical model. It would be best to delete the graphs of Y1, Y2, and Y3 by going to the Y = screen, clicking on each equation, and then clearing them.

Further IB Extension

A mysterious virus has been spreading over the last several weeks since flu season began. Dr. Murphy and her team of researchers have been watching the spread closely and has modeled the data with the following function:

$$P = 750 + 325(1.375)^t \,, t \geq 0$$

Where t is the number of days since the start of flu season and P is the number of patients who have contracted this mysterious virus.

(a) i. Find the number of patients who contracted the virus at the start of flu season.

Answer: Substituting $t = 0$ (M1)
$P = 750 + 325(1.375)^0$
$P = 1075$ patients A1

ii. Find the number of patients who contracted the virus after 6 days. [4 marks]

Answer: Substituting $t = 6$ (M1)

$P = 750 + 325(1.375)^6$
$P \approx 2946$ patients A1

(b) Find how many days it will take to reach 20,000 patients who have contracted the virus.

[3 marks]

Answer: Method 1

Setting the function equal to 20,000 (M1)

$20{,}000 = 750 + 325(1.375)^t$

$19250 = 325(1.375)^t$

$\frac{19250}{325} = 1.375^t$

Converting to log form (M1)

$\mathrm{t} = \log_{1.375}\frac{19250}{325} \approx 12.8164\ldots$

$t \approx 13\ days$ A1

Method 2

Evidence of graphing the functions:

$f(x) = 20{,}000$ **and** $f(x) = 750 + 325(1.375)^x$ (M1)

Finding the point of intersection between the functions:

$(12.8164\ldots, 20000)$ (M1)

$t \approx 13\ days$ A1