Mathematical Methods (CAS) 2003 *Examination* 1 *Part* 2 *solutions*

Note: To use *Derive* efficiently, students should be familiar with the 'tick plus equals' and 'tick plus approximately equals' evaluation buttons. These simultaneously 'author' and 'evaluate' expressions exactly and numerically respectively.For example, the 'tick plus equals' or 'author and exact evaluation' button works well for Questions 1, 2, 3, 5 and 6, while the 'tick plus approximately equals' or 'author and numerical approximation' works well for Question 4. Students should also be familiar with the use of defined functions in the form f(x):= *rule of the function*, such as in the sample solutions for Questions 1, 2 and 5.

Question 1

For this question it is useful to define the function f, and use it to write down the equations involving values of the function and its derivative, although these are not required to be written explicitly. These two equations can then be solved expressing a and b in terms of c.

#1: $f(x) := a\cdot x^2 + b\cdot x + c$

#2: $f(1) = 6$

#3: $a + b + c = 6$

#4: $f'(1) = 4$

#5: $2\cdot a + b = 4$

#6: $\text{SOLVE}([f(1) = 6, f'(1) = 4], [a, b])$

#7: $[a = c - 2 \wedge b = 2\cdot(4 - c)]$

#8: $\text{SOLVE}\left(\left[f(1) = 6, f'(1) = 4, \int_0^1 f(x)\,dx = 6\right], [a, b, c]\right)$

#9: $[a = 6 \wedge b = -8 \wedge c = 8]$

Alternatively could replace the a and b in f by the corresponding function of c and solve in one hit.

#10: $\text{SOLVE}\left(\int_0^1 ((c - 2)\cdot x^2 + 2\cdot(4 - c)\cdot x + c)\,dx = 6, c\right)$

#11: $c = 8$

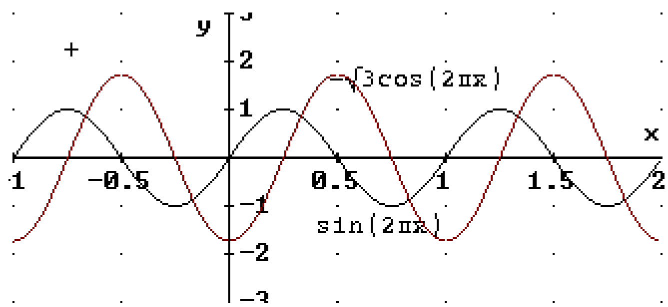
Question 2

We always have to be careful finding exact solutions to trig equations using Derive. It may not give us the ones required by the question.

#12: $\mathrm{SIN}(2\cdot\pi\cdot x) = -\sqrt{3}\cdot\mathrm{COS}(2\cdot\pi\cdot x)$

#13: $\mathrm{SOLVE}(\mathrm{SIN}(2\cdot\pi\cdot x) = -\sqrt{3}\cdot\mathrm{COS}(2\cdot\pi\cdot x), x, \mathrm{Real})$

#14: $$x = -\frac{1}{6} \vee x = -\frac{2}{3} \vee x = \frac{1}{3}$$



We can see that the curves intersect twice for $0 \le x \le 1$. Both curves have period 1, so 1 + any solution will also be a solution. Hence $-1/6 + 1 = 5/6$, giving this as the other solution.
What if instead we rearranged the equation to get an equation in terms of tan?

#15: $\mathrm{TAN}(2\cdot\pi\cdot x) = -\sqrt{3}$

#16: $\mathrm{SOLVE}(\mathrm{TAN}(2\cdot\pi\cdot x) = -\sqrt{3}, x, \mathrm{Real})$

#17: $$x = -\frac{1}{6} \vee x = -\frac{2}{3} \vee x = \frac{1}{3}$$

Just the same. We need to know that the tan function given has period 1, so again if we add 1 to any solution we will get another solution.
Overall, the easiest way to tackle this question may have been by hand, by first rearranging to a simple equation involving the tan function, which has solutions $2\pi x = -\pi/3 + n\pi$ for integers n, and then determining which lie in the appropriate interval. Answer:

#18: $$x = \frac{1}{3} \vee x = \frac{5}{6}$$

Question 3

It is useful to define the function initially, and then use "tick plus equals” to find the intercepts.

