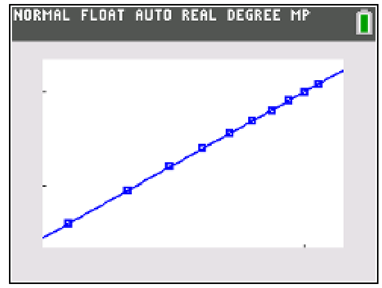
This activity explores the product property, the quotient property, and the power property of logarithms both algebraically and graphically.



For this activity, the expression used is $\log_2(x)$. The investigations also work for any base > 0 and base ≠ 1.

1. Fill in the following table, and with a classmate, discuss and answer the following questions.

(m, n)	$\log_2(m \cdot n)$	$\log_2(m) \cdot \log_2(n)$	$\log_2(m + n)$	$\log_2(m) + \log_2(n)$
$(4, 2)$				
$(4, 3)$				
$(4, 4)$				
$(4, 5)$				
$(4, 6)$				

a. Find which expressions, if any, appear to be equivalent independent of the values of *m* and *n.*

b. Using *m* = 8 and *n* = 4, substitute these values into the logarithmic expressions you found to be equivalent in part 1a, and simplify these expressions to show they are indeed equivalent.

c. Use the expressions you found in parts 1a and 1b to write a general logarithmic property for $\log_a mn$, where *a* is a real number, $a > 0$ and $a \neq 1$.

d. Explain how the operations in the logarithmic property in part 1c relate to the operations in the exponential property $a^m a^n = a^{m+n}$.

Now let's look at this same idea but graphically. Suppose you wanted to simplify the logarithm of a product, like $\log 6a$. Think about how you might go about doing this. Let's start by defining a new variable $\boldsymbol{b = 6a}$.

Step a: On your handheld, press stat, Edit, and in $\mathbf{L_1}$ enter at least 10 values for $\boldsymbol{a}$, that are in the domain of the logarithmic function.

Step b: At the top of $\mathbf{L_2}$, enter a formula that will calculate $\boldsymbol{b = 6a}$ from the values of $\mathbf{L_1}$.

Step c: Make a scatter plot of these values. Press **2nd**, **y =**, and select **Plot1**. Adjust the settings to display the a-values in $\mathbf{L_1}$ along the x-axis and the b-values in $\mathbf{L_2}$ along the y-axis.

Step d: Press **zoom**, **9: ZoomStat** to view the plot in an appropriate window.

2. Describe the shape of the graph. Discuss with a classmate if it is what you expected. Share your results with the class.

Step e: Now we will define two new variables, x and y. Let $\mathbf{x = \log \boldsymbol{a}}$ and $\mathbf{y = \log \boldsymbol{b}}$. At the top of $\mathbf{L_3}$ enter a formula that calculates x from the values of a in $\mathbf{L_1}$. At the top of $\mathbf{L_4}$ enter a formula that calculates y from the values of b in $\mathbf{L_2}$.

Step f: Make a scatter plot of y vs. x. Press **2nd**, **y =**, and select **Plot1** again. Adjust the settings to display the x-values in $\mathbf{L_3}$ along the x-axis and the y-values in $\mathbf{L_4}$ along the y-axis. Press **zoom**, **9:ZoomStat**.

3. Describe the shape of the graph. Discuss with a classmate if it is what you expected. Share your results with the class.

Step g: The data appear linear. Find the equation of a line through these points with the **LinReg(ax + b)** command. Press **stat**, **CALC**, **4: LinReg(ax + b)**. Make sure to fill in the appropriate Xlist and Ylist.

4. Write down the equation of the line through these points.

5. Find the y-intercept of the line.

Name ______________
Class ______________

You should have found that the equation of the line was $y = x + 0.778151$. Think about where this $y - intercept$ comes from. (Here's a hint: Try raising 10 to the 0.778151 power.)

6. Using logs, find what 0.778151 is.

7. Since $10^{0.778151} \approx$ __________, $\log(6) \approx$ ___________.

You have found that $y = \log 6 + x$. Think about what this means. Substitute to rewrite this as an equation in terms of $\boldsymbol{a}$. The explanation for each step is given to the right.

$y = \log 6 + x$	Equation of the line
	$x = \log a$ and $y = \log b$
	$b = 6a$

Product Property of Logarithms

For $a > 0$ and $b > 0$, $\log ab = \log a + \log b$.

Examples $\log xy$ is written in *expanded form* as $\log x + \log y$

$\log 7 + \log z$ is written as a single logarithm as $\log 7z$

8. Fill in the following table, and with a classmate, discuss and answer the following questions.

$(\boldsymbol{m}, \boldsymbol{n})$	$\log_2\left(\frac{m}{n}\right)$	$\frac{\log_2(m)}{\log_2(n)}$	$\log_2(m - n)$	$\log_2(m) - \log_2(n)$
$(4, 2)$				
$(4, 3)$				
$(4, 4)$				
$(4, 5)$				
$(4, 6)$				

a. Find which expressions, if any, appear to be equivalent independent of the values of *m* and *n.*

b. Using *m* = 8 and *n* = 4, substitute these values into the logarithmic expressions you found to be equivalent in part 8a, and simplify these expressions to show they are indeed equivalent.

c. Use the expressions you found in parts 8a and 8b to write a general logarithmic property for $\log_a\left(\frac{m}{n}\right)$ where a is a real number, $a > 0$ and $a \neq 1$.

d. Explain how the operations in the logarithmic property in part 8c relate to the operations in the exponential property $\frac{a^m}{a^n} = a^{m-n}$.

Again, let's look at this same idea but graphically. Suppose you wanted to simplify the logarithm of a quotient, like $\log\frac{8}{a}$. Think about how you might go about doing this. Let's start by defining a new variable $\boldsymbol{b = \frac{8}{a}}$.

Step a: Going back to your lists, clear the data from $\mathbf{L_2}$, $\mathbf{L_3}$, and $\mathbf{L_4}$ by going to the top of the list and pressing clear, enter. You will be leaving the data in $\mathbf{L_1}$ in as is.

Step b: At the top of $\mathbf{L_2}$, enter a formula that will calculate $\boldsymbol{b = \frac{8}{a}}$ from the values of $\mathbf{L_1}$.

Step c: Make a scatter plot of these values. Press **2nd**, **y =**, and select **Plot1**. Adjust the settings to display the a-values in $\mathbf{L_1}$ along the x-axis and the b-values in $\mathbf{L_2}$ along the y-axis.

Step d: Press **zoom**, **9: ZoomStat** to view the plot in an appropriate window.

9. Describe the shape of the graph. Discuss with a classmate if it is what you expected. Share your results with the class.

Step e: Now we will define two new variables, x and y. Let $\mathbf{x} = \boldsymbol{\log a}$ and $\mathbf{y} = \boldsymbol{\log b}$. At the top of $\mathbf{L_3}$ enter a formula that calculates x from the values of a in $\mathbf{L_1}$. At the top of $\mathbf{L_4}$ enter a formula that calculates y from the values of b in $\mathbf{L_2}$.

Step f: Make a scatter plot of y vs. x. Press **2nd**, **y =**, and select **Plot1** again. Adjust the settings to display the x-values in $\mathbf{L_3}$ along the x-axis and the y-values in $\mathbf{L_4}$ along the y-axis. Press **zoom**, **9:ZoomStat**.

Step g: The data appear linear. Find the equation of a line through these points with the **LinReg(ax + b)** command. Press **stat**, **CALC**, **4: LinReg(ax + b)**. Make sure to fill in the appropriate Xlist and Ylist.

10. Write down the equation of the line through these points.

11. Find the y-intercept of the line.

You should have found that the equation of the line was $y = 0.90309 - x$. Think about where this $y - intercept$ comes from.

12. Using logs, find what 0.90309 is.

13. Since $10^{0.90309} \approx$ ___________, $\log(8) \approx$ ____________.

You have found that $y = \log 8 - x$. Think about what this means. Substitute to rewrite this as an equation in terms of $\boldsymbol{a}$. The explanation for each step is given to the right.

$y = \log 8 - x$	Equation of the line
	$x = \log a$ and $y = \log b$
	$b = \frac{8}{a}$

Quotient Property of Logarithms

For $a > 0$ and $b > 0$, $\log ab = \log a + \log b$.