#19: $f(x) :\!= 2\cdot LN(|x + 3|) + 1$

#20: $f(0)$

#21: $2\cdot LN(3) + 1$

Hence cuts y–axis at point (0, 2ln(3)+1)

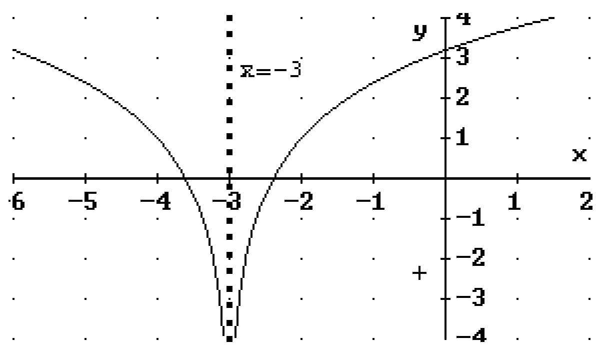
#22: SOLVE(f(x) = 0, x)

#23: $x = - e^{- 1/2} - 3 \vee x = e^{- 1/2} - 3$

#24: $x = -3.606530659 \vee x = -2.39346934$

Hence cuts x axis at the points (–exp(–0.5)–3, 0) and ((exp(–0.5)–3, 0), (or approximately
(–3.6,0) and (–2.4, 0)).
To get a vertical line dotted for the asymptote, you need to specify the equation parametrically, and then plot it for suitable values of t using points. Plotting the following this way will give us the asymptote x = –3 on the graph.

#25: [–3, t]



Question 4

Can choose to enter the expressions separately and combine as appropriate, or enter the equality and edit it for the integral later. You should always have bounds for the numerical solver, which you can get from the graph given (and determining where y = –x+1 cuts the x–axis), or draw your own graph.

#26: $-x + 1 = 1 - e^{-x}$

#27: $NSOLVE(-x + 1 = 1 - e^{-x}, x, 0, 1)$

#28: $x = 0.5671432904$

and so k = 0.567 correct to three decimal places.
The correct integral expression involves k, but you need to replace k by at least its three decimal place approximation to determine the value of the integral correct to two decimal places.

#29: $\int_0^k (-x + 1 - (1 - e^{-x}))\, dx$

#30: $\int_0^{0.567143} (-x + 1 - (1 - e^{-x}))\, dx$

#31: 0.2720309536

and so the area of the shaded region correct to two decimal places is 0.27.

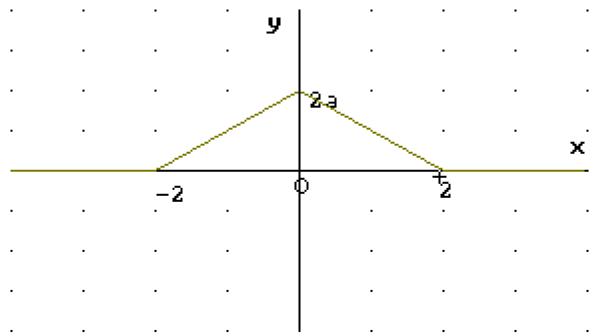
Question 5

This question can easily be done by hand. a is only a scale factor, so it is a simple matter of interpreting the function, in particular, the absolute value part. If $0 \le x < 2$, $f(x) = 2-x$ and if $-2 < x < 0$, $f(x) = 2 - (-x) = 2 + x$.
Using Derive, can define f, possibly omitting a since it is only a scale factor.

#32: $f(x) :\!= 2 - |x|$

To get an appropriate graph, we need to use embedded IF statements, and put our own labels on the axes. By hand it is a simple matter of plotting the four line segments for $x < -2$, $-2 < x < 0$, $0 \le x < 2$ and $X > 2$, and noting that it will cut the y-axis at 2a.