Examples $\log \frac{x}{y}$ is written in *expanded form* as $\log x - \log y$

$\log 7 - \log z$ is written as a single logarithm as $\log \frac{7}{z}$

14. Fill in the following table, and with a classmate, discuss and answer the following questions.

$(\boldsymbol{m}, \boldsymbol{n})$	$\mathbf{log_2}(\boldsymbol{m}^{\boldsymbol{n}})$ or $\mathbf{log_2}(\boldsymbol{m})^{\boldsymbol{n}}$	$(\mathbf{log_2}\, \boldsymbol{m})^{\boldsymbol{n}}$	$\boldsymbol{n} \cdot \mathbf{log_2}(\boldsymbol{m})$
$(4, 2)$			
$(4, 3)$			
$(4, 4)$			
$(4, 5)$			
$(4, 6)$			

a. Find which expressions, if any, appear to be equivalent independent of the values of *m* and *n.*

b. Using $m = 4$ and $n = 3$, substitute these values into the logarithmic expressions you found in part 14a, and simplify these expressions to show they are equivalent.

c. Use the expressions you found in parts 14a and 14b to write a general logarithmic property for $\log_a(m)^n$ where *a* is a real number, $a > 0$ and $a \neq 1$

d. Explain how the operations in the logarithmic property in part 14c relate to the operations in the exponential property $\left(a^m\right)^n = a^{mn}$.

e. Use the logarithmic property you proved in part 14c to show that $\log_a a = 1$ for all values of *a* where $a > 0$ and $a \neq 1$.

f. Use the logarithmic property you proved in part 14c to show that $\log_a 1 = 0$ for all values of *a* where $a > 0$ and $a \neq 1$.

One final time, let's look at this same idea but graphically. Suppose you wanted to simplify the logarithm of a power, like $\log a^2$. Think about how you might go about doing this. Let's start by defining a new variable $\boldsymbol{b = a^2}$.

Step a: Going back to your lists, clear the data from **L_2**, **L_3**, and **L_4** by going to the top of the list and pressing clear, enter. You will be leaving the data in **L_1** in as is.

Step b: At the top of **L_2**, enter a formula that will calculate $\boldsymbol{b = a^2}$ from the values of **L_1**.

Step c: Make a scatter plot of these values. Press **2nd**, **y =**, and select **Plot1**. Adjust the settings to display the a-values in **L_1** along the x-axis and the b-values in **L_2** along the y-axis.

Step d: Press **zoom**, **9: ZoomStat** to view the plot in an appropriate window.

15. Describe the shape of the graph. Discuss with a classmate if it is what you expected. Share your results with the class.

Step e: Now we will define two new variables, x and y. Let $\mathbf{x} = \mathbf{\log a}$ and $\mathbf{y} = \mathbf{\log b}$. At the top of $\mathbf{L_3}$ enter a formula that calculates x from the values of a in $\mathbf{L_1}$. At the top of $\mathbf{L_4}$ enter a formula that calculates y from the values of b in $\mathbf{L_2}$.

Step f: Make a scatter plot of y vs. x. Press **2^nd**, **y =**, and select **Plot1** again. Adjust the settings to display the x-values in $\mathbf{L_3}$ along the x-axis and the y-values in $\mathbf{L_4}$ along the y-axis. Press **zoom**, **9:ZoomStat**.

Step g: The data appear linear. Find the equation of a line through these points with the **LinReg(ax + b)** command. Press **stat**, **CALC**, **4: LinReg(ax + b)**. Make sure to fill in the appropriate Xlist and Ylist.

16. Write down the equation of the line through these points.

17. Find the y-intercept of the line.

You should have found that the equation of the line was $y = 2x$. Think about what this means.

You have found that $y = \log 6 + x$. Think about what this means. Substitute to rewrite this as an equation in terms of $\boldsymbol{a}$. The explanation for each step is given to the right.

$y = 2x$	Equation of the line
$\log b = 2\log a$	$x = \log a$ and $y = \log b$
$\log a^2 = 2\log a$	$b = a^2$

Power Property of Logarithms
For $a > 0$, $\log a^b = b\log a$.

Examples $\log x^3$ can be written as $3\log x$
$8\log x$ can be written as $\log x^8$

Further IB Math Extension

Using the properties discussed in this activity, find the solution of:

$$\log_3 x - 2\log_3 2 = 3 - \log_3 2$$