#33: IF(x < -2, 0, IF(x < 2, f(x), 0))



To get the value of a, the easiest thing to do is observe that the area under the graph is a triangle, of area 0.5 * 4 * 2a, and this must be equal to 1 for a probability density function, and hence a = 1/4, or if you must use Derive:

#34: $\text{SOLVE}(0.5\cdot4\cdot2\cdot a = 1,\ a)$

#35: $$a = \frac{1}{4}$$

Alternatively, could integrate the density function between -2 and 2 to get an expression in a, which must be equal to 1, and then solve for a.

#36: $\text{SOLVE}\left(\int_{-2}^{2} a\cdot f(x)\ dx = 1,\ a\right)$

#37: $$a = \frac{1}{4}$$

Question 6

Kim works out in the gym on Monday. The probability that she works out in the gym on Tuesday will be one third. If she works out in the gym on Tuesday, the probability that she works out in the gym on Wednesday will be one-third. hence if she works out in the gym on Monday, the probability that she works out in the gym on Tuesday and Wednesday will be

#38: $\frac{1}{3}\cdot\frac{1}{3}$

#39: $$\frac{1}{9}$$

We can use transition matrices to solve this. Firstly, enter the transition matrix. Column 1 contains the transition probabilities of going from a swim to either a swim or the gym in one transition , and column 2 contains the transition probabilities of going from the gym to either a swim or the gym in one transition. Use the matrix button on the toolbar to enter the matrix.

#40: $\begin{bmatrix} 0 & \frac{2}{3} \\ 1 & \frac{1}{3} \end{bmatrix}$

If we are interested in what happens on the Friday, that is, 4 days or transitions from the Monday, need to raise this to the power 4.

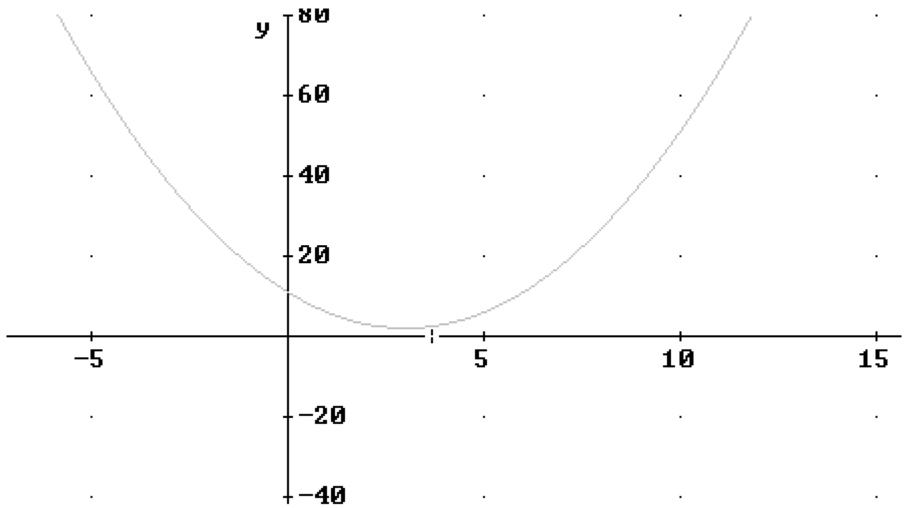
#41: $\begin{bmatrix} 0 & \frac{2}{3} \\ 1 & \frac{1}{3} \end{bmatrix}^4$

#42: $\begin{bmatrix} \frac{14}{27} & \frac{26}{81} \\ \frac{13}{27} & \frac{55}{81} \end{bmatrix}$

The element in the (i,j) position here gives the probability of going from state j to state i in 4 transitions.
Alternatively, can enter a 2 by 1 matrix (or a vector) corresponding to the initial state, working out in the gym on Monday, and left multiply this by the transition matrix raised to the power 4.

#43: $\begin{bmatrix} \frac{14}{27} & \frac{26}{81} \\ \frac{13}{27} & \frac{55}{81} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

#44: $$\begin{bmatrix} \dfrac{26}{81} \\ \dfrac{55}{81} \end{bmatrix}$$



Note that this multiplication just picks out the second column of the fourth power of the transition matrix anyway. The probability that she has a swim on Friday is 26/81.